1998/03/11–13@NAGANO

Secant varieties of Adjoint Varieties

KAJI, Hajime (Waseda University)

0. Introduction: The purpose of this talk is to illustrate the structure of adjoint varieties and their secant varieties in case of type A, and to give new results for general cases: The former is almost known; the latter is part of a joint work [KOY, KY] with O. Yasukura (Fukui University) and M. Ohno (Waseda University).

1. Definitions:

Definition 1 [B1, B2]**:** For a complex simple Lie algebra $\mathfrak{g}$, let G be a simple algebraic group with Lie algebra $\mathfrak{g}$, and denote by $\mathrm{Ad} : G \curvearrowright \mathfrak{g}$ the adjoint representation: If $G \subseteq GL$ then $\mathrm{Ad}\, g$ is just the conjugation by $g \in G$. Consider a natural action

$$G \curvearrowright \mathbb{P}_*(\mathfrak{g}) := (\mathfrak{g} \setminus \{0\})/\mathbb{C}^\times.$$

Then it is well-known [FH, H2] that since $\mathfrak{g}$ is simple, there exists a unique closed orbit, denoted by $X(\mathfrak{g})$. We call $X(\mathfrak{g})$ an ***adjoint variety*** associated to $\mathfrak{g}$. Note that since $\mathrm{Ad}\, G = \mathrm{Int}\,\mathfrak{g} \subseteq GL(\mathfrak{g})$, $X(\mathfrak{g})$ does not depend on the choice of G and is uniquely determined by $\mathfrak{g}$, where $\mathrm{Int}\,\mathfrak{g} := \langle \exp \mathrm{ad}\, Y | Y \in \mathfrak{g}\rangle$.

Take a Cartan subalgebra $\mathfrak{h}$ and a base Δ of the root system R with respect to $\mathfrak{h}$, and fix an order on R defined by Δ [H1]. Let λ be the highest root, and take a highest root vector $X_\lambda \in \mathfrak{g}$. Then we have

$$X(\mathfrak{g}) = G \cdot x_+ = \pi(G \cdot X_\lambda) \subseteq \mathbb{P}_*(\mathfrak{g}),$$

where $x_+ := \pi X_\lambda$, and $\pi : \mathfrak{g} \setminus \{0\} \to \mathbb{P}_*(\mathfrak{g}); Y \mapsto \mathbb{C} \cdot Y$ is the canonical projection.

Definition 2 [FR, LV, Z]**:** Let $X \subseteq \mathbb{P}^N_{\mathbb{C}}$ be a complex projective variety. For distinct points $x, y \in X$, we call the line joining x and y the ***secant line*** determined by x and y, denoted by $x * y$. Moreover set

$$\mathrm{Sec}\, X := \overline{\bigcup_{x,y \in X, x \neq y} x * y} \subseteq \mathbb{P}^N_{\mathbb{C}}.$$

We call $\mathrm{Sec}\, X$ the ***secant variety*** of $X \subseteq \mathbb{P}^N_{\mathbb{C}}$. This is a projective variety and usually has singularities along at least X. Furthermore set

$$S^{(k)}X := \overline{\bigcup_{x_0,\dots,x_k \in X, \dim\langle x_0,\dots,x_k\rangle = k} \langle x_0, \dots, x_k\rangle},$$

where $\langle x_0, \dots, x_k\rangle$ is the linear subspace spanned by the points $x_0, \dots, x_k$. We call $S^{(k)}X$ the ***variety of k-secants*** of $X \subseteq \mathbb{P}^N_{\mathbb{C}}$. Of course $S^{(0)}X = X$ and $S^{(1)}X = \mathrm{Sec}\, X$.

3. Purpose: The purpose of our work is to study secants and secant varieties of adjoint varieties, from view-point of projective geometry.

