SECANTS, TANGENTS AND THE HOMOGENEITY OF FREUDENTHAL VARIETIES OF CERTAIN TYPE

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0. Introduction

H. Freudenthal constructed, in a series of his papers (see [8] and its references), the exceptional Lie algebras of type E_8, E_7, E_6 and F_4, with defining various projective varieties. The purpose of our work is to study projective geometry for his varieties of certain type, which are called *varieties of planes in the symplectic geometry of Freudenthal* (see [8, 4.11], [20, 2.3]). Precisely speaking, for a given simple, graded Lie algebra $\mathfrak{g} = \Sigma \mathfrak{g}_i$ of contact type over the complex number field $\mathbb{C}$ (see [23] and §1 below), set

$$\mathcal{V} := \{x \in \mathfrak{g}_1 | x \neq 0, (\operatorname{ad} x)^2 \mathfrak{g}_{-2} = 0\},$$

and define an algebraic set V in $\mathbb{P}_*(\mathfrak{g}_1)$ to be the projectivization of $\mathcal{V}$:

$$V := \pi \mathcal{V},$$

where $\pi : \mathfrak{g}_1 \setminus \{0\} \to \mathbb{P}_*(\mathfrak{g}_1)$ is the natural projection. Then we call $V \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ (with the reduced structure) the *Freudenthal variety* associated to the graded Lie algebra $\mathfrak{g}$, which is a natural generalization of Freudenthal's varieties mentioned above: Note that V is not necessarily connected in this general setting.

Moreover set

$$\begin{aligned}
\mathcal{S} &:= \{x \in \mathfrak{g}_1 | (\operatorname{ad} x)^4 \mathfrak{g}_{-2} \neq 0\}, \\
\mathcal{T} &:= \{x \in \mathfrak{g}_1 | (\operatorname{ad} x)^3 \mathfrak{g}_{-2} \neq 0, (\operatorname{ad} x)^4 \mathfrak{g}_{-2} = 0\}, \\
\mathcal{U} &:= \{x \in \mathfrak{g}_1 | (\operatorname{ad} x)^2 \mathfrak{g}_{-2} \neq 0, (\operatorname{ad} x)^3 \mathfrak{g}_{-2} = 0\}.
\end{aligned}$$

Then we have the following stratification of $\mathfrak{g}_1$:

$$\mathfrak{g}_1 = \mathcal{S} \sqcup \mathcal{T} \sqcup \mathcal{U} \sqcup \mathcal{V} \sqcup \{0\}.$$

In the literature, several results have been known about the structure of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-space, case-by-case for each exceptional Lie algebra of types E_8, E_7, E_6 and F_4, from the view-point of the invariant theory of prehomogeneous vector spaces (see [11], [13], [16], [19]). By virtue of those results, it can be shown that the stratification gives the orbit decomposition of the $\mathfrak{g}_0$-space $\mathfrak{g}_1$ for those exceptional Lie algebras, and also that Freudenthal varieties V associated to the algebras of type E_8, E_7, E_6 and F_4 are

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respectively projectively equivalent to the 27-dimensional E_7-variety arising from the 56-dimensional irreducible representation, the orthogonal Grassmann variety of isotropic 6-planes in $\mathbb{C}^{12}$ (namely, the 15-dimensional spinor variety), the Grassmann variety of 3-planes in $\mathbb{C}^6$ and the symplectic Grassmann variety of isotropic 3-planes in $\mathbb{C}^6$, with $\dim \mathbb{P}_*(\mathfrak{g}_1) = 55, 31, 19$ and 13, respectively: for those homogeneous projective varieties, we refer to [10, §23.3].

In this article we study the Freudenthal varieties V with the stratification of the ambient space $\mathbb{P}_*(\mathfrak{g}_1)$, from the view-point of projective geometry, not individually but systematically in terms of abstract Lie algebras, without depending on the classification of simple Lie algebras as well as on the known results for each case of types E_8, E_7, E_6 and F_4.

As a consequence of our work, it turns out that the stratification of $\mathfrak{g}_1$ is closely related to the secant and tangent loci of $V \subseteq \mathbb{P}_*(\mathfrak{g}_1)$. In particular, for the exceptional types, our work yields a projective-geometric characterization for $\mathfrak{g}_0$-orbits in $\mathfrak{g}_1$. Here, the *secant locus* Σ_P as well as the *tangent locus* Θ_P of V with respect to a given point $P \in \mathbb{P}_*(\mathfrak{g}_1)$ are defined by

$$\Sigma_P^\circ := \{Q \in V | \exists R \in V \setminus \{Q\}, P \in Q * R\}, \quad \Sigma_P := \overline{\Sigma_P^\circ},$$
$$\Theta_P := \{Q \in V | P \in T_Q V\},$$

where we denote by $Q * R$ the projective line in $\mathbb{P}_*(\mathfrak{g}_1)$ through Q and R, namely, the *secant line* of V determined by Q and R, and by $T_Q V$ the embedded tangent space to V at Q in $\mathbb{P}_*(\mathfrak{g}_1)$ (see, for example, [9]).

To state the consequence, we need some notation: Let H be the characteristic element of the gradation $\mathfrak{g} = \sum \mathfrak{g}_i$, take $E_+ \in \mathfrak{g}_2$ and $E_- \in \mathfrak{g}_{-2}$ such that (E_+, H, E_-) form an $\mathfrak{sl}_2$-triple, and define a homogeneous quartic polynomial q over $\mathfrak{g}_1$ as follows:

$$2q(x)E_+ = (\operatorname{ad} x)^4 E_-.$$

Then, the consequence is

Theorem A. *We have:*

(1) *If $s \in \mathcal{S}$, then*

$$\Sigma_{\pi s} = \Sigma_{\pi s}^\circ = \left\{\pi\left((\operatorname{ad} s)^3 E_- \pm \sqrt{3q(s)}s\right)\right\}, \quad \Theta_{\pi s} = \emptyset.$$

(2) *If $t \in \mathcal{T}$, then*

$$\Sigma_{\pi t} = \emptyset, \quad \Theta_{\pi t} = \{\pi\left((\operatorname{ad} t)^3 E_-\right)\}.$$

(3) *Assume that V is irreducible. If $u \in \mathcal{U}$, then* $\dim \Sigma_{\pi u} \geq 1$ *and* $\Theta_{\pi u} \neq \emptyset$.

In particular, it turns out that non-empty Freudenthal varieties $V \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ are so-called *varieties with one apparent double point* (Corollary A2), that is, for a general point $P \in \mathbb{P}_*(\mathfrak{g}_1)$ there exists a unique secant line of V through P (see [21, IX]). We also show that *if V is neither empty nor irreducible, then V is a disjoint union of two linear subspaces of $\mathbb{P}^{2n-1}$ of dimension $n-1$, where $2n = \dim \mathfrak{g}_1$* (Corollary B2).

On the other hand, we discuss the homogeneity of V as well. Consider a subalgebra $\mathfrak{D}_0$ of $\mathfrak{g}_0$ as follows:

$$\mathfrak{D}_0 := \operatorname{Ker}(\operatorname{ad} E_+|_{\mathfrak{g}_0}),$$

and denote by G the connected, closed subgroup of $\operatorname{Int}\mathfrak{g}$, the inner automorphism group of $\mathfrak{g}$, with Lie algebra $\mathfrak{D}_0$. Then, we show

Theorem B (Cf. [18, (5.8), (2)]). *G acts transitively on each of irreducible components of V.*

Besides the results above, we discuss the dimension of V (Corollary B1 (1)) and the projective lines contained in V (Corollary B3). Moreover, for the tangent variety of V, we discuss its defining equation (Corollary A5), its singular locus (Corollary A6) and a certain duality (Corollary B1 (4)), where the *tangent variety* of V is by definition the union of embedded tangent spaces to V. On the other hand, we also discuss relationship among the strata of $\mathfrak{g}_1$ (Corollaries A1, A3 and A4).

The contents of this article are organized as follows: In §1 we prove some results on $\mathfrak{g}_1$ and the stratification above, which are used in §2. Here we use a certain ternary product in $\mathfrak{g}_1$, to avoid raging flood of Lie brackets. In the literature, several authors have introduced various ternary products on $\mathfrak{g}_1$ along with establishing their theories of triple systems ([6], [7], [17], [25]). However, those ternary products themselves are essentially same, as is easily seen. In this article we use the one introduced by K. Yamaguti and H. Asano [25], since their product seems suitable for our computation. In §2 we prove firstly the assertion for secant loci in Theorem A (1), and then we give some corollaries to this result. Using those results, we next prove Theorem B and its corollaries, before continuing the proof of Theorem A. Then, using the results obtained so far, we prove the remaining part of Theorem A.

