PROJECTIVE GEOMETRY OF FREUDENTHAL'S VARIETIES OF CERTAIN TYPE

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0. Introduction

H. Freudenthal constructed, in a series of his papers (see [10] and its references), the exceptional Lie algebras of type E_8, E_7, E_6 and F_4, with defining various projective varieties. The purpose of our work is to study projective geometry for his varieties of certain type, which are called *varieties of planes* in the symplectic geometry of Freudenthal (see [10, 4.11], [22, 2.3]).

Let $\mathfrak{g}$ be a graded, simple, finite-dimensional Lie algebra over the complex number field $\mathbb{C}$ with grades between -2 and 2, $\dim \mathfrak{g}_2 = 1$ and $\mathfrak{g}_1 \neq 0$, namely a *graded Lie algebra of contact type*: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (see §1). We set

$$\mathcal{V} := \{x \in \mathfrak{g}_1 \setminus \{0\} | (\mathrm{ad}\, x)^2 \mathfrak{g}_{-2} = 0\},$$

and define an algebraic set V in $\mathbb{P}(\mathfrak{g}_1)$ to be the projectivization of $\mathcal{V}$:

$$V := \pi(\mathcal{V}),$$

where $\pi : \mathfrak{g}_1 \setminus \{0\} \to \mathbb{P}(\mathfrak{g}_1)$ is the natural projection. Then we call $V \subseteq \mathbb{P}(\mathfrak{g}_1)$ (with the reduced structure) the *Freudenthal variety* associated to the graded Lie algebra $\mathfrak{g}$ of contact type, which is a natural generalization of Freudenthal's varieties mentioned above: Note that V is not necessarily connected in this general setting. We here consider moreover the projectivization of a closed set $\{x \in \mathfrak{g}_1 | (\mathrm{ad}\, x)^{k+1} \mathfrak{g}_{-2} = 0\}$, and denote it by V_k: we have

$$\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \mathbb{P},$$

where we set $\mathbb{P} := \mathbb{P}(\mathfrak{g}_1)$ for short. Clearly, V_3 is a quartic hypersurface, V_2 is an intersection of cubics and $V_1 = V$ is an intersection of quadrics, with a few exceptions.

In the literature, several results have been known about the structure of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-space, case-by-case for each exceptional Lie algebra of types E_8, E_7, E_6 and F_4, from the view-point of the invariant theory of prehomogeneous vector spaces (see [13], [15], [18], [21]). By virtue of those results, it can be shown, for example, that the stratification of $\mathbb{P}$ given by the differences of V_k's exactly corresponds to the orbit decomposition of the $\mathfrak{g}_0$-space $\mathfrak{g}_1$ for those exceptional Lie algebras, and also that Freudenthal varieties V associated to the algebras of type E_8, E_7, E_6 and F_4 are respectively projectively equivalent to the 27-dimensional E_7-variety arising from the 56-dimensional irreducible representation, the orthogonal Grassmann variety of isotropic 6-planes in $\mathbb{C}^{12}$ (namely, the 15-dimensional spinor variety), the Grassmann variety of 3-planes in $\mathbb{C}^6$ and the symplectic Grassmann variety of isotropic 3-planes in $\mathbb{C}^6$, with $\dim \mathbb{P} = 55, 31, 19$ and 13, respectively (see Appendix): for those homogeneous projective varieties, we refer to [12, §23.3].

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In this article we study the Freudenthal varieties V with the filtration $\{V_k\}$ of the ambient space $\mathbb{P}$, from the view-point of projective geometry, not individually but systematically in terms of abstract Lie algebras, without depending on the classification of simple Lie algebras as well as on the known results for each case of types E_8, E_7, E_6 and F_4.

Before stating the main result, we note that the Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2 \simeq \mathbb{C}$ defines a non-degenerate skew-symmetric form on $\mathfrak{g}_1$, so that this form allows us to identify $\mathfrak{g}_1$ with its dual space, hence $\mathbb{P}$ with its dual space, and $\mathfrak{g}_1$ is even-dimensional. Moreover, the quartic form on $\mathfrak{g}_1$ defining V_3 has a differential which via the symplectic form defines a vector field on $\mathfrak{g}_1$, and this vector field defines a 1-dimensional distribution on $\mathbb{P}$ away from the singular locus of V_3 (see Proposition A1). We denote by L_P the (closure of the) integral curve of this distribution passing through $P \in \mathbb{P} \setminus \operatorname{Sing} V_3$. On the other hand, we have a rational map $\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$ defined by $x \mapsto (\operatorname{ad} x)^3 \mathfrak{g}_{-2}$ with base locus V_2, which turns out to be a Cremona transformation of $\mathbb{P}$: It is deduced that $\gamma^{-1}(V) = V_3 \setminus V_2$, $\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$, $\gamma^2 = 1$ on $\mathbb{P} \setminus V_3$, and γ is explicitly given by the partial differentials of q (see Proposition A2). Note that our γ is a special case of the Cremona transformations in [7, Theorem 2.8 (ii)].