4. Exapmles of Adjoint Varieties:

4-1. Classical Type:

type	$\mathfrak{g}$	$\mathrm{Int}\,\mathfrak{g}$	λ	Ad	$X(\mathfrak{g}) \subseteq \mathbb{P}_*(\mathfrak{g})$	$\dim \mathfrak{g}$
$A_{l\geq 1}$	$\mathfrak{sl}_{l+1}$	PSL_{l+1}	$\omega_1 + \omega_l$	K	$\mathbb{P}^l \times \mathbb{P}^l \cap (1)$	$l^2 + 2l$
$B_{l\geq 2}$	$\mathfrak{so}_{2l+1}$	PSO_{2l+1}	ω_2	$\wedge^2 V$	$\mathbb{F}_1(Q^{2l-1})^{4l-5}$	$2l^2 + l$
$C_{l\geq 3}$	$\mathfrak{sp}_{2l}$	PSp_{2l}	$2\omega_1$	$S^2 V$	$v_2(\mathbb{P}^{2l-1})$	$2l^2 + l$
$D_{l\geq 4}$	$\mathfrak{so}_{2l}$	PSO_{2l}	ω_2	$\wedge^2 V$	$\mathbb{F}_1(Q^{2l-2})^{4l-7}$	$2l^2 - l$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Notations: For a group G, set $PG := G/Z(G)$, where $Z(G)$ is the center of G.

$\mathfrak{sl}_{l+1} := \{Y \in M_{l+1}\mathbb{C} | \operatorname{tr} Y = 0\}$, $SL_{l+1} := \{g \in M_{l+1}\mathbb{C} | \det g = 1\} \curvearrowright V := \mathbb{C}^{\oplus l+1}$; $Z(SL_{l+1}) = \mu_{l+1}$, where μ_n denotes the group of n-th roots of unities. $K := \operatorname{Ker}(\wedge^l V \otimes V \to \wedge^{l+1} V = \mathbb{C}) \simeq \operatorname{Ker}(V^* \otimes V \to \mathbb{C})$ via $\wedge^l V \simeq V^*$.

$\mathfrak{so}_n := \{Y \in M_n\mathbb{C} | {}^tYQ + QY = 0\}$, $SO_n := \{g \in M_n\mathbb{C} | {}^tgQg = Q, \det g = 1\} \curvearrowright V := \mathbb{C}^{\oplus n}$ with

$$Q = \begin{bmatrix} 0 & I_{[n/2]} \\ I_{[n/2]} & 0 \end{bmatrix} \text{ or } Q = \begin{bmatrix} 0 & I_{[n/2]} & 0 \\ I_{[n/2]} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$Z(SO_n) = \mu_2$ if n is even, and $Z(SO_n)$ is trivial if n is odd.

$\mathfrak{sp}_{2l} := \{Y \in M_{2l}\mathbb{C} | {}^tYQ + QY = 0\}$, $Sp_{2l} := \{g \in M_{2l}\mathbb{C} | {}^tgQg = Q\} \curvearrowright V := \mathbb{C}^{\oplus 2l}$ with

$$Q = \begin{bmatrix} 0 & I_l \\ -I_l & 0 \end{bmatrix};$$

$Z(Sp_{2l}) = \mu_2$.

We denote by $\mathbb{F}_1(Q^m)^{2m-3}$ the Fano variety of line in m-dimensional quadric hypersurface $Q \subseteq \mathbb{P}^{m+1}$, v_2 the Veronese embedding, and $\cap(1)$ cutting by a hyperplane. The notations for fundamental dominant weights ω_i are same as in [B].

4-2. Exceptional Type:

type	λ	$\dim X(\mathfrak{g})$	$\dim \mathfrak{g}$
E_6	ω_2	$20+1$	78
E_7	ω_1	$32+1$	133
E_8	ω_8	$56+1$	248
F_4	ω_1	$14+1$	52
G_2	ω_2	$4+1$	14

The adjoint variety of type G_2 appears in the work of S. Mukai [M1] as a Fano 5-fold of index 3.

