PROJECTIVE GEOMETRY OF ADJOINT VARIETIES

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Introduction

The purpose of our work is to study projective geometry of adjoint varieties, and the purpose here is to illustrate the structure of adjoint varieties and their secant varieties in case of type A, and to give new results for general cases: The former is almost known; the latter is part of joint works [KOY, KY] with O. Yasukura (Fukui University) and M. Ohno (Waseda University).

In this article, we first give fundamental definitions in §1 and some known results on adjoint varieties in §2. Then in §3 we look at A_l-case in detail and make a few observations. We next state our results for general cases in §4 and give proofs for some of those results in §5. To investigate the structure of adjoint varieties, we need to investigate that of Lie algebras in detail, and for this purpose we briefly introduce in §5 the notion of symplectic triple systems that is the key in our proofs. This notion was firstly introduced by H. Asano (Yokohama City University). In Appendix A we prove a result on certain homogeneous spaces that appear in proofs of our main results. The homogeneous spaces in question are called *cubic Veronese varieties* by S. Mukai [M2], and correspond to the *symplectic geometry* of H. Freudenthal [F]. Finally in Appendix B we give defining equations of higher secant varieties of adjoint varieties of type A. I believe that the results in Appendices are also new.

1. Definitions

Definition 1 [B1, B2]**:** For a complex simple Lie algebra $\mathfrak{g}$, let G be a simple algebraic group with Lie algebra $\mathfrak{g}$, and denote by $\mathrm{Ad} : G \curvearrowright \mathfrak{g}$ the adjoint representation: If G is a closed subgroup of some $GL_n\mathbb{C}$, then $\mathrm{Ad}\, g$ is just the conjugation by $g \in G$. Consider a natural action

$$G \curvearrowright \mathbb{P}_*(\mathfrak{g}) := (\mathfrak{g} \setminus \{0\})/\mathbb{C}^\times.$$

Then it is well-known [FH, H2] that since $\mathfrak{g}$ is simple, there exists a unique closed orbit in $\mathbb{P}_*(\mathfrak{g})$, denoted by $X(\mathfrak{g})$. We call $X(\mathfrak{g})$ an *adjoint variety* associated to $\mathfrak{g}$, which is a non-singular, non-degenerate projective variety in $\mathbb{P}_*(\mathfrak{g})$.

1998/08/25, a manuscript for proceedings of a conference "School on Commutative Algebras and Projective Varieties" (Nagano, 1998/03/12–14)

Note that since $\operatorname{Ad} G = \operatorname{Int}\mathfrak{g}$, $X(\mathfrak{g})$ does not depend on the choice of G and is uniquely determined by $\mathfrak{g}$, where $\operatorname{Int}\mathfrak{g} := \langle \exp \operatorname{ad} Y | Y \in \mathfrak{g}\rangle \subseteq GL(\mathfrak{g})$. Therefore adjoint varieties are completely classified by the Dynkin diagram (see §2 below) since so are complex simple Lie algebras, as is well-known (see, e. g. [H1]).

Take a Cartan subalgebra $\mathfrak{h}$ and a base Δ of the root system R with respect to $\mathfrak{h}$, and fix an order on R defined by Δ [H1]. Let λ be the highest root, and take a highest root vector $X_\lambda \in \mathfrak{g}$. Then we have

$$X(\mathfrak{g}) = \operatorname{Int}\mathfrak{g} \cdot x_+ = \pi(G \cdot X_\lambda) \subseteq \mathbb{P}_*(\mathfrak{g}),$$

where $x_+ := \pi X_\lambda$, and $\pi : \mathfrak{g} \setminus \{0\} \to \mathbb{P}_*(\mathfrak{g}); Y \mapsto \mathbb{C} \cdot Y$ is the canonical projection.