Our main results are summarized as follows (see Theorems A, B, C, D, E, Corollaries A2, B1, B3 and C):

Theorem. *Assume that V is irreducible. Then we have:*

(1) *V is a Legendrian subvariety of $\mathbb{P}$, that is, the projectivization of a Lagrangian subvariety of $\mathfrak{g}_1$, with $\dim V = n-1$, spans $\mathbb{P}$, and is an orbit of the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_0$, hence smooth, where $\dim \mathfrak{g}_1 = 2n$. In particular, the projective dual V^* of V is equal to the union of tangents to V via the symplectic form.*

(2) *V_2 is the singular locus of V_3, and for any $P \in \mathbb{P} \setminus V_2$, L_P is the line in $\mathbb{P}$ joining P and $\gamma(P)$. Moreover, we have:*
  - (a) *If $P \in \mathbb{P} \setminus V_3$, then L_P is a unique secant line of V passing through P, there is no tangent line to V passing through P, $L_P \cap V$ consists of harmonic conjugates with respect to P and $\gamma(P)$, and $L_P \setminus V \subseteq \mathbb{P} \setminus V_3$. Moreover, γ preserves L_P, and the automorphism of L_P induced from γ leaves each point in $L_P \cap V$ invariant and permutes P and $\gamma(P)$.*
  - (b) *If $P \in V_3 \setminus V_2$, then there is no secant line of V passing through P, L_P is a unique tangent line to V passing through P, $L_P \cap V = \gamma(P)$, and $L_P \setminus V \subseteq V_3 \setminus V_2$. Moreover, L_P is contracted by γ to the contact point $\gamma(P)$, and conversely the fibre of γ on $Q \in V$ consists of the points $P \in V_3 \setminus V_2$ such that $Q \in L_P$, or equivalently, P lies on some tangent to V at Q.*

  In particular, V is a variety with one apparent double point, and V_3 is the union of tangents to V.

(3) *For any $P \in V_2 \setminus V$, the family of secants of V passing through P is of dimension at least 1, and all of those secants are isotropic with respect to the symplectic form: In particular, $V_2 \setminus V$ is covered by isotropic secants of V.*

(4) *For any $Q, R \in V$, the secant line joining Q and R is isotropic if and only if the tangents to V at Q and at R are disjoint.*

(5) *For any $P \in V_3 \setminus V_2$ and $Q \in V$, if the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic, then there is a twisted cubic curve contained in V to which L_P and L_R are tangent at $\gamma(P)$ and at Q, respectively, where R is a point on some tangent to V at Q away from V_2, determined by P and Q.*

(6) *If $V_2 \neq V$, then V is ruled, that is, covered by lines contained in V.*

(7) *For any $P \in V$, the double projection from P gives a birational map from V onto $\mathbb{P}^{n-1}$, and by the inverse V is written as the closure of the image of a cubic Veronese embedding*

of a certain affine space $\mathbb{A}^{n-1}$ under some projection to $\mathbb{P}$.

We show also that *the three conditions, $V = \emptyset$, $V_3 = \mathbb{P}$ and $V_2 = \mathbb{P}$ are equivalent to each other* (see Corollary A1), and that *if V is neither empty nor irreducible, then $\mathfrak{g}_1$ decomposes naturally into two irreducible $\mathfrak{g}_0$-submodules of dimension n and V is the (disjoint) union of the projectivizations of those summands* (see Corollary B2).

Finally we should mention that S. Mukai announced a theorem [20, (5.8)] on cubic Veronese varieties without proofs. Our work was originated by looking for proofs of the corresponding statements for Freudenthal varieties (Corollaries A2, B1, C and Theorem D): In fact, we see from his list [20, (5.10)] of cubic Veronese varieties (and the list in Appendix) that the notion of our Freudenthal varieties coincides with that of his cubic Veronese varieties. Our result gives a partial explanation for this coincidence (see Theorem D).

This is a joint work with Osami Yasukura. For proofs of the results here, see [FV].

1. Preliminaries

For a finite-dimensional, simple Lie algebra $\mathfrak{g}$ of rank ≥ 2, a graded decomposition of contact type is obtained as follows: Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a basis Δ of the root system R with respect to $\mathfrak{h}$, and fix an order on R defined by Δ. Denote by ρ the highest root of $\mathfrak{g}$, let E_+ and E_- be highest and lowest weight vectors, respectively, and set $H := [E_+, E_-]$. By multiplying suitable scalars, one may assume that (E_+, H, E_-) form an $\mathfrak{sl}_2$-triple, that is, those vectors have the following standard relations:

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \quad [E_+, E_-] = H.$$

Then, the eigenspace decomposition of $\mathfrak{g}$ with respect to $\operatorname{ad} H$ gives $\mathfrak{g}$ a graded decomposition of contact type: In other words, if we set $\mathfrak{g}_\lambda := \{x \in \mathfrak{g} | [H, x] = \lambda x\}$ for $\lambda \in \mathbb{C}$, then it follows that $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\dim \mathfrak{g}_2 = 1$ and $\mathfrak{g}_1 \neq 0$: In fact, $\mathfrak{g}_1 = 0$ if and only if $\mathfrak{g} = \mathfrak{sl}_2$. In terms of root spaces of $\mathfrak{g}$, we have