5. A_l-case: We set $G := PSL_{l+1}, V := \mathbb{C}^{l+1}$, and consider the following subsets:

$\operatorname{Hom}(V,V)_r := \{\varphi \in \operatorname{Hom}(V,V) | \operatorname{rk}\varphi \le r\}$,

$K := \{\varphi \in \operatorname{Hom}(V,V) | \operatorname{tr}\varphi = 0\} = \operatorname{Ker}(V^* \otimes V \to \mathbb{C}) = \mathfrak{sl}_{l+1}$,

$K_r := K \cap \operatorname{Hom}(V,V)_r$

in $\operatorname{Hom}(V,V) = V^* \otimes V = M_{l+1}\mathbb{C}$. Moreover we set $H := \operatorname{diag}(1,-1,0,\dots,0) \in M_{l+1}\mathbb{C}$, and denote by $X_{(k_1\cdots k_s)}$ a nilpotent matrix that is a direct sum of Jordan cells of sizes $k_1,\dots,k_s$. Recall that the adjoint action $\operatorname{Ad} : G \curvearrowright K$ is just taking the conjugation in this case. Obviously the algebraic set $\pi(K_r \setminus \{0\}) \subseteq \mathbb{P}_*(K)$ has defining equations of degree $r+1$.

5-1. Adjoint Varieties:

Lemma 1 [FH]**:** $G \cdot \pi X_{(21\cdots1)} = \pi(K_1 \setminus \{0\})$.

Proof: The rank and trace are determined by conjugacy class, and a traceless rank one matrix has Jordan normal form $X_{(21\cdots1)}$. $\square$

Lemma 2 [FH]**:** $\pi(K_1 \setminus \{0\}) = \mathbb{P}_*(V^*) \times \mathbb{P}_*(V) \cap \mathbb{P}_*(K)$.

Proof: We have $\pi(\operatorname{Hom}(V,V)_1 \setminus \{0\}) = \mathbb{P}_*(V^*) \times \mathbb{P}_*(V)$ since $\operatorname{rk}\varphi \le 1 \Leftrightarrow \varphi = f \otimes v$ $(\exists f \in V^*, v \in V)$ for $\varphi \in \operatorname{Hom}(V,V) = V^* \otimes V$. $\square$

From the above we obtain

Proposition 1 [FH]**:** $X(A_l) = \pi(K_1 \setminus \{0\}) = \mathbb{P}_*(V^*) \times \mathbb{P}_*(V) \cap \mathbb{P}_*(K)$.

In particular $X(A_l)$ is defined by quadric equations. Note that we also have $\pi(K_1 \setminus \{0\}) = \mathbb{F}(1,l,V) = \mathbb{P}(T_{\mathbb{P}^l}(-1))$ via $\mathbb{P}_*(\operatorname{Hom}(V,V)) = \mathbb{P}_*(V^* \otimes V) = \mathbb{P}(V^* \otimes \mathcal{O}_{\mathbb{P}^l})$, where $\mathbb{F}(1,l,V)$ is a flag variety defined by the incidence correspondence of 1- and l-subspaces of V in $\mathbb{P}_*(V^*) \times \mathbb{P}_*(V)$: In fact, the first identification is given by $\varphi \leftrightarrow (\varphi(V) \subseteq \operatorname{Ker}\varphi \subseteq V)$; the second comes from the Euler quotient $V^* \otimes \mathcal{O}_{\mathbb{P}^l} \twoheadrightarrow T_{\mathbb{P}^l}(-1)$ on $\mathbb{P}^l = \mathbb{P}(V)$.

5-2. Secant Varieties:
Proposition 2: $\operatorname{Sec} X(A_l) = \pi(K_2 \setminus \{0\})$.
Proof: The inclusion ($\subseteq$) follows from Proposition 1 and an elementary fact that $\operatorname{rk}(A+B) \leq \operatorname{rk} A + \operatorname{rk} B$. For ($\supseteq$), since the roots of the characteristic polynomial of $Y \in K_2$ are $\{a, -a, 0, \ldots, 0\}$ ($a \in \mathbb{C}$), the Jordan normal form of Y is one of the following:

$$aH, \quad X_{(31\cdots1)}, \quad X_{(221\cdots1)}, \quad X_{(21\cdots1)}.$$

Each of those matrices can be written as a sum of two elements of K_1: for example,

$$2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \quad \square$$

Observation 1: We see from the proof above that $\operatorname{Sec} X(A_l)$ consists of 4 orbits through πH, $\pi X_{(31\cdots1)}$, $\pi X_{(221\cdots1)}$ and $\pi X_{(21\cdots1)}$ if $\operatorname{rk}\mathfrak{g} = l \geq 2$. Computing the stabilizers one obtains that those orbits have dimension $4l-2, 4l-3, 4l-5$ and $2l-1$, respectively.