Finally we should mention that S. Mukai announced a theorem [18, (5.8)] on cubic Veronese varieties without proofs. Our work was originated by looking for proofs of the corresponding statements for Freudenthal varieties (Theorem B, Corollaries A2, A5, B1 (3) and (4)). In fact, we see from his list [18, (5.10)] of cubic Veronese varieties that the notion of our Freudenthal varieties seems to coincide with that of cubic Veronese varieties. However, to the best of our knowledge, there is no *a priori* proof in the literature for this coincidence.

1. Preliminaries

Let $\mathfrak{g}$ be a simple Lie algebra of rank ≥ 2, let

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$$

be a graded decomposition of contact type, and let H be the characteristic element: we have

$$\mathfrak{g}_i = \{x \in \mathfrak{g} | (\operatorname{ad} H)x = ix\}$$

and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, \quad \mathfrak{g}_1 \neq 0, \quad \dim \mathfrak{g}_{\pm 2} = 1$$

(see [23]). Taking $E_+ \in \mathfrak{g}_2$ and $E_- \in \mathfrak{g}_{-2}$ such that (E_+, H, E_-) form an $\mathfrak{sl}_2$-triple with $H = [E_+, E_-]$, we define a tri-linear map, $[,,] : \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ and a skew-symmetric form, $\langle , \rangle : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{C}$ by

$$\begin{aligned} 2[a, b, c] =& [c, [b, [a, E_-]]] + [c, [a, [b, E_-]]], \\ 2\langle a, b\rangle E_+ =& [a, b] \end{aligned}$$

with $a, b, c \in \mathfrak{g}_1$. It is easily shown that the skew-symmetric form $\langle , \rangle$ is *non-degenerate* since $\mathfrak{g}$ is simple (see [23, Lemma 3.2 (2)]). Moreover, we have the following

Fact (Yamaguti-Asano)**.** *For any* $v, w, x, y, z \in \mathfrak{g}_1$*, the following formulas hold:*

(S1) $[xyz] = [yxz]$.
(S2) $[xyz] - [xzy] = \langle x, z\rangle y - \langle x, y\rangle z + 2\langle y, z\rangle x$.
(S3) $[vw[xyz]] = [[vwx]yz] + [x[vwy]z] + [xy[vwz]]$.

Obviously (S1) follows from the definition, and (S2) is not difficult to show. However, some complicated computations are necessary to show (S3). Since the only reference for the proof of this fact is [3] written in Japanese, we give a proof for (S3) (including (S2)) in Appendix for the convenience of the readers. In general, a vector space with a ternary product $\langle , \rangle$ and a skew-symmetric form $[,]$ satisfying (S1), (S2) and (S3) is called a *symplectic triple system* ([1], [3], [25]).

Now, for the subalgebra $\mathfrak{D}_0 = \mathrm{Ker}(\mathrm{ad}\, E_+|\mathfrak{g}_0)$ of $\mathfrak{g}_0$, it is easily shown that

$$\mathfrak{g}_0 = \mathfrak{D}_0 \oplus \mathbb{C}H.$$

On the other hand, for $a, b \in \mathfrak{g}_1$ we define a linear map $L(a, b) : \mathfrak{g}_1 \to \mathfrak{g}_1$ by

$$L(a, b)c := [abc]$$

with $c \in \mathfrak{g}_1$. If we define $a \times b \in \mathfrak{g}_0$ for $a, b \in \mathfrak{g}_1$ by

$$-2a \times b = [b, [a, E_-]] + [a, [b, E_-]],$$

then $[a \times b, c] = [abc] = L(a, b)c$ and $a \times b \in \mathfrak{D}_0$ (see [15, Proposition 2 (a)]). Since the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful (see [23, Lemma 3.2 (1)]), we may identify $L(a, b)$ with $a \times b$, so that we may consider $L(a, b) \in \mathfrak{D}_0$. Note that with the notation above, we have

$$\begin{aligned}
\mathcal{S} &= \{s \in \mathfrak{g}_1 | \langle s, [sss]\rangle \neq 0\},\\
\mathcal{T} &= \{t \in \mathfrak{g}_1 | [ttt] \neq 0, \langle t, [ttt]\rangle = 0\},\\
\mathcal{U} &= \{u \in \mathfrak{g}_1 | L(u, u) \neq 0, [uuu] = 0\},\\
\mathcal{V} &= \{v \in \mathfrak{g}_1 | v \neq 0, L(v, v) = 0\},
\end{aligned}$$

and $q(a) = \langle a, [aaa]\rangle$. Furthermore, we have

Lemma 1. *The subalgebra* $\mathfrak{D}_0$ *is linearly spanned by the set* $\{L(a, b) | a, b \in \mathfrak{g}_1\}$.

Proof. Since $\mathfrak{g}$ is simple, we have $[[\mathfrak{g}_1, E_-], \mathfrak{g}_1] = \mathfrak{g}_0 = \mathfrak{D}_0 \oplus \mathbb{C}H$ (see [23, Lemma 3.1]). On the other hand, according to [15, Proposition 2 (b)], we have

$$[[a, E_-], b] = a \times b + \langle a, b\rangle H$$

for $a, b \in \mathfrak{g}_1$ (see also the proof of (S2) in Appendix). Thus the result follows. □

Lemma 2. *For any* $a, b \in \mathfrak{g}_1$ *and* $D \in \mathfrak{D}_0$*, we have* $\langle Da, b\rangle + \langle a, Db\rangle = 0$.

Proof. From the definition of $\mathfrak{D}_0$ and the Jacobi identity we obtain

$$0 = 2\langle a, b\rangle D(E_+) = D([a, b]) = [Da, b] + [a, Db]. \quad \square$$

Remark. It is known that Lemmas 1 and 2 above hold for any simple, symplectic triple systems (see [1, Lemma 2.1 and Theorem 2.9]). On the other hand, all the results below in this section are deduced from Lemmas 1, 2, the non-degeneracy of $\langle , \rangle$, and the axioms of symplectic triple systems, (S1), (S2) and (S3). According to [1, Theorem 2.3] (see also [25, Theorem 2]), a symplectic triple system is simple if and only if $\langle , \rangle$ is non-degenerate. Therefore it turns out that all the results in this section hold for arbitrary simple, symplectic triple systems.

Lemma 3. *For any $a, b, c \in \mathfrak{g}_1$ and $D \in \mathfrak{D}_0$, we have $D[abc] = [Dabc] + [aDbc] + [abDc]$.*

Proof. By virtue of Lemma 1, we may assume that D is of the form $L(d, e)$ with $d, e \in \mathfrak{g}_1$. Then the claim is nothing but (S3). $\square$

Proposition 1. *For any $x \in \mathcal{V}$, we have:*

(1) (Asano [2]) *$L(a, x)x = 3\langle a, x\rangle x$ for any $a \in \mathfrak{g}_1$. In particular, if $L(a, x) = 0$, then $\langle a, x\rangle = 0$.*
(2) *$x \in \mathfrak{D}_0 x$.*

Proof. (1) It follows from (S1), (S2) and $x \in \mathcal{V}$ that

$$L(a, x)x = [xax] = [xxa] + \langle x, x\rangle a - \langle x, a\rangle x + 2\langle a, x\rangle x = 3\langle a, x\rangle x.$$

(2) This follows from (1) and the non-degeneracy of $\langle , \rangle$. $\square$

Proposition 2. *For any $a, b, c, d \in \mathfrak{g}_1$, we have $\langle L(a, b)c, d\rangle = \langle L(c, d)a, b\rangle$.*

Proof. It follows from (S2) that

$$\langle [abc], d\rangle = \langle [acb], d\rangle + \langle a, c\rangle\langle b, d\rangle - \langle a, b\rangle\langle c, d\rangle + 2\langle b, c\rangle\langle a, d\rangle,$$
$$\langle [cda], b\rangle = \langle [cad], b\rangle + \langle c, a\rangle\langle d, b\rangle - \langle c, d\rangle\langle a, b\rangle + 2\langle d, a\rangle\langle c, b\rangle.$$

Therefore, using (S1) and Lemma 2, we have

$$\langle [abc], d\rangle - \langle [cda], b\rangle = \langle [acb], d\rangle - \langle [cad], b\rangle = \langle [acb], d\rangle + \langle b, [acd]\rangle = 0. \quad \square$$

Proposition 3. *For any $x \in \mathcal{V}$ and $D, E \in \mathfrak{D}_0$, we have:*

(1) (Asano [2]) *$L(Dx, x) = 0$.*
(2) *$\langle Dx, x\rangle = 0$.*
(3) *$\langle Dx, Ex\rangle = 0$.*
(4) *$L(Dx, Ex)x = 0$.*

Proof. By virtue of Lemma 1, it suffices to show these formulas for D, $E \in \mathfrak{D}_0$ of the form $L(a, b)$ with a, $b \in \mathfrak{g}_1$.