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \setminus (R_\rho \cup \{\rho\})} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \quad \mathfrak{g}_{\pm 1} = \bigoplus_{\alpha \in R_\rho} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm\rho} = \mathbb{C}E_\pm,$$

where R^+ is the set of positive roots and $R_\rho := \{\alpha \in R^+ | \rho - \alpha \in R\}$: Indeed, let $\mathfrak{s}_\rho$ be the subalgebra of $\mathfrak{g}$ spanned by E_+, H and E_-, which is isomorphic to $\mathfrak{sl}_2$. Then the irreducible decomposition of $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module gives the decomposition above (see, for full details, [25]). Conversely, for a graded decomposition $\mathfrak{g} = \sum \mathfrak{g}_i$ of contact type, taking suitable bases E_+ for $\mathfrak{g}_2$ and E_- for $\mathfrak{g}_{-2}$ with $H := [E_+, E_-]$, one may assume that (E_+, H, E_-) form an $\mathfrak{sl}_2$-triple, as before. Then, we see that E_+ and E_- are some highest and lowest weight vectors, respectively, and each $\mathfrak{g}_i$ is recovered as an $(\operatorname{ad} H)$-eigenspace. Therefore, the graded decompositions of contact type are unique up to automorphism of $\mathfrak{g}$, so that the Freudenthal variety V is essentially unique and determined by $\mathfrak{g}$ itself (see Appendix).

Now, we define a symmetric product $\times : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ by the formula:

$$-2a \times b = [b, [a, E_-]] + [a, [b, E_-]],$$

which induces a symmetric map $L : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \operatorname{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$ and a ternary product $[,,] : \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ by

$$[a, b, c] = L(a, b)c = [a \times b, c].$$

Note that the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful since $\mathfrak{g}$ is simple (see [25, Lemma 3.2 (1)]): we may assume $\mathfrak{g}_0 \subseteq \operatorname{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$, so that we identify $L(a,b)$ with $a \times b$. We think of $\mathfrak{g}_1$ as an $\mathfrak{g}_0$-module via the adjoint action: For example, we often write Dx instead of $(\operatorname{ad} D)x$ and $[D, x]$ for $D \in \mathfrak{g}_0$ and $x \in \mathfrak{g}_1$. As the skew-symmetric form $\langle , \rangle : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{C}$ and the quartic form on $\mathfrak{g}_1$ defining V_3 mentioned in Introduction, we use the ones determined by

$$2\langle a, b\rangle E_+ = [a, b], \quad 2q(x)E_+ = (\operatorname{ad} x)^4 E_-.$$

Note that the skew-symmetric form $\langle , \rangle$ is *non-degenerate* since $\mathfrak{g}$ is simple (see [25, Lemma 3.2 (2)]).

With the notation above, it follows that

$$\begin{aligned} V = V_1 &= \pi\left(\{x \in \mathfrak{g}_1 \setminus \{0\} | x \times x = 0\}\right), \\ V_2 &= \pi\left(\{x \in \mathfrak{g}_1 \setminus \{0\} | [xxx] = 0\}\right), \\ V_3 &= \pi\left(\{x \in \mathfrak{g}_1 \setminus \{0\} | \langle x, [xxx]\rangle = 0\}\right), \end{aligned}$$

and $q(x) = \langle x, [xxx]\rangle$. Note that $V_0 = \emptyset$ since $[[x, E_-]E_+] = x$ for any $x \in \mathfrak{g}_1$: Indeed, it follows from the Jacobi identity that $[[x, E_-]E_+] = [[x, E_+], E_-] + [x[E_-, E_+]] = [x, -H] = x$ since $[x, E_+] \in \mathfrak{g}_3 = 0$. On the other hand, it follows from Lemma 1 below that $V \neq \mathbb{P}$.

Lemma 1. *Let $\mathfrak{g}_{00}$ be the subalgebra of $\mathfrak{g}_0$ defined by*

$$\mathfrak{g}_{00} := \operatorname{Ker}(\operatorname{ad} E_+|\mathfrak{g}_0) = \operatorname{Ker}(\operatorname{ad} E_-|\mathfrak{g}_0).$$

Then we have $\mathfrak{g}_0 = \mathfrak{g}_{00} \oplus \mathbb{C}H$, and $\mathfrak{g}_{00}$ is linearly spanned by the elements in $\mathfrak{g}_0$ of the form $a \times b$ with $a, b \in \mathfrak{g}_1$. In particular, $\mathfrak{g}_{00} \neq 0$, and $x \times x \neq 0$ for some $x \in \mathfrak{g}_1$.