In particular, we have
Proposition 3: $\operatorname{Sec} X(A_l) = \overline{G \cdot \pi H}$ *and* $\dim \operatorname{Sec} X(A_l) = 2 \dim X(A_l)$.
5-2A. Higher Secant Varieties: Moreover we have
Theorem 0: $S^{(k)}X(A_l) = \pi(K_{k+1} \setminus \{0\})$ *and* $\operatorname{codim}(S^{(k)}X(A_l), \mathbb{P}_*(K)) = (l+1-k)^2$, *for* $0 \leq \forall k \leq l+1$.

This follows from Proposition 1 and the following lemmas:
Lemma 3: $S^{(k)}(\mathbb{P}_*(V^*) \times \mathbb{P}_*(V)) = \pi(\operatorname{Hom}(V,V)_{k+1} \setminus \{0\})$ *for* $0 \leq \forall k \leq l+1$.
Proof: We have $\varphi = \sum_{i=0}^k f_i \otimes v_i$ ($\exists f_i \in V^*, v_i \in V$) $\Leftrightarrow \operatorname{rk}\varphi \leq k+1$ for $\varphi \in \operatorname{Hom}(V,V)$. $\square$
Lemma 4 [H]**:** $\operatorname{codim}(\operatorname{Hom}(V,V)_k, \operatorname{Hom}(V,V)) = (l+1-k)^2$ *for* $0 \leq \forall k \leq l+1$.
Lemma 5 [K]**:** *For a non-singular projective variety* $X \subseteq \mathbb{P}^N$ *and for a hyperplane* L *of* $\mathbb{P}^N$, *if* $\dim \operatorname{Sec} X < 2\dim X + 1$ *and* $X \cap L$ *is non-singular, then* $S^{(k)}(X \cap L) = S^{(k)}X \cap L$ *for* $\forall k \geq 0$.

From this theorem one can deduce
Corollary (Is this trivial?)**:** *Any traceless rank* r *matrix is written as a sum of* r *traceless rank* 1 *matrices.*

It might be interesting to compare Theorem 0 with the following: Let

$$C(m,n) := \{[a_{ij}] \in M_{m,n}\mathbb{C} | a_{ij} = a_{i+1\,j-1} \ (\forall i,j)\},$$

that is, the catalecticant space, and denote by $R_d \subseteq \mathbb{P}^d$ the rational normal curve of degree d. Then
Theorem (R. K. Wakerling) [E]**:** $S^{(k)}R_{m+n-2} = \pi(C(m,n)_{k+1} \setminus \{0\})$ *for* $\forall k \geq 0$.
5-3. Tangent Loci: For $z \in \mathbb{P}_*(\mathfrak{g})$ denote by Θ_z the tangent locus with respect to z, that is, set

$$\Theta_z := \{x \in X(\mathfrak{g}) | z \in T_x X(\mathfrak{g})\},$$

where $T_x X(\mathfrak{g}) \subseteq \mathbb{P}_*(\mathfrak{g})$ denotes the embedded tangent space to $X(\mathfrak{g})$ at x.

In the simplest case $l=1$, we have $X(A_1) = \{\xi\zeta + \eta^2 = 0\} \subseteq \mathbb{P}^2 = \mathbb{P}_*(\mathfrak{sl}_2\mathbb{C})$ via

$$(\xi : \eta : \zeta) \leftrightarrow \xi X_\lambda + \eta H + \zeta X_{-\lambda} = \begin{bmatrix} \eta & \xi \\ \zeta & -\eta \end{bmatrix}.$$