(1) It follows from (S3), (S1) and $x \in \mathcal{V}$ that

$$0 = [ab[xxc]] = [[abx]xc] + [x[abx]c] + [xx[abc]] = 2[[abx]xc].$$

(2) This follows from (1) and Proposition 1 (1). This follows from Proposition 2 as well: Indeed, we have $\langle L(a, b)x, x\rangle = \langle L(x, x)a, b\rangle = 0$.

(3) It follows from Proposition 2 and (1) that

$$\langle L(a, b)x, L(c, d)x\rangle = \langle L(x, L(c, d)x)a, b\rangle = 0.$$

(4) It follows from (S2) that

$$L([abx], [cdx])x = [[abx]x[cdx]] + \langle [abx], x\rangle[cdx] - \langle [abx], [cdx]\rangle x + 2\langle [cdx], x\rangle[abx],$$

which is equal to zero: Indeed, $L([abx], x) = 0$ follows from (1), $\langle [abx], x\rangle = \langle [cdx], x\rangle = 0$ from (2), and $\langle [abx], [cdx]\rangle = 0$ from (3). $\square$

Proposition 4. *If $\mathcal{S}=\emptyset$, then $\mathcal{T}=\emptyset$ and $\mathcal{V}=\emptyset$.*

Proof (Asano [4]). For any $a,b\in\mathfrak{g}_1$ and $\lambda,\mu\in\mathbb{C}$, we have

$$\begin{aligned}0=q(\lambda a+\mu b)=&\lambda^4 q(a)+\lambda^3\mu(\langle b,[aaa]\rangle+\langle a,[baa]\rangle+\langle a,[aba]\rangle+\langle a,[aab]\rangle)\\&+\lambda^2\mu^2(\cdots)+\cdots+\mu^4 q(b).\end{aligned}$$

In particular, we have $\langle b,[aaa]\rangle+\langle a,[baa]\rangle+\langle a,[aba]\rangle+\langle a,[aab]\rangle=0$. Since it follows from (S1) and (S2) that

$$(\dagger)\qquad [baa]=[aba]=[aab]-3\langle a,b\rangle a,$$

and it follows from Lemma 2 that $\langle a,[aab]\rangle=\langle b,[aaa]\rangle$, we have

$$(\ddagger)\qquad \langle a,[baa]\rangle=\langle a,[aba]\rangle=\langle a,[aab]\rangle=\langle b,[aaa]\rangle.$$

Therefore we have $4\langle b,[aaa]\rangle=0$ for any $b\in\mathfrak{g}_1$, so that $[aaa]=0$ since $\langle,\rangle$ is non-degenerate.

Similarly to the argument above, we obtain from $[aaa]=0$ that $[baa]+[aba]+[aab]=0$. Using ($\dagger$), we have

$$3[aab]=6\langle a,b\rangle a.$$

Therefore, if $L(a,a)=0$, then $\langle a,b\rangle a=0$ for any $b\in\mathfrak{g}_1$, so that $a=0$ since $\langle,\rangle$ is non-degenerate. $\square$

Proposition 5. *For any $a\in\mathfrak{g}_1$, we have:*

1. $L(a,[aaa])=0$.
2. $L(a,a)^2a=3q(a)a$.
3. $L([aaa],[aaa])=-3q(a)L(a,a)$. *In particular, $a\in\mathcal{T}$ if and only if $[aaa]\in\mathcal{V}$.*
4. $[[aaa][aaa][aaa]]=-9q(a)^2a$.
5. $q([aaa])=9q(a)^3$.

Proof. (1) Setting $v=w=x=y:=a$ in (S3), using (S1), we have $2[[aaa]az]=0$.

(2) It follows from (S2) and (1) that

$$[aa[aaa]]=[a[aaa]a]+\langle a,[aaa]\rangle a-\langle a,a\rangle[aaa]+2\langle a,[aaa]\rangle a=3\langle a,[aaa]\rangle a.$$

(3) It follows from (S3) that

$$[aa[[aaa]az]]=[[aa[aaa]]az]+[[aaa][aaa]z]+[[aaa]a[aaz]].$$

Then it follows from (1) and (2) that $0=3q(a)[aaz]+[[aaa][aaa]z]$.

(4) It follows from (3) and (2) that

$$L([aaa],[aaa])[aaa]=-3q(a)L(a,a)[aaa]=-9q(a)^2a.$$

(5) This follows from (4). $\square$

Proposition 6. *For any* $D \in \mathfrak{D}_0$ *and* $x \in \mathcal{V}$, *we have* $q(Dx) = 0$.

Proof. It follows from Proposition 3 (4) and Lemma 3 that

$$0 = D([DxDxx]) = [D^2xDxx] + [DxD^2xx] + [DxDxDx],$$

so that $[DxDxDx] = -2[D^2xDxx]$. Hence it follows from Proposition 3 (3) that

$$\langle Dx, [DxDxDx]\rangle = -2\langle Dx, [D^2xDxx]\rangle = 0. \quad \square$$

Proposition 7. *For* $a = u + v$ *with* $[uuu] = 0$ *and* $v \in \mathcal{V}$, *we have:*

(1) $L(a, a) = L(u, u) + 2L(u, v)$.
(2) $[aaa] = 3[uuv] - 6\langle u, v\rangle(u - v)$.
(3) $q(a) = 12\langle u, v\rangle^2$.

Proof. (1) This follows from (S1) and $v \in \mathcal{V}$.

(2) It follows from (1), $[uuu] = 0$, (S2) and Proposition 1 (1) that

$$\begin{aligned}[aaa] &= [uuu] + [uuv] + 2[uvu] + 2[uvv] \\ &= [uuv] + 2([uuv] + \langle u, u\rangle v - \langle u, v\rangle u + 2\langle v, u\rangle u) + 2(3\langle u, v\rangle v) \\ &= 3[uuv] - 6\langle u, v\rangle u + 6\langle u, v\rangle v.\end{aligned}$$

(3) It follows from (2), Lemma 2, Proposition 3 (2) and $[uuu] = 0$ that

$$\begin{aligned}q(a) &= 3\langle u, [uuv]\rangle + 3\langle v, [uuv]\rangle - 6\langle u, v\rangle\langle u + v, u - v\rangle \\ &= -3\langle [uuu], v\rangle + 12\langle u, v\rangle^2 \\ &= 12\langle u, v\rangle^2. \quad \square\end{aligned}$$

Remark. It is easily shown that $V \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ *has no trisecant line*, where a *trisecant line* of $V \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ is by definition a projective line L in $\mathbb{P}_*(\mathfrak{g}_1)$ such that $3 \leq \#(L \cap V) < \infty$. Indeed, suppose that there exist $x, y \in \mathcal{V}$ such that $\pi x \neq \pi y$ and $\{\pi x, \pi y\} \subsetneq \pi x * \pi y \cap V$, so that we may assume that $x + y \in \mathcal{V}$. Then it follows from Proposition 7 (1) that $\lambda x + \mu y \in \mathcal{V}$ for any $\lambda, \mu \in \mathbb{C}$, that is, $\pi x * \pi y \subseteq V$.

Proposition 8. *For* $u \in \mathfrak{g}_1$ *with* $[uuu] = 0$ *and* $v \in \mathcal{V}$, *we have:*

(1) $L(u, [uuv]) = 2\langle u, v\rangle L(u, u)$.
(2) $[uuv] \in \mathcal{V} \cup \{0\}$. *Moreover,* $\mathbb{C}v + \mathbb{C}[uuv] \subseteq \mathcal{V} \cup \{0\}$.

Proof. (1) Setting $a := u + v$, we have

$$\begin{aligned}0 &= L(a, [aaa]) \quad (\because \text{Proposition 5 (1)}) \\ &= L(u + v, 3[uuv] - 6\langle u, v\rangle(u - v)) \quad (\because \text{Proposition 7 (2)}) \\ &= 3L(u, [uuv]) + 3L(v, [uuv]) - 6\langle u, v\rangle L(u + v, u - v) \\ &= 3L(u, [uuv]) - 6\langle u, v\rangle L(u, u) \quad (\because \text{Proposition 3 (1), (S1)}, v \in \mathcal{V}).\end{aligned}$$

(2) Similarly to the above, setting $a := u + v$, we have

$$\begin{aligned}0 &= L([aaa], [aaa]) + 3q(a)L(a, a) \quad (\because \text{Proposition 5 (3)}) \\ &= 9\{L([uuv], [uuv]) - 4\langle u, v\rangle L(u, [uuv]) + 4\langle u, v\rangle^2(L(u, u) - 2L(u, v))\} \\ &\qquad + 36\langle u, v\rangle^2(L(u, u) + 2L(u, v)) \quad (\because \text{Propositions 7, 3 (1), (S1)}, v \in \mathcal{V}) \\ &= 9\{L([uuv], [uuv]) - 4\langle u, v\rangle L(u, [uuv]) + 8\langle u, v\rangle^2 L(u, u)\}.\end{aligned}$$

Then, using (1), we obtain $L([uuv], [uuv]) = 0$. It follows from Proposition 3 (1) that $L(v, [uuv]) = 0$, so that the latter part follows as well. $\square$

Proposition 9. *For $a \in \mathfrak{g}_1$, we have:*

(1) $3L(a,a)^2 b = 8\langle b, [aaa]\rangle a + 8\langle a, b\rangle [aaa] + \langle a, [aaa]\rangle b$ *for any* $b \in \mathfrak{g}_1$.