Lemma 2 (Asano [3])**.** *For any $a, b, c \in \mathfrak{g}_1$ and $D \in \mathfrak{g}_{00}$, we have*

(1) $\langle Da, b\rangle + \langle a, Db\rangle = 0$.
(2) $D(a \times b) = Da \times b + a \times Db$.
(3) $D[abc] = [(Da)bc] + [a(Db)c] + [ab(Dc)]$.

If we denote by G_{00} the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_{00}$, then Lemma 2 tells that the symplectic form $\langle , \rangle$, the symmetric product $\times$ and the ternary product $[, ,]$ are equivariant with respect to the action of G_{00}, so that each V_i is stable under the action of G_{00}, that is, a union of some orbits of G_{00}.

Lemma 3 (Asano [3])**.** *We have $[abc] - [acb] = \langle a, c\rangle b - \langle a, b\rangle c + 2\langle b, c\rangle a$ for any $a, b, c \in \mathfrak{g}_1$.*

2. A Line Field and a Cremona Transformation

Proposition A1.

(1) *The quartic form q on $\mathfrak{g}_1$ has a differential at $a \in \mathfrak{g}_1$ as follows:*

$$dq(a) : t_a\mathfrak{g}_1 \to \mathbb{C}; b \mapsto 4\langle b, [aaa]\rangle,$$

where $t_a\mathfrak{g}_1$ is the Zariski tangent space to $\mathfrak{g}_1$ at a, naturally identified with $\mathfrak{g}_1$.

(2) *In particular, the singular locus of V_3 is equal to V_2.*

(3) *The vector field on $\mathfrak{g}_1$ corresponding to dq via the symplectic form $\langle , \rangle$ induces a 1-dimensional distribution $\mathcal{D}$ on $\mathbb{P}$ away from $\operatorname{Sing} V_3 = V_2$, which is given by*

$$\mathcal{D} : \pi(a) \mapsto (\mathbb{C}a + \mathbb{C}[aaa])/\mathbb{C}a,$$

where $\pi(a) \in \mathbb{P} \setminus V_2$ and we naturally identify the Zariski tangent space $t_{\pi a}\mathbb{P}$ with the quotient space $\mathfrak{g}_1/\mathbb{C}a$.

Proposition A2. *Let*

$$\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$$

be a rational map induced from the cubic, $a \mapsto [aaa]$. Then we have:

1. $\gamma^{-1}(V) = V_3 \setminus V_2$.
2. $\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$.
3. $\gamma^2 = 1$ *on* $\mathbb{P} \setminus V_3$, *hence* γ *gives an automorphism of* $\mathbb{P} \setminus V_3$.
4. γ *is explicitly given by the partial differentials of* q.

In particular, γ is a Cremona transformation of $\mathbb{P}(\mathfrak{g}_1)$ with order 2 if $V_2 \neq \mathbb{P}$.

A *secant line* of V is by definition a line in $\mathbb{P}$ which passes through at least two distinct points of V and is not contained in V. We note that for a line L in $\mathbb{P}$ if the scheme-theoretic intersection $L \cap V$ has length more than 2, then $L \subseteq V$: Indeed, V is an intersection of quadric hypersurfaces.

Theorem A. *Let L_P be the closure of the integral curve of $\mathcal{D}$ through $P \in \mathbb{P} \setminus V_2$, where $\mathcal{D}$ is the 1-dimensional distribution on $\mathbb{P} \setminus V_2$ induced from the quartic form q. Then we have:*

1. *For any $P \in \mathbb{P} \setminus V_2$, L_P is the line in $\mathbb{P}$ joining P and $\gamma(P)$.*
2. *If $P \in \mathbb{P} \setminus V_3$, then we have:*
   - (a) *L_P is a secant line of V, and $L_P \cap V$ consists of harmonic conjugates with respect to P and $\gamma(P)$.*
   - (b) $L_P \setminus V \subseteq \mathbb{P} \setminus V_3$.
   - (c) *L_P is a unique secant line of V passing through P.*
   - (d) *There is no tangent line to V passing through P.*
   - (e) *$\gamma(L_P \setminus V) = L_P \setminus V$, and the automorphism of L_P induced from γ leaves each point in $L_P \cap V$ invariant and permutes P and $\gamma(P)$.*
3. *If $P \in V_3 \setminus V_2$, then we have:*
   - (a) *L_P is a tangent line to V, and $L_P \cap V = \{\gamma(P)\}$.*
   - (b) $L_P \setminus V \subseteq V_3 \setminus V_2$.
   - (c) *There is no secant line of V passing through P.*
   - (d) *L_P is a unique tangent line to V passing through P.*
   - (e) *$\gamma(L_P \setminus V) = \gamma(P)$, and $\gamma^{-1}(Q) = \{P \in V_3 \setminus V_2 | Q \in L_P\} = T_Q V \setminus V_2$ for any $Q \in V$, where $T_Q V$ is the embedded tangent space to V at Q.*

Corollary A1. *The three conditions, $V = \emptyset$, $V_3 = \mathbb{P}$ and $V_2 = \mathbb{P}$ are equivalent to each other.*

Remark A. It can be shown that $V = \emptyset$ if and only if the Lie algebra $\mathfrak{g}$ is of type C (see Appendix): In fact, using a theorem of Asano [28, 1.6.Theorem], [4], one can show that if $q \equiv 0$, then $\mathfrak{g} \simeq \mathfrak{sp}_{2n+2}$, where $\dim \mathfrak{g}_1 = 2n$; The converse is checked by an explicit computation.