Since $T_{x_+}X(A_1) = \{\zeta = 0\}$ and $T_{x_-}X(A_1) = \{\xi = 0\}$, it follows $T_{x_+}X(A_1) \cap T_{x_-}X(A_1) = \{h\}$, where $x_+ := \pi X_\lambda = (1:0:0)$, $x_- := \pi X_{-\lambda} = (0:0:1)$, and $h := \pi H = (0:1:0)$. Conversely,
Observation 2: $\Theta_h = \{x_+, x_-\}$ for $X(A_1)$. More generally for an arbitrary $z \in \operatorname{Sec} X(A_1) \setminus X(A_1)$ we have $\Theta_z = \{x, y\}$ for $\exists x \neq y \in X(A_1)$. We have $z = gh$ for $\exists g \in G$ since $\operatorname{Sec} X(A_1) \setminus X(A_1) = G \cdot h$, hence $\{x,y\} = \{gx_+, gx_-\}$. Thus for a general point $z \in \operatorname{Sec} X(A_l)$, the points x, y

of tangents through z are corresponding to the highest and lowest root vectors with respect to a Cartan subalgebra $\mathfrak{h}' = \mathbb{C}\cdot H'$, where H' is a non-zero element in $\mathfrak{g}$ such that $\pi H' = z$.

6. General Cases:

Graded Lie Algebras of Contact Type [A1, A2]**:** For highest and lowest root vectors, X_λ and $X_{-\lambda}$, there exists $H \in \mathbb{C}\cdot[X_\lambda, X_{-\lambda}]$ such that

$$\mathfrak{g} = \mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2, \quad \mathfrak{g}_{\pm 2} = \mathbb{C}\cdot X_{\pm\lambda},$$

where $\mathfrak{g}_i := \{Y \in \mathfrak{g} | [H,Y] = iY\}$ an eigenspace of $\operatorname{ad} H$. It follows from the Jacobi identity that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. This element $H \in \mathfrak{h}$ is called a ***characteristic element***.

We see $\dim X(\mathfrak{g}) = \dim \mathfrak{g}_1 + 1$ since it follows $T_{x_+}X(\mathfrak{g}) = \mathbb{P}_*([\mathfrak{g}, X_\lambda]) = \mathbb{P}_*(\mathbb{C}\cdot H \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2)$.

Dynkin's Normal Forms [Y1]**:** Let $D : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ be a symmetric bilinear form on $\mathfrak{g}$ defined by a scalar multiple of the Killing form such that $D(H,H) = 2$, where H is the characteristic element above. We call D the Dynkin's normal form. For example, it turns out that $D(Y,Z) = \operatorname{tr}(YZ)$ $(Y, Z \in K = \mathfrak{sl}_{l+1})$ in A_l-case .

Theorem 1 (defining equations of adjoint varieties) [KOY]**:** $X(\mathfrak{g}) = \pi(W \setminus \{0\})$, *where*

$$W := \{Y \in \mathfrak{g} | (\operatorname{ad} Y)^2 Z + 2D(Y,Z)Y = 0 \ (\forall Z \in \mathfrak{g})\}.$$

Theorem 2 (dimension of secant varieties) [KOY]**:**

$$\dim \operatorname{Sec} X(\mathfrak{g}) = 2\dim X(\mathfrak{g}).$$

From this one sees that $\operatorname{codim}(\operatorname{Sec} X(\mathfrak{g}), \mathbb{P}_*(\mathfrak{g})) = \dim \mathfrak{g}_0 - 1 \geq \operatorname{rk}\mathfrak{g} - 1$, and in particular, $\operatorname{Sec} X(\mathfrak{g}) \neq \mathbb{P}_*(\mathfrak{g})$ if $\operatorname{rk}\mathfrak{g} \geq 2$.

Theorem 3 (orbits in secant varieties): $\operatorname{Sec} X(\mathfrak{g}) = \overline{G\cdot h}$ ([KOY]). Moreover, we have

$$\operatorname{rk}\mathfrak{g} \geq 2 \Rightarrow \operatorname{Sec} X(\mathfrak{g}) \supsetneqq X(\mathfrak{g}) \sqcup G\cdot h,$$

where $h := \pi H$.

Theorem 4 (tangent loci):

$$\Theta_h = \{x_+, x_-\},$$

where $x_\pm := \pi X_{\pm\lambda}$.

Therefore for a general $z \in \operatorname{Sec} X(\mathfrak{g})$ we have $z = gh$ for $\exists g \in G$, and since $\Theta_z = \{gx_+, gx_-\}$, the points of tangents through z are corresponding to graded pieces $\mathfrak{g}'_{\pm 2}$ with respect to a characteristic element $H' = gH$ with $\pi H' = z$.