(2) *If* $a \in \mathcal{S}$, *then the linear map* $L(a,a)$ *has full rank.*

Proof. (1) It follows from (S2) that

$$L(a,a)^2 b = [aa[aba]] + 3\langle a, b\rangle [aaa].$$

Moreover it follows from (S2) that $[aa[aba]] = [a[aba]a] + 3\langle a, [aba]\rangle a$. Since $\langle a, [aba]\rangle = \langle b, [aaa]\rangle$ by Proposition 2, we have

$$(\star) \qquad [aa[aba]] = [a[aba]a] + 3\langle b, [aaa]\rangle a.$$

On the other hand, it follows from (S1) and (S3) that $[ab[aaa]] = 2[a[aba]a] + [aa[aba]]$, while it follows from (S2) and Proposition 5 (1) that $[ab[aaa]] = \langle a, [aaa]\rangle b - \langle a, b\rangle [aaa] + 2\langle b, [aaa]\rangle a$. Therefore, we have

$$(\star\star) \qquad 2[a[aba]a] + [aa[aba]] = \langle a, [aaa]\rangle b - \langle a, b\rangle [aaa] + 2\langle b, [aaa]\rangle a.$$

Thus it follows from $(\star)$ and $(\star\star)$ that

$$3[aa[aba]] = \langle a, [aaa]\rangle b - \langle a, b\rangle [aaa] + 8\langle b, [aaa]\rangle a.$$

Therefore, combining this with the first formula, we obtain the required result.

(2) (Asano [5]) Note that $\mathfrak{g}_1 = \mathbb{C}a \oplus \mathbb{C}[aaa] \oplus (\mathbb{C}a \oplus \mathbb{C}[aaa])^\perp$: Indeed, it follows from $\langle a, [aaa]\rangle \neq 0$ that $\mathbb{C}a \cap \mathbb{C}[aaa] = (\mathbb{C}a + \mathbb{C}[aaa]) \cap (\mathbb{C}a + \mathbb{C}[aaa])^\perp = \{0\}$, and from the non-degeneracy of $\langle , \rangle$ that $\dim(\mathbb{C}a + \mathbb{C}[aaa]) + \dim(\mathbb{C}a + \mathbb{C}[aaa])^\perp = \dim \mathfrak{g}_1$.

Now, it is clear from (1) that

$$\begin{aligned} L(a,a)^2|_{\mathbb{C}a+\mathbb{C}[aaa]} &= 3\langle a, [aaa]\rangle 1_{\mathbb{C}a+\mathbb{C}[aaa]}, \\ L(a,a)^2|_{(\mathbb{C}a+\mathbb{C}[aaa])^\perp} &= \frac{1}{3}\langle a, [aaa]\rangle 1_{(\mathbb{C}a+\mathbb{C}[aaa])^\perp}. \end{aligned}$$

Therefore $L(a,a)^2$ has full rank if $\langle a, [aaa]\rangle \neq 0$, so does $L(a,a)$. □

2. Proofs and Corollaries

Proof of Theorem A. (1) We first show the assertion for secant loci: we prove the following

Theorem A'. *For any* $s \in \mathcal{S}$ *there exists a unique pair* $\{x, y\} \subseteq \mathcal{V}$ *such that* $s = x + y$, *and we have*

$$\{x, y\} = \left\{ \frac{1}{2}s + \frac{1}{2\sqrt{3q(s)}}[sss], \frac{1}{2}s - \frac{1}{2\sqrt{3q(s)}}[sss] \right\}.$$

Proof. If $z = \lambda s + \mu [sss]$ with $\lambda, \mu \in \mathbb{C}$, then it follows from (S1), Proposition 5 (1) and (3) that

$$\begin{aligned} L(z,z) &= \lambda^2 L(s,s) + 2\lambda\mu L(s, [sss]) + \mu^2 L([sss],[sss]) \\ &= (\lambda^2 - 3\mu^2 q(s)) L(s,s). \end{aligned}$$

Taking $\lambda := 1/2$ and $\mu := \pm 1/2\sqrt{3q(s)}$, we see that $\lambda^2 - 3\mu^2 q(s) = 0$, so that $x, y \in \mathcal{V}$ with $s = x + y$.

For the uniqueness, it follows from Proposition 7 (2) and (3) that

$$x - y = \pm \frac{1}{\sqrt{3q(s)}}[sss]$$

for such a pair $\{x, y\}$. Therefore $\{x, y\}$ is uniquely determined by s. $\square$

Before proving the remaining part of Theorem A, we give some corollaries to Theorem A', and show Theorem B and its corollaries.

Corollary A1. $\mathcal{S} = \{x + y \in \mathfrak{g}_1 | x, y \in \mathcal{V}, \langle x, y\rangle \neq 0\}$.

Proof. This follows from Proposition 7 (3) and Theorem A'. $\square$

Corollary A2 (Cf. [18, (5.8), (3)]). *If $V \neq \emptyset$, then V is non-degenerate in $\mathbb{P}_*(\mathfrak{g}_1)$, and moreover V is a variety with one apparent double point. In particular, for any $a \in \mathfrak{g}_1 \setminus \{0\}$, there exists $x \in \mathcal{V}$ such that $\langle a, x\rangle \neq 0$ if $\mathcal{V} \neq \emptyset$.*

Proof. Recall that V is said to be *non-degenerate* if there is no hyperplane in $\mathbb{P}_*(\mathfrak{g}_1)$ containing V. To prove the non-degeneracy of V, suppose that there exists a proper subspace $\mathfrak{s}$ of $\mathfrak{g}_1$ such that $V \subseteq \mathbb{P}_*(\mathfrak{s})$. Since $\pi\mathcal{S}$ is a non-empty (Proposition 4), Zariski open subset of an irreducible space $\mathbb{P}_*(\mathfrak{g}_1)$, $\pi\mathcal{S}$ is dense in $\mathbb{P}_*(\mathfrak{g}_1)$. Therefore, there exists a point $R \in \pi\mathcal{S} \setminus \mathbb{P}_*(\mathfrak{s})$, and, according to Theorem A', we have $R \in P * Q$ for some $P, Q \in V$. Then we have $P * Q \subseteq \mathbb{P}_*(\mathfrak{s})$, so that $R \in \mathbb{P}_*(\mathfrak{s})$. This is a contradiction, and V is non-degenerate. Moreover it follows from Theorem A' that for any point R in the dense open $\pi\mathcal{S}$, there exists a unique secant line through R, that is, $P * Q$ above, so that V is a variety with one apparent double point.

