ON THE DUALS OF SEGRE VARIETIES

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ABSTRACT. The reflexivity, the (semi-)ordinariness, the dimension of dual varieties and the structure of Gauss maps are discussed for Segre varieties, where a *Segre variety* is the image of the product of two or more projective spaces under Segre embedding. A generalization is given to a theorem of A. Hefez and A. Thorup on Segre varieties of two projective spaces. In particular, a new proof is given to a theorem of F. Knop, G. Menzel, I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky that states a necessary and sufficient condition for Segre varieties to have codimension one duals. On the other hand, a negative answer is given to a problem raised by S. Kleiman and R. Piene as follows: For a projective variety of dimension at least two, do the Gauss map and the natural projection from the conormal variety to the dual variety have the same inseparable degree?

0. INTRODUCTION

Segre varieties are one of the most fundamental examples of projective varieties. The purpose of this article is to study the Segre varieties from the view-point of projective geometry, where a *Segre variety* is the image of the product of two or more projective spaces under the Segre embedding. Precisely speaking, we discuss four points, namely, the reflexivity, the (semi-)ordinariness, the dimension of dual varieties, and the structure of Gauss maps for Segre varieties (see §1 for the terminology).

A. Hefez and A. Thorup [3] studied the first three points for Segre varieties of just *two* projective spaces. According to their result [3, Theorem 2], such varieties are all reflexive for any characteristic of the ground field, for instance. However, this does not hold for general cases. Our main result gives a generalization of the result of Hefez and Thorup, as follows:

Theorem. *Let X be the image of the Segre embedding of $\prod_{i=1}^{r} \mathbb{P}^{n_i}$ over the ground field K of characteristic $p \geq 0$ with dimension $n = \sum_{i=1}^{r} n_i$, and assume that $n_i \geq 1$ for all i and $r \geq 2$. Then we have:*

    (1) *The dual variety X^* is a hypersurface if and only if $2n_i \leq n$ for all i.*

(2) *If $2n_i > n$, then*
    (a) *X is reflexive with* $\operatorname{codim} X^* = 1 + 2n_i - n$, *and*
    (b) *X is semi-ordinary if and only if $2n_i = n + 1$.*
(3) *If $2n_i \leq n$ for all i, then the following conditions are equivalent:*
    (a) *X is not reflexive (or equivalently, not ordinary).*
    (b) *X is semi-ordinary.*
    (c) *$p = 2$ and n is odd.*

I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky [1] proved the statement (1) over the complex number field $\mathbb{C}$ (see [1, Theorem 0.1]), which follows also from more general results by F. Knop and G. Menzel [9]. Our proof is quite different from the known ones and seems to be much simpler.

For the fourth point, we have

Proposition 1. *The Gauss map γ of the Segre embedding of $\prod_{i=1}^{r} \mathbb{P}^{n_i}$ with $n_i \geq 1$ for all i and $r \geq 2$ is always an embedding in any characteristic.*

Thus it turns out that, combining Theorem (3) and Proposition 1, one obtains a *negative* answer to a problem raised by S. Kleiman and R. Piene [8] as follows:

Problem (Kleiman-Piene [8, pp. 108–109]). *For a projective variety $X \subseteq \mathbb{P}^N$ of dimension $n \geq 2$, let $\gamma : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$ be the Gauss map, CX the conormal variety, X^* the dual variety, and $\pi : CX \to X^*$ the natural projection. Then do γ and π have the same inseparable degree?*

Note that it is well-known that X is reflexive if and only if π is separable (see [7, (4.4)]), and that the answer for the case $n = 1$ is known to be affirmative (see [2], [4], [5], [10]).

Related to this problem, although it has a slightly different flavour from the subject of the present paper, we include the following result here since it is proved in a quite similar way to the case of Proposition 1:

Proposition 2. *Let $Y \subseteq \mathbb{P}^N$ be a projective variety of dimension $n \geq 2$, and let v_d be the Veronese embedding of $\mathbb{P}^N$ of degree $d \geq 2$. Then the Gauss map γ of $v_d(Y)$ is generically finite and separable, and moreover it is finite and unramified if Y is smooth.*

It is known that in characteristic 2 the Gauss map of every projective curve has inseparable degree at least 2 (for a plane curve, this is a classical result, and for general cases, see [5, Corollaries 2.2 and 2.3]). But, it turns out from Proposition 2 that this statement does not hold for a higher dimensional case.