Definition 2 [FR, LV, Z]**:** Let $X \subseteq \mathbb{P}^N_{\mathbb{C}}$ be a complex projective variety. For distinct points $x, y \in X$, we call the line joining x and y the *secant line* determined by x and y, which is denoted by $x * y$. Moreover set

$$\operatorname{Sec} X := \overline{\bigcup_{x,y \in X, x \neq y} x * y} \subseteq \mathbb{P}^N_{\mathbb{C}}$$

This is an irreducible algebraic set in $\mathbb{P}^N_{\mathbb{C}}$, as is easily seen, and considered as a projective variety with reduced structure. We call $\operatorname{Sec} X$ the *secant variety* of $X \subseteq \mathbb{P}^N_{\mathbb{C}}$. Secant varieties of projective varieties usually have singularities along at least X.

2. Table of Adjoint Varieties

The facts given in this section are well-known (see, e. g. [FH]).

Classical Type.

type	$\mathfrak{g}$	$\operatorname{Int}\mathfrak{g}$	λ	Ad	$X(\mathfrak{g}) \subseteq \mathbb{P}_*(\mathfrak{g})$	$\dim \mathfrak{g}$
$A_{l\geq 1}$	$\mathfrak{sl}_{l+1}$	PSL_{l+1}	$\omega_1 + \omega_l$	K	$\mathbb{P}^l \times \mathbb{P}^l \cap (1)$	$l^2 + 2l$
$B_{l\geq 2}$	$\mathfrak{so}_{2l+1}$	PSO_{2l+1}	ω_2	$\wedge^2 V$	$\mathbb{F}_1(Q^{2l-1})$	$2l^2 + l$
$C_{l\geq 3}$	$\mathfrak{sp}_{2l}$	PSp_{2l}	$2\omega_1$	$S^2 V$	$v_2(\mathbb{P}^{2l-1})$	$2l^2 + l$
$D_{l\geq 4}$	$\mathfrak{so}_{2l}$	PSO_{2l}	ω_2	$\wedge^2 V$	$\mathbb{F}_1(Q^{2l-2})$	$2l^2 - l$

Notation. For a group G, set $PG := G/Z(G)$, where $Z(G)$ is the center of G. Moreover for each type, set

(A) $\mathfrak{sl}_{l+1} := \{Y \in M_{l+1}\mathbb{C} |\operatorname{tr} Y = 0\}$, $SL_{l+1} := \{g \in GL_{l+1}\mathbb{C} | \det g = 1\} \curvearrowright V := \mathbb{C}^{\oplus l+1}$; $Z(SL_{l+1}) = \mu_{l+1}$, where μ_n denotes the group of n-th roots of unities. $K := \operatorname{Ker}(\wedge^l V \otimes V \to \wedge^{l+1} V = \mathbb{C}) \simeq \operatorname{Ker}(V^* \otimes V \to \mathbb{C})$ via $\wedge^l V \simeq V^*$.

(B, D) $\mathfrak{so}_n := \{Y \in M_n\mathbb{C} | {}^t Y Q + QY = 0\}$, $SO_n := \{g \in GL_n\mathbb{C} | {}^t g Q g = Q, \det g = 1\} \curvearrowright V := \mathbb{C}^{\oplus n}$ with

$$Q = \begin{bmatrix} 0 & I_{[n/2]} \\ I_{[n/2]} & 0 \end{bmatrix} \quad \text{or} \quad Q = \begin{bmatrix} 0 & I_{[n/2]} & 0 \\ I_{[n/2]} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$Z(SO_n) = \mu_2$ if n is even, and $Z(SO_n)$ is trivial if n is odd.

(C) $\mathfrak{sp}_{2l} := \{Y \in M_{2l}\mathbb{C} | {}^tYQ + QY = 0\}$, $Sp_{2l} := \{g \in GL_{2l}\mathbb{C} | {}^tgQg = Q\} \curvearrowright V := \mathbb{C}^{\oplus 2l}$ with

$$Q = \begin{bmatrix} 0 & I_l \\ -I_l & 0 \end{bmatrix};$$

$Z(Sp_{2l}) = \mu_2$.