For the last part, since $\langle , \rangle$ is non-degenerate, $a^\perp := \{b \in \mathfrak{g}_1 | \langle a, b\rangle = 0\}$ is a proper subspace of $\mathfrak{g}_1$. Therefore we have $\mathcal{V} \setminus a^\perp \neq \emptyset$ since V is non-degenerate. $\square$

Corollary A3. *$\mathcal{S} = \emptyset$ if and only if $\mathcal{V} = \emptyset$, that is, $V = \emptyset$, and in that case $\mathcal{T} = \emptyset$.*

Proof. This follows from Proposition 4 and Theorem A'. $\square$

Proof of Theorem B. We prove

Theorem B'. *We have:*

(1) *G acts transitively on each of irreducible components of $\mathcal{V}$, and $t_x\mathcal{V} = \mathfrak{D}_0 x$ for any $x \in \mathcal{V}$, where $t_x\mathcal{V}$ is the Zariski tangent space to $\mathcal{V}$ at x.*

(2) *$\mathfrak{D}_0 x = (\mathfrak{D}_0 x)^\perp$ with $2\dim \mathfrak{D}_0 x = \dim \mathfrak{g}_1$ for any $x \in \mathcal{V}$, and $\mathfrak{g}_1 = \mathfrak{D}_0 x \oplus \mathfrak{D}_0 y$ for any $x, y \in \mathcal{V}$ with $\langle x, y\rangle \neq 0$.*

Proof. (1) Let G_0 be the connected, closed subgroup of $\operatorname{Int}\mathfrak{g}$ with Lie algebras $\mathfrak{g}_0$. We first show

Claim. *G_0 acts transitively on $\mathcal{S}$.*

Proof of Claim. We may assume that $\mathcal{S} \neq \emptyset$, so that $\mathcal{S}$ is irreducible since it is a Zariski open subset of an irreducible space $\mathfrak{g}_1$. Moreover, since $\mathcal{S}$ is stable under the action of G_0, it suffices to show that

$$[\mathfrak{g}_0, s] = \mathfrak{g}_1$$

for any $s \in \mathcal{S}$. Take an arbitrary $a \in \mathfrak{g}_1$. It follows from Proposition 9 (2) that there exists $b \in \mathfrak{g}_1$ such that $a = L(s,s)b$. On the other hand, it follows from (S2) that $L(s,s)b = (L(s,b) + 3\langle s,b\rangle \operatorname{ad} H)s \in [\mathfrak{g}_0, s]$. Thus, we have $a \in [\mathfrak{g}_0, s]$. $\square$

Now, it follows from $\mathfrak{g}_0 = \mathfrak{D}_0 \oplus \mathbb{C}H$ that $G_0 = G \cdot \mathbb{C}^\times$. Taking account of Proposition 1 (2), we see that to show (1) of Theorem B' it suffices to show that G_0 acts transitively on each of irreducible components of $\mathcal{V}$. Take an arbitrary $x \in \mathcal{V}$. There exists $y \in \mathcal{V}$ such that $\langle x, y\rangle \neq 0$ by Corollary A2. Then we show that for any x' in a Zariski open neighborhood $\mathcal{V} \setminus y^\perp$ of x in $\mathcal{V}$, there exists $g \in G_0$ such that $gx = x'$, which implies the required result. Since it follows from Proposition 7 (3) that $x+y, x'+y \in \mathcal{S}$, there exists $g \in G_0$ such that $g(x+y) = x'+y$ by Claim, where we note that $gx, gy \in \mathcal{V}$ since $\mathcal{V}$ is G_0-stable. Therefore, it follows from Theorem A' that

$$\{gx, gy\} = \{x', y\}.$$

If $gx = x'$, then there is nothing to prove: Otherwise, we have $gx = y$ and $gy = x'$, so that $g^2x = x'$. This completes the proof.

(2) We note that

$$(\clubsuit) \qquad \dim \mathfrak{D}_0 x \leq \frac{1}{2} \dim \mathfrak{g}_1$$

holds for any $x \in \mathcal{V}$: Indeed, it follows from Proposition 3 (3) that $\mathfrak{D}_0 x \subseteq (\mathfrak{D}_0 x)^\perp$, and from the non-degeneracy of $\langle , \rangle$ that $\dim(\mathfrak{D}_0 x)^\perp = \dim \mathfrak{g}_1 - \dim \mathfrak{D}_0 x$.

Now, we show that

$$(\spadesuit) \qquad \mathfrak{g}_1 = \mathfrak{D}_0 x + \mathfrak{D}_0 y.$$

Take an arbitrary $a \in \mathfrak{g}_1$, and set $s := x+y$. Since $\langle x, y\rangle \neq 0$, it follows from Proposition 7 (3) that $s \in \mathcal{S}$. Then, it follows from Proposition 9 (2) that there exists $b \in \mathfrak{g}_1$ such that $a = L(s,s)b$. On the other hand, it follows from $x, y \in \mathcal{V}$ and (S2) that

$$\begin{aligned} L(s,s)b &= [xyb] + [yxb] \\ &= [xby] + \langle x,b\rangle y - \langle x,y\rangle b + 2\langle y,b\rangle x \\ &\qquad\qquad + [ybx] + \langle y,b\rangle x - \langle y,x\rangle b + 2\langle x,b\rangle y \\ &= ([byx] + 3\langle y,b\rangle x) + ([bxy] + 3\langle x,b\rangle y), \end{aligned}$$

which is contained in $\mathfrak{D}_0 x + \mathfrak{D}_0 y$ by Proposition 1 (2). Thus we have $a \in \mathfrak{D}_0 x + \mathfrak{D}_0 y$.

Combining ($\clubsuit$) and ($\spadesuit$), we obtain that $\mathfrak{D}_0 x \cap \mathfrak{D}_0 y = 0$ for any $y \in \mathcal{V}$ with $\langle x, y\rangle \neq 0$: We note that for any $x \in \mathcal{V}$ there exists $y \in \mathcal{V}$ such that $\langle x, y\rangle \neq 0$ (Corollary A2). Thus the equality holds in ($\clubsuit$), and the required results follow. $\square$

Remark 1. It follows from Claim above that $\mathfrak{D}_0 s = [sss]^\perp$ for any $s \in \mathcal{S}$: Indeed, note that it follows from Lemma 1, Propositions 2 and 5 (1) that $\mathfrak{D}_0 a \subseteq [aaa]^\perp$ for any $a \in \mathfrak{g}_1$ since $\langle L(b,c)a, [aaa]\rangle = \langle L(a,[aaa])b, c\rangle = 0$. Since $[sss]^\perp$ has codimension 1 in $\mathfrak{g}_1$, the assertion follows from $\mathfrak{D}_0 s + \mathbb{C}s = \mathfrak{g}_0 s = \mathfrak{g}_1$.

Remark 2. One can deduce Theorem B' (1) from Linear Section Theorem [15, Theorem B], using a generalization of a theorem of Richardson [24, Lemma, p. 469], as well as from Theorem B' (2), using the finiteness theorem for the number of nilpotent orbits

[24, Proposition 2, p. 469]: Note that both of those proofs depend essentially on the argument [24, Lemma, p. 469]. On the other hand, (2) follows from (1) in Theorem B' by Theorem A', as follows:

Proof of (1) $\Rightarrow$ (2) in Theorem B'. Similarly to the proof of (2) above, it suffices to show (♠) for $x, y \in \mathcal{V}$ with $\langle x, y\rangle \neq 0$. Take an arbitrary $a \in \mathfrak{g}_1$, and consider a line in $\mathfrak{g}_1$ as follows:

$$\sigma : \mathbb{C} \to \mathfrak{g}_1; \lambda \mapsto (x + y) + \lambda a.$$

Since $\sigma(0) = x + y \in \mathcal{S}$ by Proposition 7 (3), for a sufficiently small $\varepsilon > 0$ we have $\sigma(\lambda) \in \mathcal{S}$ for any $\lambda \in \Delta$, where we set $\Delta := \{\lambda \in \mathbb{C} | |\lambda| < \varepsilon\}$. Then it follows from Theorem A' that there exist curves $\xi, \eta : \Delta \to \mathcal{V}$ such that $\xi(0) = x, \eta(0) = y$ and

$$\sigma(\lambda) = \xi(\lambda) + \eta(\lambda)$$

for any $\lambda \in \Delta$. Then we have

$$a = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \sigma(\lambda) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \xi(\lambda) + \left.\frac{d}{d\lambda}\right|_{\lambda=0} \eta(\lambda) \in t_x\mathcal{V} + t_y\mathcal{V}.$$

According to (1), we have $t_x\mathcal{V} = \mathfrak{D}_0 x$ and $t_y\mathcal{V} = \mathfrak{D}_0 y$, so that $a \in \mathfrak{D}_0 x + \mathfrak{D}_0 y$. □

Recall that the *tangent variety* of V, denoted by $\operatorname{Tan} V$, is the union of embedded tangent spaces to V, and the *projective dual* of V, denoted by $V^\vee$, is the set of hyperplanes tangent to V (see, for example, [9, §3]): In other words, we set

$$\operatorname{Tan} V := \bigcup_{P\in V} T_P V \subseteq \mathbb{P}_*(\mathfrak{g}_1), \quad V^\vee := \bigcup_{P\in V} (T_P V)^\vee \subseteq \mathbb{P}_*(\mathfrak{g}_1)^*,$$

where we define $\mathbb{P}_*(\mathfrak{g}_1)^*$ to be $\mathbb{P}_*(\mathfrak{g}_1^*)$, that is, the set of hyperplanes in $\mathbb{P}_*(\mathfrak{g}_1)$, and for a linear subspace $\mathfrak{s} \subseteq \mathfrak{g}_1$, we define $\mathbb{P}_*(\mathfrak{s})^\vee$ to be $\mathbb{P}_*(\operatorname{Ker}(\mathfrak{g}_1^* \twoheadrightarrow \mathfrak{s}^*)) \subseteq \mathbb{P}_*(\mathfrak{g}_1)^*$, that is, the set of hyperplanes containing $\mathbb{P}_*(\mathfrak{s})$. From Theorem B' we immediately obtain

Corollary B1. *Assume that $V \neq \emptyset$. Then we have:*

1. *V is equi-dimensional with $2\dim V + 1 = \dim \mathbb{P}_*(\mathfrak{g}_1)$.*
2. *$T_P V = \mathbb{P}_*(\mathfrak{D}_0 x)$ for any $P \in V$ with $x \in \pi^{-1}P$.*
3. (Cf. [18, (5.8), (4)]) *$T_P V = (T_P V)^\vee$ for any $P \in V$.*
4. (Cf. [18, (5.8), (1)]) *$\operatorname{Tan} V = V^\vee$.*

Here we identify $\mathbb{P}_*(\mathfrak{g}_1)$ with its dual space $\mathbb{P}_*(\mathfrak{g}_1)^*$ by means of the non-degenerate, skew-symmetric form $\langle , \rangle$.