The contents of this article are organized as follows: In §1 we give basic definitions and state a known fact due to Hefez and Thorup [3]. In §2 we show Propositions 1 and 2, along with introducing some notation. In §3 we study the rank of symmetric matrices of a certain form (see Lemma in §3), which plays the key role in our proof of Theorem. Finally in §4 we prove Theorem.

To simplify the notation and arguments, we work over an algebraically closed field K. But, all the results here are easily verified to be true over an arbitrary field by trivial modification of the setup and proofs given here: In that case, Y in Proposition 2 should be a closed subscheme of $\mathbb{P}^N$ geometrically integral over K, and smooth over K in the latter assertion.

1. Preliminaries

For a projective variety $X \subseteq \mathbb{P}^N$, let CX° be the set of pairs (P, H) such that P is a smooth point of X and H is a hyperplane tangent to X at P. The *conormal variety* of X, denoted by CX, is by definition the closure of CX° in $X \times \check{\mathbb{P}}^N$, and the *dual variety* of X, denoted by X^*, is the image of CX in $\check{\mathbb{P}}^N$ under the natural projection, where $\check{\mathbb{P}}^N$ is the dual space of $\mathbb{P}^N$. On the other hand, the *Gauss map* of X, denoted by γ, is by definition the rational map $X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$ that sends a smooth point P of X to the embedded tangent space to X at P, where n is the dimension of X and $\mathbb{G}(n, \mathbb{P}^N)$ is the Grassmann variety of n-dimensional linear spaces in $\mathbb{P}^N$.

A projective variety $X \subseteq \mathbb{P}^N$ is said to be *reflexive* if we have $CX \simeq C(X^*)$ via $\mathbb{P}^N \times \check{\mathbb{P}}^N \simeq \check{\mathbb{P}}^N \times \mathbb{P}^N$, and *ordinary* if X is reflexive and X^* has codimension 1.

Moreover, for a general (hence, smooth) point P of X and for a rational function h on $\mathbb{P}^N$ defining a general hyperplane tangent to X at P by its zero locus, denote by $H(X)$ the Hessian matrix of the function h_X on X at P, where h_X is the restriction of h to X. In general we have the following inequalities:

$$1 \leq \operatorname{codim} X^* \leq \operatorname{cork} H(X) + 1.$$

A projective variety $X \subseteq \mathbb{P}^N$ is said to be *semi-ordinary* if $H(X)$ has corank 1.

For full details on those notions, we refer to [2], [6], [7], [8].

Fact (Hefez-Thorup [3, Lemma 3]). *Let $X \subseteq \mathbb{P}^N_{\mathbb{Z}}$ be a closed, integral subscheme, smooth over $\mathbb{Z}$, let K be an algebraically closed field, and denote by X_K the fibre product $X \times_{\mathbb{Z}} K$. Then we have:*

(1) $\operatorname{codim}(X_K)^* = \operatorname{cork} H(X_{\mathbb{C}}) + 1$.
(2) X_K *is reflexive if and only if* $\operatorname{cork} H(X_K) = \operatorname{cork} H(X_{\mathbb{C}})$.

Remark. Hefez and Thorup [3] proved a corresponding statement for an arbitrary field K with a more general setup.

2. The Gauss map

Proof of Proposition 1. Let $(x_{i1}, x_{i2}, \ldots, x_{in_i}) = (1 : x_{i1} : x_{i2} : \cdots : x_{in_i})$ be affine coordinates of $\mathbb{P}^{n_i}$ for each i. For $P = (x_{ia})_{1 \leq i \leq r, 1 \leq a \leq n_i} \in \prod_{i=1}^r \mathbb{P}^{n_i}$, the Segre embedding σ is given by

$$\sigma(P) = (1 : x_{11} : x_{12} : \cdots : x_{1n_1} : x_{21} : \cdots : x_{rn_r} : x_{11}x_{21} : \cdots : x_{1n_1} \cdots x_{rn_r}) \in \mathbb{P}^N,$$