We denote by $\mathbb{F}_1(Q^m)$ the Fano variety of lines in m-dimensional quadric hypersurface $Q \subseteq \mathbb{P}^{m+1}$, which is a $(2m-3)$-dimensional subvariety of a Grassmann variety $\mathbb{G}(2, m+2)$ and the polarization here is defined by the Plücker embedding of $\mathbb{G}(2, m+2)$. We also denote by v_2 the Veronese embedding, and by $\cap(1)$ cutting by a hyperplane. The notation for fundamental dominant weights ω_i are same as in [B].

Exceptional Type.

type	λ	$\dim X(\mathfrak{g})$	$\dim \mathfrak{g}$
E_6	ω_2	$20+1$	78
E_7	ω_1	$32+1$	133
E_8	ω_8	$56+1$	248
F_4	ω_1	$14+1$	52
G_2	ω_2	$4+1$	14

The adjoint variety of type G_2 appears in the work of S. Mukai [M1] as a Fano 5-fold of index 3.

3. The case of type A

We set $G := PSL_{l+1}, V := \mathbb{C}^{l+1}$, and consider the following subsets of $\mathrm{Hom}(V, V) = V^* \otimes V = M_{l+1}\mathbb{C}$:

$$\begin{aligned} \mathrm{Hom}(V,V)_r &:= \{\varphi \in \mathrm{Hom}(V,V) | \operatorname{rk}\varphi \leq r\} \\ K &:= \{\varphi \in \mathrm{Hom}(V,V) | \operatorname{tr}\varphi = 0\} = \mathrm{Ker}(V^* \otimes V \to \mathbb{C}) = \mathfrak{sl}_{l+1} \\ K_r &:= K \cap \mathrm{Hom}(V,V)_r. \end{aligned}$$

Moreover we set

$$H := \mathrm{diag}(1, -1, 0, \dots, 0) \in M_{l+1}\mathbb{C},$$

and denote by

$$X_{(k_1 \cdots k_s)}$$

a nilpotent matrix in $M_{l+1}\mathbb{C}$ that is a direct sum of Jordan cells of sizes $k_1, \dots, k_s$. Recall that the adjoint action $\mathrm{Ad} : G \curvearrowright K$ is just taking the conjugation in this case. Obviously the algebraic set $\pi(K_r \setminus \{0\}) \subseteq \mathbb{P}_*(K)$ has defining equations of degree $r+1$.

3-1. Adjoint Varieties. We first recall a few descriptions of adjoint varieties of type A: The results in this subsection are well-known or easily deduced (see, e. g., [FH, p. 389]).

Lemma 1: $G \cdot \pi X_{(21\cdots1)} = \pi(K_1 \setminus \{0\})$.

Proof: Note that orbits are considered with the reduced structure, as usual, and it follows from [E, Theorem 2.1] that the right hand side is reduced: In fact K is l-generic. Therefore it suffices to show that the equality holds set-theoretically.

The inclusion $\subseteq$ follows since the rank and trace are determined by conjugacy class. The converse $\supseteq$ follows since a traceless rank one matrix has Jordan normal form $X_{(21\cdots1)}$. $\square$

Lemma 2: $\pi(K_1 \setminus \{0\}) = \mathbb{P}_*(V^*) \times \mathbb{P}_*(V) \cap \mathbb{P}_*(K)$.