Proof. (1) follows directly from Theorem B'. For (2), it follows from Theorem B' (1) that $T_P V = \mathbb{P}_*(\mathbb{C}x + \mathfrak{D}_0 x)$ (see, for example, [14, Lemma 2.1]), which is equal to $\mathbb{P}_*(\mathfrak{D}_0 x)$ by Proposition 1 (2). Now, (3) follows from Theorem B' (2) since we have $(\mathfrak{D}_0 x)^\perp = \operatorname{Ker}(\mathfrak{g}_1^* \twoheadrightarrow (\mathfrak{D}_0 x)^*)$, and (4) follows as well. □

Remark. Since V is not necessarily irreducible, Corollary B1 (1) does not follow directly from Corollary A2.

Corollary B2. *If V is neither empty nor irreducible, then $V \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ is a disjoint union of two linear varieties of dimension $n-1$ in $\mathbb{P}^{2n-1}$:*

$$V \simeq \mathbb{P}^{n-1} \sqcup \mathbb{P}^{n-1},$$

where $2n = \dim \mathfrak{g}_1$.

Proof. Let $\{V_i\}_{1\le i\le k}$ be the set of irreducible components of V with $k \ge 2$, and let $\mathfrak{s}_i$ be the linear subspace of $\mathfrak{g}_1$ spanned by $\pi^{-1}V_i$. It follows from Theorem B that each V_i is an orbit of G, so that $\mathfrak{s}_i$ is G-stable. Moreover, by virtue of an argument by Zak [26, pp. 49–50] we see that $\mathfrak{s}_i$ is an irreducible G-module. Since $\mathbb{P}_*(\mathfrak{s}_i)$ has a unique closed orbit of G (see, for example, [22, Ch. 1, §4.6.1, Lemma]), we see that $\mathfrak{s}_i \cap \mathfrak{s}_j = 0$ if $i \ne j$. Taking account of the non-degeneracy of V (Corollary A2), we obtain

$$\mathfrak{g}_1 = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k.$$

On the other hand, we have $\dim \mathfrak{s}_i \ge \dim \pi^{-1}V_i = n$ by Corollary B1 (1). Therefore, it follows that $\mathfrak{g}_1 = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ with $k = 2$ and $V_i = \mathbb{P}_*(\mathfrak{s}_i) \simeq \mathbb{P}^{n-1}$. $\square$

Corollary B3. *If $V \ne \emptyset$ and $\mathcal{U} \ne \emptyset$, then V is covered by the lines in $\mathbb{P}_*(\mathfrak{g}_1)$ contained in V.*

Proof. We may assume that V is irreducible: Otherwise the claim is obvious from Corollary B2. Take an arbitrary $u \in \mathcal{U}$. It suffices to show that

$$\mathcal{V} \setminus (L(u,u)\mathcal{V} \cup \operatorname{Ker} L(u,u)) \ne \emptyset.$$

Indeed, if $v \in \mathcal{V} \setminus (L(u,u)\mathcal{V} \cup \operatorname{Ker} L(u,u))$, then $\pi v \ne \pi[uuv] \in \mathbb{P}_*(\mathfrak{g}_1)$, and it follows from Proposition 8 (2) that the secant line $\pi v * \pi[uuv]$ is contained in V. Taking account of the homogeneity of $\mathcal{V}$ (Theorem B' (1)), we see that this holds for any $v \in \mathcal{V}$.

Now, we have $L(u,u)\mathcal{V} \subseteq L(u,u)\mathfrak{g}_1 \subseteq u^\perp$ since $\langle [uua], u\rangle = -\langle a, [uuu]\rangle = 0$ for any $a \in \mathfrak{g}_1$ by Lemma 2. Therefore we have

$$\mathcal{V} \setminus (L(u,u)\mathcal{V} \cup \operatorname{Ker} L(u,u)) \supseteq \mathcal{V} \setminus (u^\perp \cup \operatorname{Ker} L(u,u)).$$

On the other hand, we have $\mathcal{V} \setminus (u^\perp \cup \operatorname{Ker} L(u,u)) \ne \emptyset$: Indeed, $u^\perp$ as well as $\operatorname{Ker} L(u,u)$ are proper subspaces of $\mathfrak{g}_1$, and V is irreducible and non-degenerate (Corollary A2). Thus the claim follows. $\square$

Corollary B4. *If V is irreducible, then the linear map $L(u,u)$ has rank at least 2 for any $u \in \mathcal{U}$.*

Proof. We have $\operatorname{rk} L(u,u) \ge 1$ since $L(u,u) \ne 0$. Suppose $\operatorname{rk} L(u,u) = 1$: we set

$$Q := \mathbb{P}_*(L(u,u)\mathfrak{g}_1) \in \mathbb{P}_*(\mathfrak{g}_1).$$

Since $\pi[uuv] = Q$ for any $v \in \mathcal{V} \setminus \operatorname{Ker} L(u,u)$, it follows from Proposition 8 (2) that $P * Q \subseteq V$ for any $P \in V \setminus \mathbb{P}_*(\operatorname{Ker} L(u,u))$ with $P \ne Q$. Since V is irreducible, V is a cone with vertex Q. On the other hand, it follows from Theorem B that V is smooth. Therefore V is a linear variety. This contradicts to Corollaries A2 and B1 (1). $\square$

Now we return to proving Theorem A.

Proof of Theorem A (continued). (1) For the tangent locus, suppose that $\pi s \in T_{\pi x}V$ for some $x \in \mathcal{V}$. Then it follows from Corollary B1 (2) that $s = Dx$ for some $D \in \mathfrak{D}_0$, and from Proposition 6 that $q(s) = q(Dx) = 0$. This contradicts to $s \in \mathcal{S}$.

(2) For the secant locus, the assertion easily follows from Proposition 7: Indeed, suppose that $\pi t \in \pi x * \pi y$ for some $x, y \in \mathcal{V}$. Then we may assume that $t = x + y$. We see from Proposition 7 (2) and (3) that $[ttt] \neq 0$ implies $\langle x, y\rangle \neq 0$, which implies $q(t) = 12\langle x, y\rangle^2 \neq 0$. This contradicts to $t \in \mathcal{T}$.

Next for the tangent locus, it follows from Proposition 5 (3) that $[ttt] \in \mathcal{V}$. Moreover, we have $t \in \mathfrak{D}_0[ttt]$: Indeed, for any $a, b \in \mathfrak{g}_1$, it follows from Propositions 2 and 5 (1) that $\langle L(a, b)[ttt], t\rangle = \langle L([ttt], t)a, b\rangle = 0$, so that $\langle \mathfrak{D}_0[ttt], t\rangle = 0$ by Lemma 1. Now, our claim follows from Theorem B' (2). Therefore it follows from Corollary B1 (2) that $\pi[ttt] \in \Theta_{\pi t}$.