and the embedded tangent space at $\sigma(P)$ is spanned by $n + 1$ row vectors as follows:

$$\sigma(P), \frac{\partial \sigma}{\partial x_{11}}(P), \ldots, \frac{\partial \sigma}{\partial x_{1n_1}}(P), \frac{\partial \sigma}{\partial x_{21}}(P), \ldots, \frac{\partial \sigma}{\partial x_{rn_r}}(P),$$

where $N + 1 = \prod_{i=1}^r (n_i + 1)$ and $n = \sum_{i=1}^r n_i$. Consider an $(n+1) \times (N+1)$-matrix formed by those row vectors . If we denote by $\Delta(j_0, \ldots, j_n)$ its minor of degree $n + 1$ consisting of columns $j_0, \ldots, j_n$, then the Gauss map γ is given by

$$\gamma(P) = (\Delta(j_0, \ldots, j_n))_{1 \leq j_0 < \cdots < j_n \leq N+1} \in \mathbb{P}^M$$

via the Plücker embedding, where $M+1 = \binom{N+1}{n+1}$. Then we see that $\Delta(1,\ldots,n,n+1) = 1$, and moreover that for any i with $1 \leq i \leq r$ and for any a with $1 \leq a \leq n_i$, there exist integers l and m such that

$$\Delta(1,2,\ldots,l-1,m,l+1,\ldots,n+1) = x_{ia},$$

where $1 \leq l \leq n+1 < m \leq N+1$: Indeed, for such i and a, there exists a column vector in the $(n+1) \times (N+1)$-matrix whose coordinates are given by

$$(x_{ia}x_{jb}, 0, \ldots, 0, x_{jb}, 0, \ldots, 0, x_{ia}, 0, \ldots)$$

with $i < j$ or

$$(x_{jb}x_{ia}, 0, \ldots, 0, x_{ia}, 0, \ldots, 0, x_{jb}, 0, \ldots)$$

with $j < i$. Thus, we see that γ is an embedding. $\square$

Proof of Proposition 2. Let P_0 be a smooth point of Y, let $\{x_1,\ldots,x_n\}$ be local parameters of Y at P_0, and denote by ι the embedding $Y \hookrightarrow \mathbb{P}^N$: one may assume that ι around P_0 is given by

$$\iota(x_1,\ldots,x_n) = (1 : x_1 : \cdots : x_n : \cdots) \in \mathbb{P}^N,$$

so that $v_d \circ \iota$ is given by

$$v_d \circ \iota(x_1,\ldots,x_n) = (1 : x_1 : \cdots : x_n : q_1 : \cdots : q_M : \cdots) \in \mathbb{P}^{N_d},$$

where $q_1,\ldots,q_M$ are the monomials in the x_i of degree 2, $M = \binom{n+1}{2}$ and $N_d+1 = \binom{N+d}{d}$. Then, we see that the Gauss map of $v_d(Y)$ is locally an embedding around P_0: Indeed, a similar argument to the last part of the proof of Proposition 1 works by choosing reduced monomials from the q_i, where we need the assumption $n \geq 2$. $\square$

3. Symmetric matrices

Let $n_1,\ldots,n_r$ be positive integers, and set $n := \sum_{i=1}^r n_i$. Let M_{kl} be an $n_k \times n_l$-matrix with $k < l$ such that each component is an indeterminate, set $M_{lk} :=^t M_{kl}$, and consider an $n \times n$-matrix as follows:

$$M(n_1,\ldots,n_r) := \begin{bmatrix} 0 & M_{12} & M_{13} & \cdots & M_{1r} \\ M_{21} & 0 & M_{23} & \cdots & M_{2r} \\ M_{31} & M_{32} & 0 & \cdots & M_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & M_{r3} & \cdots & 0 \end{bmatrix}.$$

Denote by z_{ij} the (i,j)-th component of $M(n_1,\ldots,n_r)$ with $i < j$. Then $M(n_1,\ldots,n_r)$ is a symmetric matrix over a polynomial ring $\mathbb{Z}[\{z_{ij} | 1 \leq i < j \leq n\}]$.