Recall that a projective variety $V \subseteq \mathbb{P}$ is called a *variety with one apparent double point* if for a general point $P \in \mathbb{P}$ there exists a unique secant line of V passing through P (see [23, IX]).

Corollary A2. *If $V \neq \emptyset$, then V is a variety with one apparent double point. In particular, V is non-degenerate in $\mathbb{P}$.*

3. The Homogeneity

Theorem B. *Let G_{00} be the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_{00}$, where $\mathfrak{g}_{00}$ is the subalgebra of $\mathfrak{g}_0$ defined by $\mathfrak{g}_{00} := \mathrm{Ker}(\mathrm{ad}\, E_\pm | \mathfrak{g}_0)$. Then we have:*

1. *G_{00} acts transitively on each of irreducible components of $\mathcal{V}$. In particular, we have $t_x \mathcal{V} = \mathfrak{g}_{00} x$ for any $x \in \mathcal{V}$, where $t_x \mathcal{V}$ is the Zariski tangent space to $\mathcal{V}$ at x.*

(2) $\mathfrak{g}_{00}x = (\mathfrak{g}_{00}x)^\perp$ *with* $2\dim \mathfrak{g}_{00}x = \dim \mathfrak{g}_1$ *for any* $x \in \mathcal{V}$, *and* $\mathfrak{g}_1 = \mathfrak{g}_{00}x \oplus \mathfrak{g}_{00}y$ *for any* $x, y \in \mathcal{V}$ *with* $\langle x, y\rangle \neq 0$.

Recall that the *tangent variety* of V, denoted by $\operatorname{Tan} V$, is the union of embedded tangent spaces to V, and the *projective dual* of V, denoted by V^*, is the set of hyperplanes tangent to V (see, for example, [11, §3]).

Corollary B1. *Assume that* $V \neq \emptyset$. *Then we have:*

(1) G_{00} *acts transitively on each of irreducible components of* V, *and* V *is smooth, equidimensional of dimension* $n-1$, *where* $\dim \mathfrak{g}_1 = 2n$.

(2) *Denote by* L^* *the set of hyperplanes containing a linear subspace* $L \subseteq \mathbb{P}$. *Then we have* $(T_Q V)^* = T_Q V$ *for any* $Q \in V$, *hence*

$$\operatorname{Tan} V = V^*,$$

where we identify $\mathbb{P}$ *with its dual space* $\mathbb{P}^\vee := \mathbb{P}(\mathfrak{g}_1^*)$ *via the symplectic form* $\langle , \rangle$.

Corollary B2. *If* V *is neither empty nor irreducible, then there are irreducible* $\mathfrak{g}_{00}$*-modules* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *of dimension* n *such that* $\mathfrak{g}_1 = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, *and we have*

$$V = \mathbb{P}(\mathfrak{s}_1) \sqcup \mathbb{P}(\mathfrak{s}_2),$$

where $\dim \mathfrak{g}_1 = 2n$.

Remark B1. It is known that V is irreducible unless $\mathfrak{g}$ is of type A or C (see Appendix): In fact, if $\mathfrak{g} = \mathfrak{so}_m$, then V is a Segre embedding of $\mathbb{P}^1 \times Q$ in $\mathbb{P}^{2m-9}$, where Q is a quadric hypersurface in $\mathbb{P}^{m-5}$; if $\mathfrak{g}$ is of type G_2, then V is a cubic Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$; for other exceptional Lie algebras $\mathfrak{g}$, see Introduction. Conversely, it follows from a direct computation that we are in the case above if $\mathfrak{g} = \mathfrak{sl}_{n+2}$ with $n \geq 1$.

Corollary B3. *If* $V \neq \emptyset$ *and* $V_2 \neq V$, *then* V *is ruled, that is, covered by lines contained in* V.

Remark B2. It can be shown that $V = V_2$ if and only if $\mathfrak{g}$ is of type G_2.

4. Isotropic Secants

Proposition C. *For* $P = \pi(u) \in \mathbb{P}$, *let* $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$ *be a rational map induced from* $L(u, u)$ *with base locus* $B_P = \mathbb{P}(\operatorname{Ker} L(u, u))$. *If* V *is irreducible and* $P \in V_2 \backslash V$, *then* $\dim \Phi_P(V \backslash B_P) \geq 1$, *hence* $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ *and* $\operatorname{codim} B_P \geq 2$.