7. Proofs of Theorems 3 and 4: The key ingredient in the proofs is

Symplectic Triple Systems [A1, A2, YA]**:** For $P, Q, R \in \mathfrak{g}_1$, define $2\langle P, Q\rangle X_\lambda := [P,Q]$, $2P\times Q := [P[Q, X_{-\lambda}]] + [Q[P, X_{-\lambda}]]$, and $[P,Q,R] := [R, P\times Q]$: One obtains a skew-symmetric form $\langle,\rangle$, a symmetric product $\times$, and, triple product $[,,]$ on $\mathfrak{g}_1$ as follows:

$$\langle,\rangle : \mathfrak{g}_1\times\mathfrak{g}_1 \to \mathbb{C}, \quad \times : \mathfrak{g}_1\times\mathfrak{g}_1 \to \mathfrak{g}_0, \quad [,,] : \mathfrak{g}_1\times\mathfrak{g}_1\times\mathfrak{g}_1 \to \mathfrak{g}_1.$$

Then

$$(\mathfrak{g}_1, [,,], \langle,\rangle)$$

has the structure of a symplectic triple system, that is, for $P, Q, R, S, T \in \mathfrak{g}_1$ the following holds:

(STS1) $[PQR] = [QPR]$;

(STS2) $[PQR] - [PRQ] = \langle P, R\rangle Q - \langle P, Q\rangle R + 2\langle Q, R\rangle P$;

(STS3) $[ST[PQR]] = [[STP]QR] + [P[STQ]R] + [PQ[STR]]$.

The notion of symplectic triple system was introduced by H. Asano [A1, A2] (Yokohama City Univesity).

Now consider

$$M := \{Y \in \mathfrak{g}_1 | Y \times Y = 0\} = \{Y \in \mathfrak{g}_1 | (\operatorname{ad} Y)^2 X_{-\lambda} = 0\}.$$

Using Theorem 1 one can prove
Lemma 6:

$$W \cap \mathfrak{g}_1 \subseteq M.$$

Proof of the main part of Theorem 3: Since it follows $\operatorname{Sec} X(\mathfrak{g}) \supseteq T_{x_+} X(\mathfrak{g}) \supseteq \mathbb{P}_*(\mathfrak{g}_1)$ and $\mathbb{P}_*(\mathfrak{g}_1) \cap G \cdot h = \emptyset$, we have

$$\operatorname{Sec} X(\mathfrak{g}) \setminus (X(\mathfrak{g}) \sqcup G \cdot h) \supseteq \mathbb{P}_*(\mathfrak{g}_1) \setminus X(\mathfrak{g}).$$

If $\mathbb{P}_*(\mathfrak{g}_1) \subseteq X(\mathfrak{g})$, then it follows from Lemma 6 that $\mathfrak{g}_1 \subseteq M$, which implies that $[\mathfrak{g}_1 \mathfrak{g}_1 \mathfrak{g}_1] = 0$. But this contradicts to the fact [A1, A2] that $(\mathfrak{g}_1, [,,], \langle,\rangle)$ is simple as a symplectic triple system if $\operatorname{rk} \mathfrak{g} \geq 2$. Thus $\mathbb{P}_*(\mathfrak{g}_1) \setminus X(\mathfrak{g}) \neq \emptyset$. $\square$

For the proof of Theorem 4, consider moreover the decomposition of a reductive Lie algebra $\mathfrak{g}_0$ into a direct sum of its semi-simple part, denoted by $\mathfrak{D}_0$, and its abelian part, $\mathbb{C} \cdot H$:

$$\mathfrak{g}_0 = \mathfrak{D}_0 \oplus \mathbb{C} \cdot H.$$

It can be shown that $\mathfrak{D}_0 = \{Z \in \mathfrak{g}_0 | (\operatorname{ad} Z) X_\lambda = 0\} = \{Z \in \mathfrak{g}_0 | (\operatorname{ad} Z) X_{-\lambda} = 0\}$, and the decomposition is then explicitly obtained from an exact sequence,

$$0 \quad \rightarrow \quad \mathfrak{D}_0 \quad \rightarrow \quad \mathfrak{g}_0 \quad \xrightarrow{\operatorname{ad} X_{\pm\lambda}} \quad \mathbb{C} \cdot X_{\pm\lambda} \quad \rightarrow \quad 0,$$

which splits by the map, $\operatorname{ad} X_{\mp\lambda} : \mathfrak{g}_0 \leftarrow \mathbb{C} \cdot X_{\pm\lambda}$, with $\operatorname{ad} X_{\mp\lambda}(\mathbb{C} \cdot X_{\pm\lambda}) = \mathbb{C} \cdot H$. It follows from the Jacobi identity that