Lemma. *For given positive integers $n_1, \ldots, n_r$ with $n := \sum_{i=1}^{r} n_i$, and for an arbitrary field K of characteristic $p \geq 0$, we have:*

(1) *If $2n_i > n$, then $M(n_1, \ldots, n_r)$ has corank $2n_i - n$ over K.*

(2) *Assume $2n_i \leq n$ for all i. If $p = 2$, and n is odd, then $M(n_1, \ldots, n_r)$ has corank 1 over K; Otherwise, $M(n_1, \ldots, n_r)$ has corank 0 over K.*

Proof. (1) Renumbering the indexes, one may assume that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$. Consider a partition of $M(n_1, \ldots, n_r)$ as follows:

$$M(n_1, \ldots, n_r) = \begin{bmatrix} M'_{11} & M'_{12} & M'_{13} \\ M'_{21} & 0 & 0 \\ M'_{31} & 0 & 0 \end{bmatrix},$$

where M'_{11} and M'_{12} are $(n - n_r) \times (n - n_r)$-matrices, M'_{13} is an $(n - n_r) \times (2n_r - n)$-matrix and $M'_{lk} =^t M'_{kl}$. Then, the first principal minor of degree $2(n - n_r)$ is equal to

$$(-1)^{n-n_r} \det M'_{12} \det M'_{21} = (-1)^{n-n_r} (\det M'_{12})^2,$$

whose coefficient in

$$z^2_{1,n-n_r+1} z^2_{2,n-n_r+2} \cdots z^2_{n-n_r,2n-2n_r}$$

is equal to ± 1. On the other hand, any minor of degree $2(n - n_r) + 1$ is obviously zero. Therefore, $M(n_1, \ldots, n_r)$ has rank $2(n - n_r)$ over $\mathbb{Q}$ as well as over $\mathbb{Z}/p\mathbb{Z}$ for all prime p, hence over an arbitrary field K.

(2) It suffices to show:

(a) for odd $n = 2m + 1$, all the coefficient of $\det M(n_1, \ldots, n_r)$ are divisible by 2, and there exist integers $i(1), \ldots, i(m), j(1), \ldots, j(m), k$ with $i(t) < j(t)$ for all t and $j(m) < k$ such that

$$\{i(1), \ldots, i(m), j(1), \ldots, j(m), k\} = \{1, \ldots, n\}$$

and the coefficient of $\det M(n_1, \ldots, n_r)$ in

$$z^2_{i(1)j(1)} \cdots z^2_{i(m-1)j(m-1)} z_{i(m)j(m)} z_{i(m)k} z_{j(m)k}$$

is equal to ± 2.

(b) for even $n = 2m$, there exist integers $i(1), \ldots, i(m), j(1), \ldots, j(m)$ with $i(t) < j(t)$ for all t such that

$$\{i(1), \ldots, i(m), j(1), \ldots, j(m)\} = \{1, \ldots, n\}$$

and the coefficient of $\det M(n_1, \ldots, n_r)$ in

$$z^2_{i(1)j(1)} \cdots z^2_{i(m)j(m)}$$

is equal to ± 1;

Indeed, if $p \neq 2$ or n is even, then it clearly follows from (a) and (b) that $M(n_1, \ldots, n_r)$ has rank n over an arbitrary field K. If $p = 2$ and n is odd, then it follows from (a) that $M(n_1, \ldots, n_r)$ has rank at most $n - 1$. Moreover we see from (b) that the rank is actually equal to $n - 1$, as follows: Renumbering the indexes, we may assume that $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. Consider an $(n-1) \times (n-1)$-matrix M' by taking off the 1-st row and column from $M(n_1, \ldots, n_r)$, which has the same form as $M(n_1 - 1, n_2, \ldots, n_r)$. Then it follows from (b) that M' has rank $n - 1$: Indeed, the sizes $n_1 - 1$, n_2, $\ldots$, n_r of blocks in M' satisfy the assumption of (2), where we omit $n_1 - 1$ if it is zero. To see this, it suffices to verify the condition for n_2: this is trivial for $n_1 - 1$ by the assumption of (2). Suppose the contrary, that is, suppose

$$n_2 > (n_1 - 1) + n_3 + \cdots + n_r.$$

Then we have $n_2 \geq n_1 + n_3 \geq n_1$, so that $n_1 = n_2$ and $n_3 = 0$ with $r = 2$, which contradicts our assumption of n being odd.