Proof: We have $\pi(\mathrm{Hom}(V,V)_1 \setminus \{0\}) = \mathbb{P}_*(V^*) \times \mathbb{P}_*(V)$ since for $\varphi \in \mathrm{Hom}(V,V) = V^* \otimes V$, $\operatorname{rk}\varphi \leq 1$ if and only if $\varphi = f \otimes v$ for some $f \in V^*$ and $v \in V$. $\square$

From the above we obtain

Proposition 1: $X(A_l) = \pi(K_1 \setminus \{0\}) = \mathbb{P}_*(V^*) \times \mathbb{P}_*(V) \cap \mathbb{P}_*(K)$. $\square$

In particular $X(A_l)$ is defined by quadric equations. Note that we also have

$$\pi(K_1 \setminus \{0\}) = \mathbb{F}(1, l, V) = \mathbb{P}(T_{\mathbb{P}^l}(-1))$$

via $\mathbb{P}_*(\mathrm{Hom}(V,V)) = \mathbb{P}_*(V^* \otimes V) = \mathbb{P}(V^* \otimes \mathcal{O}_{\mathbb{P}^l})$, where $\mathbb{P}(*)$ is the dual of $\mathbb{P}_*(*)$, and $\mathbb{F}(1,l,V)$ is a flag variety defined by the incidence correspondence of 1- and l-subspaces of V in $\mathbb{P}_*(V^*) \times \mathbb{P}_*(V)$: In fact, the first identification is given by $\varphi \leftrightarrow (\varphi(V) \subseteq \operatorname{Ker}\varphi \subseteq V)$; the second comes from the Euler quotient $V^* \otimes \mathcal{O}_{\mathbb{P}^l} \twoheadrightarrow T_{\mathbb{P}^l}(-1)$ on $\mathbb{P}^l = \mathbb{P}(V)$.

3-2. Secant Varieties. Using the description given in the previous subsection, let us look at secant varieties of adjoint varieties of type A in detail. We first have

Proposition 2: $\operatorname{Sec} X(A_l) = \pi(K_2 \setminus \{0\})$.

Proof: As in the proof of Lemma 1, it suffices to show that the equality holds set-theoretically. The inclusion $\subseteq$ follows from Proposition 1 and an elementary fact that $\operatorname{rk}(A+B) \leq \operatorname{rk}A + \operatorname{rk}B$. For the converse $\supseteq$, since the roots of the characteristic polynomial of $Y \in K_2$ are $a, -a$ ($a \in \mathbb{C}$) and 0 with multiplicity $l-1$, the Jordan normal form of Y is one of the following:

$$aH, \quad X_{(31\cdots1)}, \quad X_{(221\cdots1)}, \quad X_{(21\cdots1)}.$$

Each of those matrices can be written as a sum of two elements of K_1: For example, we have

$$2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \quad \square$$

Observation 1: We see from the proof above that $\operatorname{Sec} X(A_l)$ *consists of* 4 *orbits through* πH, $\pi X_{(31\cdots1)}$, $\pi X_{(221\cdots1)}$ *and* $\pi X_{(21\cdots1)}$ *if* $\operatorname{rk}\mathfrak{g} = l \geq 2$. Computing the stabilizers one obtains that *those orbits have dimension* $4l-2, 4l-3, 4l-5$ *and* $2l-1$, *respectively.*

Since secant varieties are irreducible, this implies

Proposition 3: $\operatorname{Sec} X(A_l) = \overline{G \cdot \pi H}$ *and* $\dim \operatorname{Sec} X(A_l) = 2 \dim X(A_l)$. $\square$

3-3. Tangent Loci. For $z \in \mathbb{P}_*(\mathfrak{g})$ denote by Θ_z the tangent locus with respect to z, that is, set

$$\Theta_z := \{x \in X(\mathfrak{g}) | z \in T_x X(\mathfrak{g})\},$$

where $T_x X(\mathfrak{g}) \subseteq \mathbb{P}_*(\mathfrak{g})$ denotes the embedded tangent space to $X(\mathfrak{g})$ at x.