Remark C1. The irreducibility condition for V is essential in Proposition C: In fact, there is an example of u satisfying the assumption above such that $\operatorname{rk} L(u, u) = 1$ in case of $\mathfrak{g} = \mathfrak{sl}_m$, where V is not irreducible (see Remark B3).

Remark C2. It follows easily from Proposition 6 that $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ if $P \notin V_2$, and $\operatorname{codim} \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ if $P \in V_3$, though we do not use these facts in this article.

Recall that the *secant locus* Σ_P as well as the *tangent locus* Θ_P of V with respect to a given point $P \in \mathbb{P}$ are defined by

$$\Sigma_P := \overline{\{Q \in V | \exists R \in V \setminus \{Q\}, P \in Q * R\}}, \quad \Theta_P := \{Q \in V | P \in T_Q V\},$$

where we denote by $Q * R$ the line in $\mathbb{P}$ joining Q and R, and by $T_Q V$ the embedded tangent space to V at Q in $\mathbb{P}$ (see, for example, [11]).

Theorem C. *Assume that V is irreducible. Then we have:*

(1) *For any $x, y \in \mathcal{V}$, $\langle x, y\rangle = 0$ if and only if $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y \neq 0$. In particular, a secant line joining $Q, R \in V$ is isotropic with respect to the symplectic form if and only if $T_QV \cap T_RV \neq \emptyset$.*

(2) *$V_2 \setminus V$ is covered by isotropic secants of V. More precisely, for any $u \in \mathfrak{g}_1$, we have that $[uuu] = 0$ and $u \times u \neq 0$ if and only if $u = x + y$ for some $x, y \in \mathcal{V}$ such that $\langle x, y\rangle = 0$ and $x \times y \neq 0$.*

(3) *If $P \in V_2 \setminus V$, then*

$$\Phi_P(V \setminus B_P) \subseteq \Sigma_P, \quad \Phi_P(V \cap P^\perp \setminus B_P) \subseteq \Theta_P,$$

where $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$ is the rational map induced from $L(u, u)$ with base locus $B_P = \mathbb{P}(\operatorname{Ker} L(u, u))$ and $P^\perp = \mathbb{P}(u^\perp)$ with $P = \pi(u)$.

(4) *We have $\dim \Sigma_P \geq 1$ for any $P \in V_2 \setminus V$.*

Remark C3. The irreducibility condition for V is essential in (1) above: In fact, it is easily seen that the conclusion does not hold in case of $\mathfrak{g} = \mathfrak{sl}_m$.

Corollary C. *If V is irreducible, then $V_3 = \operatorname{Tan} V$.*

5. Double Projections

Proposition D. *For any $x, y \in \mathcal{V}$, let $\Psi_{xy} : \mathfrak{g}_1 \to \mathfrak{g}_1$ be a linear map defined by*

$$\Psi_{xy}(a) := [axy] + \langle a, x\rangle y.$$

(1) *If $\langle x, y\rangle \neq 0$, then $\operatorname{Ker} \Psi_{xy} = \mathfrak{g}_{00}x$ and $\Psi_{xy}(\mathfrak{g}_1) = \mathfrak{g}_{00}y$. In particular, a rational map $\Psi_{PQ} : \mathbb{P} \dashrightarrow \mathbb{P}$ induced from Ψ_{xy} is a double projection from P with image T_QV, that is, a projection with center T_PV onto T_QV, hence defines a morphism*

$$\Psi_{PQ} : \mathbb{P} \setminus T_PV \to T_QV,$$

where T_PV is the embedded tangent space to V at P with $P = \pi(x)$ and $Q = \pi(y)$.

(2) *Moreover for any $R \in V$, the four points R, $[PQR]$, $\Psi_{PR}(Q)$ and $\Psi_{QR}(P)$ are collinear, and $[PQR]$ is the harmonic conjugate of R with respect to $\Psi_{PR}(Q)$, $\Psi_{QR}(P)$, where we set $[PRR] := \pi([xyz])$ with $R = \pi(z)$. In particular, this holds for general $P, Q, R \in V$ and gives a geometric meaning of our ternary product.*

Remark D1. In terms of the Lie bracket, we have $\Psi_{ab}(c) = [b[a[c, E_-]]]$.

Theorem D. *For any $P, Q \in V$, if the secant line joining P and Q is not isotropic, that is, $T_PV \cap T_QV = \emptyset$, then we have:*

(1) *$V \setminus P^\perp = (\Psi_{PQ}|_{V \setminus T_PV})^{-1}(T_QV \setminus P^\perp)$.*

(2) *The double projection Ψ_{PQ} gives an isomorphism $V \setminus P^\perp \to T_QV \setminus P^\perp$. In fact, a rational map $\Gamma_{QP} : T_QV \dashrightarrow V$ induced from a map $\Gamma_{yx} : \mathfrak{g}_{00}y \to \mathcal{V} \cup \{0\}$ defined by*

$$\Gamma_{yx}(t) := \langle x, [ttt]\rangle x + 3\langle x, t\rangle [ttx] + 12\langle x, t\rangle^2 t$$

gives the inverse of $\Psi_{PQ}|_{V \setminus P^\perp}$, where $P = \pi(x)$ and $Q = \pi(y)$.

