Projective geometry of Freudenthal's varieties of certain type

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Freudenthal's Magic Square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	projective variety
elliptic geometry	B_1	A_2	C_3	F_4	Severi variety$\cap(1)$
projective geometry	A_2	$A_2 + A_2$	A_5	E_6	Severi variety
symplectic geometry	C_3	A_5	D_6	E_7	*Freudenthal variety* V
metasymplectic geometry	F_4	E_6	E_7	E_8	adjoint variety X

Freudenthal Varieties

$\mathfrak{g}$	$\mathfrak{g}_{00}$	$V \subseteq \mathbb{P}(\mathfrak{g}_1)$	references
$\mathfrak{e}_8$	$\mathfrak{e}_7$	$E_7(\omega_6) \subseteq \mathbb{P}^{55}$	[S.Harris] (1971)
$\mathfrak{e}_7$	$\mathfrak{so}_{12}$	$S_5 = \mathbb{G}_{\text{orthog.}}(6, 12) \subseteq \mathbb{P}^{2^5-1}$	[J.-i.Igusa] (1970)
$\mathfrak{e}_6$	$\mathfrak{sl}_6$	$\mathbb{G}(3, 6) \subseteq \mathbb{P}^{19}$	[T.Kimura] (1982)
$\mathfrak{f}_4$	$\mathfrak{sp}_6$	$\mathbb{G}_{\text{sympl.}}(3, 6) \subseteq \mathbb{P}^{13}$	[J.-i.Igusa] (1970)
$\mathfrak{g}_2$	$\mathfrak{sl}_2$	$v_3\mathbb{P}^1 \subseteq \mathbb{P}^3$	[M.Sato-M.Kashiwara-T.Kimura-T.Oshima] (1980)
$\mathfrak{so}_m$	$\mathfrak{sl}_2 \oplus \mathfrak{so}_{m-4}$	$\mathbb{P}^1 \times Q^{m-6} \subseteq \mathbb{P}^{2m-9}$	[M.Sato-M.Kashiwara-T.Kimura-T.Oshima] (1980)
$\mathfrak{sp}_{2m}$	$\mathfrak{sp}_{2m-2}$	$\emptyset \subseteq \mathbb{P}^{2m-3}$	
$\mathfrak{sl}_m$	$\mathfrak{gl}_1 \oplus \mathfrak{sl}_{m-2}$	$\mathbb{P}^{m-3} \sqcup \mathbb{P}^{m-3} \subseteq \mathbb{P}^{2m-5}$	

Notation: We denote by $\cap(1)$ cutting by a general hyperplane, and by v_d the Veronese embedding of degree d. We denote by $\mathbb{G}(r, m)$ a Grassmann variety of r-planes in $\mathbb{C}^m$, and denote by $\mathbb{G}_{\text{orthog.}}(r, m)$ and by $\mathbb{G}_{\text{symp.}}(r, m)$ respectively an orthogonal and a symplectic Grassmann varieties of isotropic r-planes in $\mathbb{C}^m$. A simple exceptional Lie algebra of Dynkin type G is denoted by the lowercase of G in the German character, a simple algebraic group of type G is denoted by just G, and for a dominant integral weight ω of G, the minimal closed orbit of G in $\mathbb{P}(V_\omega)$ is denoted by $G(\omega)$, where V_ω is the irreducible representation space of G with highest weight ω: For example, $\mathfrak{g}_2$ in the list is the simple Lie algebra of type G_2, and $G_2(\omega_2)$ is the minimal closed orbit of an algebraic group of type G_2 in $\mathbb{P}(V_{\omega_2})$, where ω_2 is the second fundamental dominant weight with the standard notation of Bourbaki.

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Aim: study projective geometry of Freudenthal varieties *systematically* with *unified proofs*, not depending of the classification of simple Lie algebras

Notation:

$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, a simple, graded Lie algebra of contact type/$\mathbb{C}$
(E_+, H, E_-), the $\mathfrak{sl}_2$-triple associated to the decomposition $\mathfrak{g} = \sum \mathfrak{g}_i$ above
$\mathfrak{g}_{00} := \mathrm{Ker}(\mathrm{ad}\, E_+|_{\mathfrak{g}_0})$, a subalgebra of $\mathfrak{g}_0$ with $\mathfrak{g}_0 = \mathfrak{g}_{00} \oplus \mathbb{C}$
G_{00}, a connected, algebraic group with Lie algebra $\mathfrak{g}_{00}$
$\mathcal{V} := \{x \in \mathfrak{g}_1 \setminus \{0\} | (\mathrm{ad}\, x)^2 \mathfrak{g}_{-2} = 0\}$
$V := \pi(\mathcal{V}) \subseteq \mathbb{P}(\mathfrak{g}_1)$, the *Freudenthal variety* associated to $\mathfrak{g}$
$\pi : \mathfrak{g}_1 \setminus \{0\} \to \mathbb{P}(\mathfrak{g}_1) =: \mathbb{P}$, the natural projection
$V_k := \pi(\{x \in \mathfrak{g}_1 \setminus \{0\} | (\mathrm{ad}\, x)^{k+1} \mathfrak{g}_{-2} = 0\})$ with $\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \mathbb{P}$ and $V = V_1$
q, a quartic polynomial on $\mathfrak{g}_1$ defining V_3, defined by $(\mathrm{ad}\, x)^4 E_- = 2q(x) E_+$
$\langle , \rangle : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2 \simeq \mathbb{C}$, a non-degenerate skew-symmetric form defined by $[x, y] = 2\langle x, y \rangle E_+$
$\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$, the Cremona transformation defined by $x \mapsto (\mathrm{ad}\, x)^3 \mathfrak{g}_{-2}$ with base locus V_2
$\mathcal{D}$, the 1-dimensional distribution on $\mathbb{P} \setminus \mathrm{Sing}\, V_3$ defined by the differential dq
L_P, the (closure of the) integral curve of $\mathcal{D}$ passing through $P \in \mathbb{P} \setminus \mathrm{Sing}\, V_3$

Theorem (H.Kaji-O.Yasukura). *Assume that V is irreducible. Then we have:*

(1) *V is a Legendre subvariety of $\mathbb{P}$, that is, the projectivization of a Lagrangian subvariety of $\mathfrak{g}_1$, with $\dim V = n - 1$, spans $\mathbb{P}$, and is an orbit of G_{00} hence smooth, where $\dim \mathfrak{g}_1 = 2n$. In particular, the projective dual V^* of V is equal to the union of tangents to V via the symplectic form.*

(2) *V_2 is the singular locus of V_3, and for any $P \in \mathbb{P} \setminus V_2$, L_P is the line in $\mathbb{P}$ joining P and $\gamma(P)$. Moreover, we have:*
   (a) *If $P \in \mathbb{P} \setminus V_3$, then L_P is a unique secant line of V passing through P, there is no tangent line to V passing through P, $L_P \cap V$ consists of harmonic conjugates with respect to P and $\gamma(P)$, and $L_P \setminus V \subseteq \mathbb{P} \setminus V_3$. Moreover, γ preserves L_P, and the automorphism of L_P induced from γ leaves each point in $L_P \cap V$ invariant and permutes P and $\gamma(P)$.*
   (b) *If $P \in V_3 \setminus V_2$, then there is no secant line of V passing through P, L_P is a unique tangent line to V passing through P, $L_P \cap V = \gamma(P)$, and $L_P \setminus V \subseteq V_3 \setminus V_2$. Moreover, L_P is contracted by γ to the contact point $\gamma(P)$, and conversely the fibre of γ on $Q \in V$ consists of the points $P \in V_3 \setminus V_2$ such that $Q \in L_P$, or equivalently, P lies on some tangent to V at Q. Furthermore, for any $Q \in V$, if the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic with respect to the symplectic form, then there is a twisted cubic curve C contained in V such that L_P and L_R are tangent to C at $\gamma(P)$ and at Q, respectively, where R is a point in $T_Q V \setminus V_2$ uniquely determined by P and Q.*

   In particular, V is a variety with one apparent double point, and V_3 is the union of tangents to V.

(3) *For any $P \in V_2 \setminus V$, the family of secants of V passing through P is of dimension at least 1 and a tangent to V passing through P exists as a degeneration of those secants. Moreover, a secant line joining $Q, R \in V$ is isotropic if and only if $T_Q V \cap T_R V \neq \emptyset$.*

(4) *If $V_2 \neq V$, then V is ruled, that is, covered by lines contained in V.*

(5) *For any $P \in V$, a double projection from P gives a birational map from V onto $\mathbb{P}^{n-1}$, and by the inverse V is written as the closure of the image of a cubic Veronese embedding of a certain affine space $\mathbb{A}^{n-1}$ under some projection to $\mathbb{P}$.*

It can be shown also that *the three conditions, $V = \emptyset$, $V_3 = \mathbb{P}$ and $V_2 = \mathbb{P}$ are equivalent to each other*, and that *if V is neither empty nor irreducible, then $\mathfrak{g}_1$ decomposes naturally into two irreducible $\mathfrak{g}_0$-submodules of dimension n and V is the (disjoint) union of the projectivizations of those summands.*