In the simplest case $l = 1$, we see that there are exactly two orbits, $G \cdot \pi X_{(2)} = X(A_l)$ and $G \cdot \pi H = \mathbb{P}_*(\mathfrak{g}) \setminus X(A_1)$. In terms of homogeneous coordinates, we have

$$X(A_1) = \{(\xi : \eta : \zeta) | \xi\zeta + \eta^2 = 0\} \subseteq \mathbb{P}^2 = \mathbb{P}_*(\mathfrak{sl}_2\mathbb{C})$$

via

$$(\xi : \eta : \zeta) \leftrightarrow \xi X_\lambda + \eta H + \zeta X_{-\lambda} = \begin{bmatrix} \eta & \xi \\ \zeta & -\eta \end{bmatrix}$$

with $X_\lambda = X_{(2)}$ and $X_{-\lambda} = {}^t X_{(2)}$. Since $T_{x_+} X(A_1) = \{\zeta = 0\}$ and $T_{x_-} X(A_1) = \{\xi = 0\}$, it follows that $T_{x_+} X(A_1) \cap T_{x_-} X(A_1) = \{h\}$, where $x_+ := \pi X_\lambda = (1 : 0 : 0)$, $x_- := \pi X_{-\lambda} = (0 : 0 : 1)$, and $h := \pi H = (0 : 1 : 0)$.

Observation 2: We have $\Theta_h = \{x_+, x_-\}$ for $X(A_1)$ since $X(A_1)$ is a conic. More generally *for an arbitrary $z \in \operatorname{Sec} X(A_1) \setminus X(A_1)$ we have $\Theta_z = \{x, y\}$ for some points $x \neq y \in X(A_1)$.* We have $z = gh$ for some $g \in G$ since $\operatorname{Sec} X(A_1) \setminus X(A_1) = G \cdot h$, hence $\{x, y\} = \{gx_+, gx_-\}$. Thus *for a general point $z \in \operatorname{Sec} X(A_1)$, the points x, y of tangents through z are corresponding to the highest and lowest root vectors with respect to a Cartan subalgebra $\mathfrak{h}' = \mathbb{C} \cdot H'$, where H' is a non-zero element in $\mathfrak{g}$ such that $\pi H' = z$.*

4. General Cases

To state our results for general cases, we need

Graded Decompositions of Lie Algebras of Contact Type and Characteristic Elements [A1, A2]**.** For highest and lowest root vectors, X_λ and $X_{-\lambda}$, of a simple Lie algebara $\mathfrak{g}$, there exists $H \in \mathbb{C} \cdot [X_\lambda, X_{-\lambda}]$ such that

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \mathfrak{g}_{\pm 2} = \mathbb{C} \cdot X_{\pm\lambda},$$

where we set

$$\mathfrak{g}_i := \{Y \in \mathfrak{g} | [H, Y] = iY\},$$

an eigenspace of $\operatorname{ad} H$. It follows from the Jacobi identity that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. It can be shown that $\mathfrak{g}_1 \neq 0$ if and only if $\operatorname{rk} \mathfrak{g} \geq 2$, and in this case the decomposition above is said to be *of contact type*. The element $H \in \mathfrak{h}$ above is called a *characteristic element* of the gradation.

We see $\dim X(\mathfrak{g}) = \dim \mathfrak{g}_1 + 1$ since we have

$$T_{x_+} X(\mathfrak{g}) = \mathbb{P}_*([\mathfrak{g}, X_\lambda]) = \mathbb{P}_*(\mathbb{C} \cdot H \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2).$$

Dynkin's Normal Forms [Y1]**.** Let

$$D : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

be a symmetric bilinear form on $\mathfrak{g}$ defined by a scalar multiple of the Killing form such that $D(H, H) = 2$, where H is the characteristic element above. We call D the

Dynkin's normal form. Similarly to the case of Killing forms, we have $D(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$. For example, we have $D(Y, Z) = \mathrm{tr}(YZ)$ $(Y, Z \in K = \mathfrak{sl}_{l+1})$ in A_l-case .