Now, let us show the statements (a) and (b). To show the former assertion of (a), denote by $\mathfrak{S}_n$ the permutation group of the set $\{1, 2, \ldots, n\}$, and set

$$\begin{aligned}
F &:= \{g \in \mathfrak{S}_n \mid \exists i, g(i) = i\}, \\
G_+ &:= \{g \in \mathfrak{S}_n \setminus F \mid \#\{i \mid g(i) > i\} > n/2\}, \\
G_- &:= \{g \in \mathfrak{S}_n \setminus F \mid \#\{i \mid g(i) > i\} < n/2\}.
\end{aligned}$$

Since n is odd, we have $\mathfrak{S}_n = F \sqcup G_+ \sqcup G_-$ and

$$\begin{aligned}
\det M(n_1, \ldots, n_r) &= \sum_{g \in \mathfrak{S}_n} \operatorname{sign}(g) z_{1g(1)} z_{2g(2)} \cdots z_{ng(n)} \\
&= \sum_{g \in F}(\cdots) + \sum_{g \in G_+}(\cdots) + \sum_{g \in G_-}(\cdots).
\end{aligned}$$

Now, the sum over F is zero since the diagonal components of $M(n_1, \ldots, n_r)$ are zero. On the other hand, the sum over G_- is equal to the sum over G_+ since $M(n_1, \ldots, n_r)$ is symmetric: Indeed, since a map $G_+ \to G_-; g \mapsto g^{-1}$ is bijective and $\operatorname{sign}(g) = \operatorname{sign}(g^{-1})$, we have

$$\begin{aligned}
\sum_{g \in G_-} \operatorname{sign}(g) z_{1g(1)} z_{2g(2)} \cdots z_{ng(n)} &= \sum_{g \in G_+} \operatorname{sign}(g^{-1}) z_{1g^{-1}(1)} z_{2g^{-1}(2)} \cdots z_{ng^{-1}(n)} \\
&= \sum_{g \in G_+} \operatorname{sign}(g) z_{g(1)1} z_{g(2)2} \cdots z_{g(n)n},
\end{aligned}$$

which is equal to that sum over G_+ since $z_{g(i)i} = z_{ig(i)}$. This proves the former assertion of (a).

For the remaining part, we use induction on n, where we allow the integers n_i to be zero. Assume $n \geq 4$: Note that the determinant is equal to $-z_{12}^2$ for $n = 2$, and equal to $2z_{12} z_{13} z_{23}$ for $n = 3$. Assume that $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, as before, and consider an

$(n-2) \times (n-2)$-matrix M'' by taking off the 1-st and (n_1+1)-th rows and columns from $M(n_1, \ldots, n_r)$, which has the same form as $M(n_1 - 1, n_2 - 1, n_3, \ldots, n_r)$. Then, for M'' we may assume the hypothesis of induction: Indeed, the sizes $n_1 - 1$, $n_2 - 1$, n_3, $\ldots$, n_r of blocks of M'' satisfy the assumption of (2). To see this, it suffices to verify the condition for n_3 with $r \geq 3$, as before. Suppose the contrary, that is, suppose

$$n_3 > (n_1 - 1) + (n_2 - 1) + n_4 + \cdots + n_r.$$

Then we have $n_3 \geq n_1 + n_2 - 1 \geq 2n_3 - 1$, so that $n_3 = 1$, which implies $n_1 = n_2 = 1$ with $r = 3$ and $n = 3$. This contradicts our assumption $n \geq 4$.