(3) *The base locus of Γ_{QP} is $T_QV \cap P^\perp \cap V_2$.*

In particular, if V is irreducible, then Ψ_{PQ} gives a birational map from V to T_QV, and V is the closure of the image of a composition of a cubic Veronese embedding of the affine space $T_QV \setminus P^\perp$ with some projection to $\mathbb{P}$.

Remark D2. The morphism $\Psi_{PQ} : V \setminus T_PV \to T_QV$ is not necessarily surjective: In fact, if $\mathfrak{g}$ is of type G_2, then for any $P \in V$, $P^\perp$ is the osculating plane to the twisted cubic $V \subseteq \mathbb{P}^3$ at P, $V \cap P^\perp = \{P\}$, and $\Psi_{PQ}(V \setminus T_PV) = T_QV \setminus P^\perp$ for any $Q \in V$ with $P \neq Q$.

6. Twisted Cubic Curves

Proposition E. *For any $P \in V_3 \setminus V_2$ and $Q \in V$, if the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic, then we have:*

1. $Q \in L_{\Phi_P(Q)}$ *and* $\Phi_P^3(Q) = \gamma(P) \in L_P = L_{\Phi_P^2(Q)}$ *with* $\Phi_P(Q), \Phi_P^2(Q) \in V_3 \setminus V_2$.
2. $L_P \cap L_{\Phi_P(Q)} = \emptyset$, *hence* $Q, \Phi_P(Q), \Phi_P^2(Q)$ *and* $\Phi_P^3(Q)$ *are linearly independent in* $\mathbb{P}$.

Theorem E. *For any $P \in V_3 \backslash V_2$ and $Q \in V$ such that the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic, that is, $T_Q V \cap T_{\gamma(P)} V = \emptyset$, let $\mathbb{P}_{PQ}$ be the linear subspace of dimension 3 in $\mathbb{P}$ spanned by Q, $\Phi_P(Q)$, $\Phi_P^2(Q)$ (or equivalently P) and $\Phi_P^3(Q) = \gamma(P)$, that is, spanned by L_P and $L_{\Phi_P(Q)}$, the unique tangent lines to V passing through P and $\Phi_P(Q)$. Then we have:*

1. *The intersection $V \cap \mathbb{P}_{PQ}$ is a twisted cubic curve in $\mathbb{P}_{PQ} \simeq \mathbb{P}^3$ given explicitly by the image of L_P under the cubic map $\Gamma_{\gamma(P)Q}$:*

$$V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P).$$

2. *The twisted cubic curve in $\mathbb{P}_{PQ}$ above has the following properties:*
   (a) *L_P and $L_{\Phi_P(Q)}$ are respectively the tangent lines at $\gamma(P)$ and at Q, and*
   (b) *$\gamma(P)^\perp \cap \mathbb{P}_{PQ}$ and $Q^\perp \cap \mathbb{P}_{PQ}$ are respectively the osculating planes at $\gamma(P)$ and at Q, which are spanned by L_P and $\Phi_P(Q)$ and by $L_{\Phi_P(Q)}$ and $\Phi_P^2(Q)$, respectively.*

Remark E. Set $D := L(t,t)$, $E := L(Dx, Dx)$ and $F := [D, E]$ with $P = \pi(t)$ and $Q = \pi(x)$, and denote by $\mathfrak{g}_{00PQ}$ the subalgebra of $\mathfrak{g}_{00}$ generated by D, E and F. Then it follows that

$$[F, D] = \frac{4}{3}\langle D^3 x, x\rangle D, \quad [F, E] = -\frac{4}{3}\langle D^3 x, x\rangle E,$$

so that $\mathfrak{g}_{00PQ}$ is isomorphic to the Lie algebra $\mathfrak{sl}_2$. If we denote by $\mathfrak{g}_{1PQ}$ the subspace of $\mathfrak{g}_1$ spanned by x, Dx, D^2x and D^3x, then we see that $\mathfrak{g}_{1PQ}$ is an irreducible $\mathfrak{g}_{00PQ}$-module of dimension 4 with

$$F(D^k x) = (2k-3)\frac{2}{3}\langle D^3 x, x\rangle D^k x,$$

and the twisted cubic curve $V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P)$ is a unique closed orbit in $\mathbb{P}_{PQ} = \mathbb{P}(\mathfrak{g}_{1PQ})$ under the natural action of the group of inner automorphisms of $\mathfrak{g}_{00}$ with Lie algebra $\mathfrak{g}_{00PQ}$. Thus, for any $P \in V_3 \setminus V_2$ and $Q \in V$ with $T_{\gamma(P)}V \cap T_Q V = \emptyset$, a subalgebra $\mathfrak{g}_{00PQ}$ of $\mathfrak{g}_{00}$ isomorphic to $\mathfrak{sl}_2$ and an irreducible $\mathfrak{g}_{00PQ}$-submodule $\mathfrak{g}_{1PQ}$ of $\mathfrak{g}_1$ with dimension 4 are associated to P and Q. If $\mathfrak{g}$ is of type G_2, then $\mathfrak{g}_{00PQ}$ and $\mathfrak{g}_{1PQ}$ are respectively equal to $\mathfrak{g}_{00}$ and $\mathfrak{g}_1$ themselves.