Now we can state our results:

Theorem 1 (defining equations of adjoint varieties) [KOY]**:**

$$X(\mathfrak{g}) = \pi(W \setminus \{0\}),$$

where

$$W := \{Y \in \mathfrak{g} | (\mathrm{ad}\, Y)^2 Z + 2D(Y, Z)Y = 0 \ (\forall Z \in \mathfrak{g})\}.$$

Theorem 2 (dimension of secant varieties) [KOY]**:**

$$\dim \mathrm{Sec}\, X(\mathfrak{g}) = 2 \dim X(\mathfrak{g}).$$

From this one sees that

$$\mathrm{codim}(\mathrm{Sec}\, X(\mathfrak{g}), \mathbb{P}_*(\mathfrak{g})) = \dim \mathfrak{g}_0 - 1 \geq \mathrm{rk}\, \mathfrak{g} - 1,$$

and in particular, $\mathrm{Sec}\, X(\mathfrak{g}) \neq \mathbb{P}_*(\mathfrak{g})$ if $\mathrm{rk}\, \mathfrak{g} \geq 2$.

Theorem 3 (orbits in secant varieties): $\mathrm{Sec}\, X(\mathfrak{g}) = \overline{\mathrm{Int}\, \mathfrak{g} \cdot h}$ ([KOY]). *Moreover, we have*

$$\mathrm{rk}\, \mathfrak{g} \geq 2 \Rightarrow \mathrm{Sec}\, X(\mathfrak{g}) \supsetneqq X(\mathfrak{g}) \sqcup \mathrm{Int}\, \mathfrak{g} \cdot h,$$

where $h := \pi H$.

Theorem 4 (tangent loci):

$$\Theta_h = \{x_+, x_-\},$$

where $x_\pm := \pi X_{\pm\lambda}$.

This result tells us that for a general point $z \in \mathrm{Sec}\, X(\mathfrak{g})$, the relationship among z and the points in Θ_z is described as follows: *For a general* $z \in \mathrm{Sec}\, X(\mathfrak{g})$ *there exists a characteristic element* $H' \in \mathfrak{g}$ *of graded decomposition of contact type such that* $z = \pi H'$, *and* $\Theta_z = \mathbb{P}_*(\mathfrak{g}'_{-2}) \cup \mathbb{P}_*(\mathfrak{g}'_2)$, *where* $\mathfrak{g}'_{\pm 2}$ *are graded piece of the decomposition of degree* ± 2 *with respect to* H'. Because for a general $z \in \mathrm{Sec}\, X(\mathfrak{g})$ it follows from the former part of Theorem 3 that $z = gh$ for some $g \in \mathrm{Int}\, \mathfrak{g}$, and it follows from Threorem 4 that $\Theta_z = \{gx_+, gx_-\}$ with $\{gx_\pm\} = \mathbb{P}_*(\mathfrak{g}'_{\pm 2})$.

5. Proofs of Theorems

We here give proofs of the latter part of Theorem 3 and Theorem 4. For proofs of Theorems 1, 2 and the former part of Theorem 3, see [KOY].

The key ingredient in the proofs here is

Symplectic Triple Systems [A1, A2, YA]**.** Choosing the highest and lowest root vectors $X_{\pm\lambda}$ so that $[X_\lambda, X_{-\lambda}] = H$, we define

$$\begin{aligned} 2\langle P, Q\rangle X_\lambda :=& [P, Q], \\ 2P \times Q :=& [P[Q, X_{-\lambda}]] + [Q[P, X_{-\lambda}]], \\ [P, Q, R] :=& [R, P \times Q]. \end{aligned}$$

for $P, Q, R \in \mathfrak{g}_1$. One obtains a skew-symmetric form $\langle , \rangle$, a symmetric product $\times$ and a triple product $[, ,]$ on $\mathfrak{g}_1$ as follows:

$$\langle , \rangle : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{C}, \quad \times : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0, \quad [, ,] : \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1.$$