Now, it follows from the hypothesis of the induction that there exist integers $i(2)$, $\ldots$, $i(m)$, $j(2)$, $\ldots$, $j(m)$, k satisfying

$$\{i(2), \ldots, i(m), j(2), \ldots, j(m), k\} = \{1, \ldots, n\} \setminus \{1, n_1 + 1\}$$

for $n = 2m + 1$ (respectively, there exist integers $i(2)$, $\ldots$, $i(m)$, $j(2)$, $\ldots$, $j(m)$ satisfying

$$\{i(2), \ldots, i(m), j(2), \ldots, j(m)\} = \{1, \ldots, n\} \setminus \{1, n_1 + 1\}$$

for $n = 2m$) such that the above property on the coefficient for M'' is satisfied. Then for $n = 2m + 1$ the coefficient of $\det M(n_1, \ldots, n_r)$ in

$$z_{1n_1+1}^2 z_{i(2)j(2)}^2 \cdots z_{i(m-1)j(m-1)}^2 z_{i(m)j(m)} z_{i(m)k} z_{j(m)k}$$

is equal to ± 2, and for $n = 2m$ the coefficient of $\det M(n_1, \ldots, n_r)$ in

$$z_{1n_1+1}^2 z_{i(2)j(2)}^2 \cdots \cdots z_{i(m)j(m)}^2$$

is equal to ± 1, as is required. $\square$

4. THE HESSIAN MATRIX

Let P_0 be the point of the Segre variety X with all x_{ia} being zero, and let h be a rational function on $\mathbb{P}^N$ corresponding to a general hyperplane tangent to X at P_0, with the same notation as in §2. Then we have

$$\sigma^* h = \sum_{1 \leq i < j \leq r, 1 \leq a \leq n_i, 1 \leq b \leq n_j} h_{ij}^{ab} x_{ia} x_{jb} + \text{(terms of higher degree in } x_{kc}),$$

where the h_{ij}^{ab} are general elements of K. Therefore, the Hessian matrix at P_0 is given as follows:

$$H(X) = \left[\frac{\partial^2 \sigma^* h}{\partial x_{ia} \partial x_{jb}}(P_0) \right] = \begin{bmatrix} 0 & H_{12} & H_{13} & \cdots & H_{1r} \\ H_{21} & 0 & H_{23} & \cdots & H_{2r} \\ H_{31} & H_{32} & 0 & \cdots & H_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{r1} & H_{r2} & H_{r3} & \cdots & 0 \end{bmatrix},$$

where $H_{ij} = [\, h_{ij}^{ab}\,]_{1 \leq a \leq n_i, 1 \leq b \leq n_j}$ and $H_{ji} =^t H_{ij}$ with $1 \leq i < j \leq r$.

Proof of Theorem. (1) Using Fact (1), we see that the if-part follows from Lemma (2), and the only-if-part follows from Lemma (1).

(2) The assertion (a) follows from Fact (2) and Lemma (1), and (b) also follows from Lemma (1).

(3) Note that since codim $X^* = 1$ by (1), X is reflexive if and only if X is ordinary. Then, the equivalence (a) $\Leftrightarrow$ (c) follows from Fact (2) and Lemma (2), and (b) $\Leftrightarrow$ (c) also follows from Lemma (2). $\square$

References

1. I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Hyperdeterminants*, Adv. Math. **96** (1992), 226–263.
2. A. Hefez, S. Kleiman, *Notes on the duality of projective varieties*, "Geometry Today", Prog. Math., vol. 60, Birkhäuser, Boston, 1985, pp. 143–183.
3. A. Hefez, A. Thorup, *Reflexivity of Grasmannians and Segre verieties*, Comm. Alg. **15 (6)** (1987), 1095–1108.
4. A. Hefez, J. Voloch, *Frobenius non classical curves*, Arch. Math. **54** (1990), 263–273.
5. H. Kaji, *On the Gauss maps of space curves in characteristic p*, Compos. Math. **70** (1989), 177–197.
6. H. Kaji, *On the inseparable degrees of the Gauss map and the projection of the conormal variety to the dual of higher order for space curves*, Math. Ann. **292** (1992), 529–532.
7. S. Kleiman, *About the conormal scheme*, Springer Lecture Notes in Math. **1092** (1984), 161–197.
8. S. Kleiman, R. Piene, *On the inseparability of the Gauss map*, Contemp. Math. **123** (1991), 107–129.
9. F. Knop, G. Menzel, *Duale Varietäten von Fahnenvarietäten*, Comment. Math. Helvetici **62** (1987), 38–61.
10. J. Voloch, *On the geometry of space curves*, Proceedings of the 10th School of Algebra, Vitoria, Brazil, 1989, pp. 121–122.

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