Appendix. A Classification of Freudenthal Varieties

We here give a classification of Freudenthal varieties V in terms of the root data of $\mathfrak{g}$. It would be interesting to compare V with the adjoint variety associated to $\mathfrak{g}$ since those varieties are closely related to each other: In fact, for a simple graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_i$ of contact type, denote by V the Freudenthal variety associated to $\mathfrak{g}$, as before, and denote by X the orbit of the inner automorphism group of $\mathfrak{g}$ through $\pi(E_+)$ in $\mathbb{P}(\mathfrak{g})$, which is the minimal closed orbit in $\mathbb{P}(\mathfrak{g})$, called the *adjoint variety* associated to $\mathfrak{g}$ (see [16]). Then, according to [17, Theorem B], we have $V = X \cap \mathbb{P}(\mathfrak{g}_1)$.

Adjoint Varieties and Freudenthal Varieties

$\mathfrak{g}$	$X \subseteq \mathbb{P}(\mathfrak{g})$	$\mathfrak{g}_{00}$	$V \subseteq \mathbb{P}(\mathfrak{g}_1)$
$\mathfrak{sl}_m$	$(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}) \cap (1) \subseteq \mathbb{P}^{m^2-2}$	$\mathfrak{gl}_1 \oplus \mathfrak{sl}_{m-2}$	$\mathbb{P}^{m-3} \sqcup \mathbb{P}^{m-3} \subseteq \mathbb{P}^{2m-5}$
$\mathfrak{so}_m$	$\mathbb{G}_{\text{orthog.}}(2, m) \subseteq \mathbb{P}^{\binom{m}{2}-1}$	$\mathfrak{sl}_2 \oplus \mathfrak{so}_{m-4}$	$\mathbb{P}^1 \times Q^{m-6} \subseteq \mathbb{P}^{2m-9}$
$\mathfrak{sp}_{2m}$	$v_2\mathbb{P}^{2m-1} \subseteq \mathbb{P}^{\binom{2m+1}{2}-1}$	$\mathfrak{sp}_{2m-2}$	$\emptyset \subseteq \mathbb{P}^{2m-3}$
$\mathfrak{e}_6$	$E_6(\omega_2)^{21} \subseteq \mathbb{P}^{77}$	$\mathfrak{sl}_6$	$\mathbb{G}(3,6) \subseteq \mathbb{P}^{19}$
$\mathfrak{e}_7$	$E_7(\omega_1)^{33} \subseteq \mathbb{P}^{132}$	$\mathfrak{so}_{12}$	$S_5 = \mathbb{G}_{\text{orthog.}}(6, 12) \subseteq \mathbb{P}^{2^5-1}$
$\mathfrak{e}_8$	$E_8(\omega_8)^{57} \subseteq \mathbb{P}^{247}$	$\mathfrak{e}_7$	$E_7(\omega_6) \subseteq \mathbb{P}^{55}$
$\mathfrak{f}_4$	$F_4(\omega_1)^{15} \subseteq \mathbb{P}^{51}$	$\mathfrak{sp}_6$	$\mathbb{G}_{\text{sympl.}}(3, 6) \subseteq \mathbb{P}^{13}$
$\mathfrak{g}_2$	$G_2(\omega_2)^5 \subseteq \mathbb{P}^{13}$	$\mathfrak{sl}_2$	$v_3\mathbb{P}^1 \subseteq \mathbb{P}^3$

Notation: We denote by $\cap(1)$ cutting by a general hyperplane, and by v_d the Veronese embedding of degree d. We denote by $\mathbb{G}(r, m)$ a Grassmann variety of r-planes in $\mathbb{C}^m$, and denote by $\mathbb{G}_{\text{orthog.}}(r, m)$ and by $\mathbb{G}_{\text{symp.}}(r, m)$ respectively an orthogonal and a symplectic Grassmann varieties of isotropic r-planes in $\mathbb{C}^m$. A simple exceptional Lie algebra of Dynkin type G is denoted by the lowercase of G in the German character, as in [12], a simple algebraic group of type G is denoted by just G, and for a dominant integral weight ω of G, the minimal closed orbit of G in $\mathbb{P}(V_\omega)$ is denoted by $G(\omega)$, where V_ω is the irreducible representation space of G with highest weight ω: For example, $\mathfrak{g}_2$ in the list is the simple Lie algebra of type G_2, and $G_2(\omega_2)$ is the minimal closed orbit of an algebraic group of type G_2 in $\mathbb{P}(V_{\omega_2})$, where ω_2 is the second fundamental dominant weight with the standard notation of Bourbaki [6].

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