Then, according to [A1, A2], if $\operatorname{rk}\mathfrak{g} \geq 2$, a triplet

$$(\mathfrak{g}_1, [, ,], \langle , \rangle)$$

has the structure of a *symplectic triple system*, that is, the skew-symmetric form $\langle , \rangle$ is not trivial and for $P, Q, R, S, T \in \mathfrak{g}_1$ the following holds:

$$\text{(STS1)} \qquad [PQR] = [QPR];$$
$$\text{(STS2)} \qquad [PQR] - [PRQ] = \langle P, R\rangle Q - \langle P, Q\rangle R + 2\langle Q, R\rangle P;$$
$$\text{(STS3)} \qquad [ST[PQR]] = [[STP]QR] + [P[STQ]R] + [PQ[STR]].$$

In general, a vector space with a triple product $[, ,]$ and a non-trivial skew-symmetric form $\langle , \rangle$ satisfying the conditions (STS1-3) above is called a *symplectic triple system*. The notion of symplectic triple systems was firstly introduced by H. Asano [A1, A2] (Yokohama City Univesity).

Remark (O. Yasukura): It can be shown that for the graded piece $\mathfrak{g}_1$ of a complex simple Lie algebra $\mathfrak{g}$, the following are equivalent:

(1) $\mathfrak{g}_1 \neq 0$.
(2) the skew-symmetric form $\langle , \rangle$ is non-trivial;
(3) the triple product $[, ,]$ is non-trivial;

The implication (1)⇒(2) is involved in [A1, A2] though there is no proof; (2)⇒(3) follows from (STS2); and, (3)⇒(1) is trivial.

Now consider the following subset of $\mathfrak{g}_1$:

$$M := \{Y \in \mathfrak{g}_1 | Y \times Y = 0\} = \{Y \in \mathfrak{g}_1 | (\operatorname{ad} Y)^2 X_{-\lambda} = 0\}.$$

Lemma 3:

$$W \cap \mathfrak{g}_1 \subseteq M.$$

Proof: If $Y \in W$, then we have

$$(\operatorname{ad} Y)^2 X_{-\lambda} + 2D(Y, X_{-\lambda})Y = 0.$$

Moreover if $Y \in \mathfrak{g}_1$, then we have $D(Y, X_{-\lambda}) = 0$ since $X_{-\lambda} \in \mathfrak{g}_{-2}$. Thus we obtain $(\operatorname{ad} Y)^2 X_{-\lambda} = 0$, and hence $Y \times Y = 0$, that is, $Y \in M$. $\square$

Proof of the latter part of Theorem 3: Since $\operatorname{ad} Y$ is nilpotent on $\mathfrak{g}$ for $Y \in \mathfrak{g}_1$ but $\operatorname{ad} H$ is not, we see that

$$\mathbb{P}_*(\mathfrak{g}_1) \cap \operatorname{Int}\mathfrak{g} \cdot h = \emptyset.$$

On the other hand, we have

$$\operatorname{Sec} X(\mathfrak{g}) \supseteq T_{x_+} X(\mathfrak{g}) \supseteq \mathbb{P}_*(\mathfrak{g}_1).$$

Therefore we have

$$\operatorname{Sec} X(\mathfrak{g}) \setminus (X(\mathfrak{g}) \sqcup \operatorname{Int}\mathfrak{g} \cdot h) \supseteq \mathbb{P}_*(\mathfrak{g}_1) \setminus X(\mathfrak{g}).$$

If $\mathbb{P}_*(\mathfrak{g}_1) \subseteq X(\mathfrak{g})$, then it follows from Lemma 3 that $\mathfrak{g}_1 \subseteq M$. This implies that $[PPQ] = 0$ for any $P, Q \in \mathfrak{g}_1$, hence we have $[\mathfrak{g}_1\mathfrak{g}_1\mathfrak{g}_1] = 0$ since $[, ,]$ is symmetric in the first and second variables. According to Remark above, we have $\operatorname{rk}\mathfrak{g} \leq 1$. Therefore if $\operatorname{rk}\mathfrak{g} \geq 2$, then $\mathbb{P}_*(\mathfrak{g}_1) \setminus X(\mathfrak{g})$ is not empty, so is $\operatorname{Sec} X(\mathfrak{g}) \setminus (X(\mathfrak{g}) \sqcup \operatorname{Int}\mathfrak{g} \cdot h)$, as required. □