To complete the proof, it suffices to show that for any $t \in \mathcal{T}$, if $t \in t_x\mathcal{V}$ for some $x \in \mathcal{V}$, then $\pi x = \pi[ttt]$. We may assume that there is a curve $\xi : \Delta \to \mathcal{V}$ such that $\xi(0) = x$ and

$$\lim_{\lambda\to 0} \frac{1}{\lambda}\{\xi(\lambda) - \xi(0)\} = t,$$

where $\Delta := \{\lambda \in \mathbb{C} | |\lambda| < \varepsilon\}$ with a sufficiently small $\varepsilon > 0$. Setting

$$\tau(\lambda) := \frac{1}{\lambda}\{\xi(\lambda) - \xi(0)\}$$

with $\lambda \in \Delta \setminus \{0\}$, we have $\lim_{\lambda\to 0} \tau(\lambda) = t$. Moreover, by the assumption $[ttt] \neq 0$, taking ε smaller if necessary, we may assume that $[\tau(\lambda)\tau(\lambda)\tau(\lambda)] \neq 0$ for any $\lambda \in \Delta\setminus\{0\}$. On the other hand, it follows from Proposition 7 (2) that

$$\lambda^3[\tau(\lambda)\tau(\lambda)\tau(\lambda)] = -6\langle \xi(\lambda), -\xi(0)\rangle(\xi(\lambda) + \xi(0)).$$

Thus it follows that $\langle \xi(\lambda), -\xi(0)\rangle \neq 0$, so that $\tau(\lambda) \in \mathcal{S}$ by Proposition 7 (3). Now, it follows from Theorem A (1) that

$$\{\pi\xi(\lambda), \pi\xi(0)\} = \Sigma^\circ_{\pi\tau(\lambda)} = \left\{\pi\left([\tau(\lambda)\tau(\lambda)\tau(\lambda)] \pm \sqrt{3q(\tau(\lambda))}\tau(\lambda)\right)\right\}.$$

Therefore, using $\lim_{\lambda\to 0} q(\tau(\lambda)) = q(t) = 0$, we have

$$\begin{aligned}\pi x = \lim_{\lambda\to 0} \pi\xi(\lambda) &= \lim_{\lambda\to 0} \pi\left([\tau(\lambda)\tau(\lambda)\tau(\lambda)] \pm \sqrt{3q(\tau(\lambda))}\tau(\lambda)\right)\\ &= \lim_{\lambda\to 0} \pi\left([\tau(\lambda)\tau(\lambda)\tau(\lambda)]\right)\\ &= \pi[ttt]. \quad \square\end{aligned}$$

(3) It follows from Corollary A2 that there exists $v \in \mathcal{V}$ such that $\langle u, v\rangle \neq 0$. If $z = \lambda u + \mu[uuv]$ with $\lambda, \mu \in \mathbb{C}$, then it follows from Proposition 8 that

$$\begin{aligned}L(z, z) &= \lambda^2 L(u, u) + 2\lambda\mu L(u, [uuv]) + \mu^2 L([uuv], [uuv])\\ &= \lambda(\lambda + 4\mu\langle u, v\rangle)L(u, u).\end{aligned}$$

Taking $(\lambda, \mu) = (1, -1/4\langle u, v\rangle), (0, 1/4\langle u, v\rangle)$ with $\lambda(\lambda + 4\mu\langle u, v\rangle) = 0$, we set

$$\{x, y\} := \left\{u - \frac{1}{4\langle u, v\rangle}[uuv], \frac{1}{4\langle u, v\rangle}[uuv]\right\}.$$

Then we have that $u = x + y$ with $x, y \in \mathcal{V}$: Note that it follows from Proposition 8 (1) that $[uuv] \neq 0$ since $L(u,u) \neq 0$ and $\langle u, v\rangle \neq 0$.

Thus we see that $\pi[uuv] \in \Sigma^\circ_{\pi u}$ for any $v \in \mathcal{V}$ with $\langle u, v\rangle \neq 0$, so that

$$\pi(L(u,u)(\mathcal{V} \setminus u^\perp) \setminus \{0\}) \subseteq \Sigma^\circ_{\pi u}.$$

Since $\mathcal{V}$ is irreducible, we obtain $\pi(L(u,u)\mathcal{V} \setminus \{0\}) \subseteq \Sigma_{\pi u}$. On the other hand, it follows from Corollary A2 that $\pi(L(u,u)\mathcal{V} \setminus \{0\})$ is non-degenerate in $\mathbb{P}_*(L(u,u)\mathfrak{g}_1)$, so that $\dim \pi(L(u,u)\mathcal{V} \setminus \{0\}) \geq 1$ since $\dim \mathbb{P}_*(L(u,u)\mathfrak{g}_1) \geq 1$ by Corollary B4 and $\pi(L(u,u)\mathcal{V} \setminus \{0\})$ is irreducible. Thus we obtain $\dim \Sigma_{\pi u} \geq 1$.

Next for the tangent locus, we note that for $u \in \mathcal{U}$ and $v \in \mathcal{V}$ it follows

$$(\bigstar) \qquad \langle u, v\rangle = 0, [uuv] \neq 0 \Rightarrow u \in \mathfrak{D}_0[uuv].$$

Indeed, for any $a, b \in \mathfrak{g}_1$ it follows from Propositions 2 and 8 (1) that

$$\langle L(a,b)[uuv], u\rangle = \langle L([uuv], u)a, b\rangle = 2\langle u, v\rangle\langle L(u,u)a, b\rangle = 0.$$

Therefore, we have $\langle \mathfrak{D}_0[uuv], u\rangle = 0$ by Lemma 1, so that $u \in \mathfrak{D}_0[uuv]$ by Theorem B' (2), where one should note that $[uuv] \in \mathcal{V} \cup \{0\}$ by Proposition 8 (2).

Now, we see from ($\bigstar$) that

$$\pi(L(u,u)(\mathcal{V} \cap u^\perp) \setminus \{0\}) \subseteq \Theta_{\pi u}.$$

On the other hand, since V is irreducible and non-degenerate in $\mathbb{P}_*(\mathfrak{g}_1)$ as above, $V \cap \mathbb{P}_*(u^\perp)$ is also non-degenerate in $\mathbb{P}_*(u^\perp)$. Therefore, it follows from Corollary B4 that $V \cap \mathbb{P}_*(u^\perp) \not\subseteq \mathbb{P}_*(\operatorname{Ker} L(u,u))$, so that $\pi(L(u,u)(\mathcal{V} \cap u^\perp) \setminus \{0\}) \neq \emptyset$. Thus we obtain $\Theta_{\pi u} \neq \emptyset$. $\square$

Corollary A4. *If $V \neq \emptyset$, then $\mathcal{U} = \{x + y \in \mathfrak{g}_1 | x, y \in \mathcal{V}, \langle x, y\rangle = 0, L(x,y) \neq 0\}$.*

Proof. This follows from Proposition 7 (1), (2) and the first paragraph of the proof of Theorem A (3). $\square$

Corollary A5 (Cf. [18, (5.8), (1)])**.** *If V is irreducible, then* $\operatorname{Tan} V$ *is a quartic hypersurface in $\mathbb{P}_*(\mathfrak{g}_1)$ defined by q.*

Proof. Denote by T the quartic hypersurface:

$$T := \pi(\{t \in \mathfrak{g}_1 | q(t) = 0, t \neq 0\}) = \pi\mathcal{T} \sqcup \pi\mathcal{U} \sqcup V.$$

Note that it follows from Proposition 4 that q is not identically zero since $V \neq \emptyset$. Then, we see that $\operatorname{Tan} V \subseteq T$ follows from Proposition 6 and Corollary B1 (2), and $\operatorname{Tan} V \supseteq T$ follows from Theorem A (2) and (3). $\square$

Corollary A6. *If V is irreducible, then the singular locus of* $\operatorname{Tan} V$ *is an algebraic set in $\mathbb{P}_*(\mathfrak{g}_1)$ defined by the system of cubic equations, $[uuu] = 0$, with variables in u over $\mathfrak{g}_1$.*

Proof. By virtue of Corollary A5, it suffices to show that the hypersurface defined by $q = 0$ has a singularity at $a \in \mathfrak{g}_1$ if and only if $[aaa] = 0$, that is,

$$dq(a) = 0 \Leftrightarrow [aaa] = 0,$$

for $a \in \mathfrak{g}_1$. Using (‡) in the proof of Proposition 4, for $\lambda \in \mathbb{C}$ and $b \in \mathfrak{g}_1$ we have

$$\begin{aligned}\frac{1}{\lambda}\{q(a+\lambda b)-q(a)\} &= \langle b,[aaa]\rangle + \langle a,[baa]\rangle + \langle a,[aba]\rangle + \langle a,[aab]\rangle + \lambda(\cdots)\\ &= 4\langle b,[aaa]\rangle + \lambda(\cdots)\\ &\to 4\langle b,[aaa]\rangle \quad (\lambda \to 0).\end{aligned}$$

Therefore $dq(a) = 0$ if and only if $\langle b,[aaa]\rangle = 0$ for any $b \in \mathfrak{g}_1$, which is equivalent to $[aaa] = 0$ since $\langle , \rangle$ is non-degenerate. □

Remark. It follows from Proposition 5 (4) that the rational map

$$\gamma : \mathbb{P}_*(\mathfrak{g}_1) \dashrightarrow \mathbb{P}_*(\mathfrak{g}_1)$$

given by $a \mapsto [aaa]$ is a Cremona transformation of $\mathbb{P}_*(\mathfrak{g}_1)$ of order 2 (see, for example, [21, III]), that is, a birational automorphism of $\mathbb{P}_*(\mathfrak{g}_1)$ with $\gamma^2 = 1$, and the base locus is $\pi\mathcal{U}$. It follows from Proposition 5 that $\gamma\pi\mathcal{T} \subseteq \pi\mathcal{V}$ and $\pi\mathcal{T} = \gamma^{-1}\pi\mathcal{V}$, and that γ gives an automorphism of $\pi\mathcal{S}$. Moreover it follows from Proposition 5 (3) and Theorem A (2) that $\gamma^{-1}P = T_PV \cap \pi\mathcal{T}$ for any $P \in V$. From the proof of Corollary A6 we see that γ is explicitly given by the partial differentials of q.