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{D}_0, \quad \mathfrak{g}_1 \times \mathfrak{g}_1 \subseteq \mathfrak{D}_0.$$

The key in our proof is the following lemmas (for proofs see [KY]):
Lemma 7 (decomposition formula): *For $Y \in \mathfrak{g}_{-1}, P \in \mathfrak{g}_1$, we have*

$$[Y, P] = Y^+ \times P - \langle Y^+, P \rangle H,$$

where $Y^+ := [X_\lambda, Y]$.
Lemma 8: *For $P, Q \in \mathfrak{g}_1$, we have*

$$P \times Q = 0, P \in M \Rightarrow \langle P, Q \rangle = 0.$$

Proof of Theorem 4: It suffices to show that if $h \in T_{gx_+} X$ with $g \in G$, then $gx_+ \in \{x_+, x_-\}$. Since $T_{gx_+} X(\mathfrak{g}) = \mathbb{P}_*([\mathfrak{g}, gX_\lambda])$, in terms of Lie algebra this is equivalent to saying that for $g \in G, Y \in \mathfrak{g}$,

$$H = [Y, gX_\lambda] \Rightarrow gX_\lambda \in \mathfrak{g}_2 \cup \mathfrak{g}_{-2}.$$

Using Theorem 1 one can show that $gX_\lambda \in \mathfrak{g}_i$ for $\exists i$, and may assume that $Y \in \mathfrak{g}_{-i}$. Our claim is now $i = \pm 2$.

First of all $i \neq 0$. Suppose $i = 0$: it follows

$$H = [Y, gX_\lambda] \in [\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{D}_0,$$

that is, $H \in \mathfrak{D}_0$. This contradicts to $[H, X_\lambda] = 2X_\lambda \neq 0$. Thus we have $i \neq 0$.

Next $i \neq \pm 1$. Suppose $i = 1$: we have $Y \in \mathfrak{g}_{-1}, gX_\lambda \in \mathfrak{g}_1$, and it follows from Lemma 7 that

$$H = [Y, gX_\lambda] = Y^+ \times gX_\lambda - \langle Y^+, gX_\lambda \rangle H.$$

Taking account of the decomposition $\mathfrak{g}_0 = \mathfrak{D}_0 \oplus \mathbb{C} \cdot H$, comparing both sides above, one gets

$$Y^+ \times gX_\lambda = 0, \quad \langle Y^+, gX_\lambda \rangle = -1.$$

Since $gX_\lambda \in \mathfrak{g}_1 \cap W \subseteq M$ (Lemma 6), by Lemma 8 one obtains from the former that $\langle Y^+, gX_\lambda \rangle = 0$. But this contradicts to the latter. Thus, $i \neq 1$. Similarly we see $i \neq -1$.

Therefore $i = 2$ or $i = -2$. $\square$

8. Appendix: It can be shown that $\pi(M \setminus \{0\}) \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ is a projective variety that is a homogeneous space of (an algebraic group with) Lie algebra $\mathfrak{D}_0$ [Y2]. On the other hand, it is known [A, M2, YA] that the adjoint varieties $X(\mathfrak{g})$ correspond to the meta-symplectic geometry while the Severi varieties [FR, LV, Z] correspond to the projective geometry, in the magic square of H. Freudenthal [F] as follows:

elliptic geometry	B_1	A_2	C_3	F_4
projective geometry	A_2	$A_2 + A_2$	A_5	E_6
symplectic geometry	C_3	A_5	D_6	E_7
metasymplectic geometry	F_4	E_6	E_7	E_8

In this context $\pi(M \setminus \{0\}))$ correspond to the symplectic geometry, and are called cubic Veronese varieties in [M2].

Now our result is

Theorem 5 (homogeneous spaces M of $\mathfrak{D}_0$):

$$X(\mathfrak{g}) \cap \mathbb{P}_*(\mathfrak{g}_1) = \pi(M \setminus \{0\}).$$

Note that $\mathbb{P}_*(\mathfrak{g}_1)$ is a linear subspace of $T_{x_+}X$ of codimension 2: In fact, $T_{x_+}X$ is spanned by the point x_+ of contact x_+, the point h corresponding to the characteristic element, and this linear space $\mathbb{P}_*(\mathfrak{g}_1)$.

In our proof for this result, Theorem 1 as well as Lemma 8 are essential (for details see [KY]).

References

A. K. Atsuyama, *Freudenthal's projective geometries (in Japanese)*, Proc. Symp. Algebraic Geometry, "Projective Varieties/Projective Geometry of Algebraic Varieties 2 + Singularities", Waseda University, JAPAN, 1997, pp. 1–62.

A1. H. Asano, *On triple systems (in Japanese)*, Yokohama City Univ. Ronso, Ser. Natural Sci. **27** (1975), 7–31.

A2. H. Asano, *Symplectic triple systems and simple Lie algebras (in Japanese)*, RIMS Kokyuroku, Kyoto Univ. **308** (1977), 41–54.

B. N. Bourbaki, *Éléments de Mathématique, Groupes et algèbres de Lie , Chapitres 4,5 et 6*, Hermann, Paris, 1968.

B1. W. Boothby, *Homogneous complex contact manifolds*, Amer. Math. Soc. Proc. Symp. Pure Math. **3** (1961), 144–154.

B2. W. Boothby, *A note on homogeneous complex contact manifolds*, Proc. Amer. Math. Soc. **13** (1962), 276–280.

E. D. Eisenbud, *Linear sections of determinantal varieties*, Amer. J. Math. **110** (1988), 541–575.

F. H. Freudenthal, *Lie groups in the foundations of geometry*, Advances in Math. **1** (1964), 145–190.

FR. T. Fujita, J. Roberts, *Varieties with small secant varieties: The extremal case*, Amer. J. Math. **103** (1981), 953–976.

FH. W. Fulton, J. Harris, *Representation Theory: A First Course*, GTM **129**, Springer-Verlag, New York, 1991.

H. J. Harris, *Algebraic Geometry: A First Course*, GTM **133**, Springer-Verlag, New York, 1993.

H1. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, GTM **9**, Springer-Verlag, New York, 1972.

H2. J. E. Humphreys, *Linear Algebraic Groups*, GTM **21**, Springer-Verlag, New York, 1975.

K. H. Kaji, *Higher secant varieties*, in preparation.

KOY. H. Kaji, M. Ohno and O. Yasukura, *Adjoint varieties and their secant varieties*, to appear in Indag. Math. (Preprint 1996).

KY. H. Kaji and O. Yasukura, *Adjoint varieties and their secant varieties, II*, in preparation.

LV. R. Lazarsfeld and A. Van de Ven, *Topics in the geometry of projective space. Recent work of F. L. Zak*, DMV Sem. 4, Birkhäuser Verlag, Basel and Boston, 1984.

M1. S. Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Japan Academy **86** (1989), 3000–3002.

M2. S. Mukai, *Projective geometry of homogeneous spaces (in Japanese)*, Proc. Symp. Algebraic Geometry, "Projective Varieties/Projective Geometry of Algebraic Varieties", Waseda University, JAPAN, 1994, pp. 1–52.

YA. K. Yamaguti and H. Asano, *On Freudenthal's construction of exceptional Lie algebras*, Proc. Japan Academy **51** (1975), 253–258.

Y1. O. Yasukura, *Symplectic triple systems and graded Lie algebras*, International Symposium on Nonassociative Algebras and Related Topics, Hiroshima, JAPAN 1990, ed. Yamaguti-Kawamoto, World Scientific, Singapore, New Jersey, London, Hong Kong, 1991, pp. 345–351.

Y2. O. Yasukura, *The list of irreducible quaternionic symmetric spaces*, unpublished.

Z. F. L. Zak, *Tangents and Secants of Algebraic Varieties*, Translations of Math. Monographs, vol. **127**, AMS, Providence, 1993.

e-mail address: `kaji@mn.waseda.ac.jp`