Appendix. Proof of (S3)

We firstly show (S2), since it is partly used in showing (S3) below.

Proof of (S2) (Asano [3]). It follows from the Jacobi identity that $[[x,y]E_-] = [[x,E_-]y] + [x[y,E_-]]$, so that $[x[y,E_-]] = [y[x,E_-]] + 2\langle x,y\rangle H$. Adding $[y[x,E_-]]$ on the both sides, applying $\frac{1}{2}\operatorname{ad} z$, we get

$$[xyz] = [z[y[x,E_-]]] - \langle x,y\rangle z. \tag{*}$$

Therefore it follows from the Jacobi identity that

$$\begin{aligned}[xyz]-[xzy] &= \{[z[y[x,E_-]]] - \langle x,y\rangle z\} - \{[y[z[x,E_-]]] - \langle x,z\rangle y\}\\ &= \langle x,z\rangle y - \langle x,y\rangle z + [[x,E_-],[y,z]]\\ &= \langle x,z\rangle y - \langle x,y\rangle z + 2\langle y,z\rangle[[x,E_-]E_+].\end{aligned}$$

Thus, we get the formula since we have $[[x,E_-]E_+] = x$. □

Proof of (S3) (Asano [3]). It suffices to show that

$$[L(x,x),L(y,y)] = 2L(L(x,x)y,y). \tag{♡}$$

Indeed, setting $x := v + w$ in (♡), and using (♡) again, we obtain

$$[L(v,w),L(y,y)] = 2L(L(v,w)y,y).$$

Moreover setting $y := x + y$ in this formula, and using (♡) again, we obtain

$$[L(v,w),L(x,y)] = L(L(v,w)x,y) + L(x,L(v,w)y).$$

Then we have

$$\begin{aligned}&[L(v,w),L(x,y)] = L(L(v,w)x,y) + L(x,L(v,w)y)\\ &\quad\Leftrightarrow L(v,w)L(x,y) = L(L(v,w)x,y) + L(x,L(v,w)y) + L(x,y)L(v,w),\end{aligned}$$

and the latter is nothing but (S3).

To prove (♡), we show the following

Lemma. *Denote by* xy^* *the map* $z \mapsto \langle z, y\rangle x$*, and set* $\widetilde{x} := [x[x, E_-]]$*. Then:*

(1) $L(x, y) = -\operatorname{ad} y \circ \operatorname{ad} x \circ \operatorname{ad} E_- - 2xy^* - 2yx^* - \langle x, y\rangle 1$.
(2) $y(L(x, x)y)^* = -yy^* \circ L(x, x)$.
(3) $\operatorname{ad} L(x, x)y = [\operatorname{ad} y, \operatorname{ad} \widetilde{x}]$.
(4) $\operatorname{ad} \widetilde{x} \circ \operatorname{ad} E_- = \operatorname{ad} E_- \circ \operatorname{ad} \widetilde{x}$.

Proof. (1) This follows from (*) and the Jacobi identity: Indeed, we have

$$\begin{aligned} L(x, y)z &= [z[y[x, E_-]]] - \langle x, y\rangle z \\ &= [[z, y][x, E_-]] + [y[z[x, E_-]]] - \langle x, y\rangle z \\ &= 2\langle z, y\rangle [E_+[x, E_-]] + [y[[z, x]E_-]] + [y[x[z, E_-]]] - \langle x, y\rangle z \\ &= -2\langle z, y\rangle x + 2\langle z, x\rangle [y[E_+, E_-]] + [y[x[z, E_-]]] - \langle x, y\rangle z \\ &= -2\langle z, y\rangle x - 2\langle z, x\rangle y - [y[x[E_-, z]]] - \langle x, y\rangle z. \end{aligned}$$

(2) Using Lemma 2 in §1, we have

$$y(L(x, x)y)^* z = \langle z, L(x, x)y\rangle y = -\langle L(x, x)z, y\rangle y = -yy^*(L(x, x)z).$$

(3) Since it follows from (*) that $L(x, x) = -(\operatorname{ad} \widetilde{x})|\mathfrak{g}_1$, we have

$$[L(x, x)y, z] = -[[\widetilde{x}, y]z] = [[y, z]\widetilde{x}] + [[z, \widetilde{x}]y] = [\operatorname{ad} y, \operatorname{ad} \widetilde{x}]z.$$

(4) Since $[[x, E_-]E_-] = 0$, we have $[\widetilde{x}, E_-] = 0$, so that $[\widetilde{x}[E_-, z]] = [E_-[\widetilde{x}, z]]$. □

Proof of (♡).

$$\begin{aligned} &[L(x, x), L(y, y)] - \{L(L(x, x)y, y) + L(y, L(x, x)y)\} \\ &\quad = [L(x, x), -(\operatorname{ad} y)^2 \circ \operatorname{ad} E_- - 4yy^*] \\ &\qquad - \{-\operatorname{ad} y \circ \operatorname{ad} L(x, x)y \circ \operatorname{ad} E_- \\ &\qquad\qquad - 2(L(x, x)y)y^* - 2y(L(x, x)y)^* - \langle L(x, x)y, y\rangle 1\} \\ &\qquad - \{-\operatorname{ad} L(x, x)y \circ \operatorname{ad} y \circ \operatorname{ad} E_- \\ &\qquad\qquad - 2y(L(x, x)y)^* - 2(L(x, x)y)y^* - \langle y, L(x, x)y\rangle 1\} \\ &\qquad (\because \text{Lemma (1)}) \\ &\quad = [\operatorname{ad} \widetilde{x}, (\operatorname{ad} y)^2 \circ \operatorname{ad} E_-] - [L(x, x), 4yy^*] \\ &\qquad + \operatorname{ad} y \circ [\operatorname{ad} y, \operatorname{ad} \widetilde{x}] \circ \operatorname{ad} E_- + [\operatorname{ad} y, \operatorname{ad} \widetilde{x}] \circ \operatorname{ad} y \circ \operatorname{ad} E_- \\ &\qquad\qquad + 4y(L(x, x)y)^* + 4(L(x, x)y)y^* \\ &\qquad (\because L(x, x) = -\operatorname{ad} \widetilde{x}|\mathfrak{g}_1 \text{ and Lemma (3)}) \\ &\quad = \operatorname{ad} \widetilde{x} \circ (\operatorname{ad} y)^2 \circ \operatorname{ad} E_- - (\operatorname{ad} y)^2 \circ \operatorname{ad} E_- \circ \operatorname{ad} \widetilde{x} \\ &\qquad - 4L(x, x) \circ yy^* + 4yy^* \circ L(x, x) \\ &\qquad\qquad + (\operatorname{ad} y)^2 \circ \operatorname{ad} \widetilde{x} \circ \operatorname{ad} E_- - \operatorname{ad} y \circ \operatorname{ad} \widetilde{x} \circ \operatorname{ad} y \circ \operatorname{ad} E_- \\ &\qquad\qquad\qquad + \operatorname{ad} y \circ \operatorname{ad} \widetilde{x} \circ \operatorname{ad} y \circ \operatorname{ad} E_- - \operatorname{ad} \widetilde{x} \circ (\operatorname{ad} y)^2 \circ \operatorname{ad} E_- \\ &\qquad\qquad\qquad\qquad + 4y(L(x, x)y)^* + 4(L(x, x)y)y^* \\ &\quad = -4L(x, x) \circ yy^* + 4yy^* \circ L(x, x) + 4y(L(x, x)y)^* + 4(L(x, x)y)y^* \end{aligned}$$

$$\begin{aligned} &(\because \text{Lemma } (4)) \\ &= -4L(x,x) \circ yy^* + 4(L(x,x)y)y^* \quad (\because \text{Lemma } (2)). \end{aligned}$$

The last term is equal to zero, since we have

$$(L(x,x) \circ yy^*)(z) = L(x,x)(\langle z,y\rangle y) = \langle z,y\rangle L(x,x)y = (L(x,x)y)y^*(z). \quad \square$$

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