To prove Theorem 4, consider moreover

$$\mathfrak{D}_0 := \{Z \in \mathfrak{g}_0 | (\operatorname{ad} Z) X_\lambda = 0\} = \{Z \in \mathfrak{g}_0 | (\operatorname{ad} Z) X_{-\lambda} = 0\},$$

and a decomposition of $\mathfrak{g}_0$ as follows:

$$\mathfrak{g}_0 = \mathfrak{D}_0 \oplus \mathbb{C} \cdot H.$$

The decomposition is obtained from an exact sequence,

$$0 \quad \to \quad \mathfrak{D}_0 \quad \to \quad \mathfrak{g}_0 \quad \xrightarrow{\operatorname{ad} X_{\pm\lambda}} \quad \mathfrak{g}_{\pm 2} \quad \to \quad 0,$$

which splits by the map, $\operatorname{ad} X_{\mp\lambda} : \mathfrak{g}_0 \leftarrow \mathfrak{g}_{\pm 2}$, with $\operatorname{ad} X_{\mp\lambda}(\mathfrak{g}_{\pm 2}) = \mathbb{C} \cdot H$.

Then we have

Lemma 4: $[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{D}_0$.

Proof: Since $[\mathfrak{g}_0, H] = 0$, we have $[\mathfrak{g}_0, \mathfrak{g}_0] = [\mathfrak{D}_0 \oplus \mathbb{C} \cdot H, \mathfrak{D}_0 \oplus \mathbb{C} \cdot H] = [\mathfrak{D}_0, \mathfrak{D}_0]$. On the other hand, it follows from the Jacobi identity that for $Z_1, Z_2 \in \mathfrak{D}_0$ we have

$$[[Z_1, Z_2]X_\lambda] = -[[Z_2, X_\lambda]Z_1] - [[X_\lambda, Z_1]Z_2] = -[0, Z_1] - [0, Z_2] = 0,$$

hence, $[Z_1, Z_2] \in \mathfrak{D}_0$. Therefore $[\mathfrak{D}_0, \mathfrak{D}_0] \subseteq \mathfrak{D}_0$. □

In fact the above is the decomposition of a reductive Lie algebra $\mathfrak{g}_0$ into a direct sum of its semi-simple part, denoted by $\mathfrak{D}_0$, and its abelian part, $\mathbb{C} \cdot H$ (see [A1, H1]). So we have moreover that $[\mathfrak{D}_0, \mathfrak{D}_0] = \mathfrak{D}_0$. But we do not use these facts.

Lemma 5: $\mathfrak{g}_1 \times \mathfrak{g}_1 \subseteq \mathfrak{D}_0$.

Proof: It follows from the Jacobi identity that for $P_1, P_2 \in \mathfrak{g}_1$ we have

$$\begin{aligned}
[[P_i[P_j, X_{-\lambda}]]X_\lambda] &= -[[[P_j, X_{-\lambda}], X_\lambda]P_i] - [[X_\lambda, P_i], [P_j, X_{-\lambda}]] \\
&= -[P_j, P_i] - [0, [P_j, X_{-\lambda}]] \\
&= [P_i, P_j],
\end{aligned}$$

so that

$$2[P_1 \times P_2, X_\lambda] = [[P_1[P_2, X_\lambda]] + [P_2[P_1, X_\lambda]], X_\lambda] = [P_1, P_2] + [P_2, P_1] = 0.$$

Therefore we have $P_1 \times P_2 \in \mathfrak{D}_0$. $\square$

The key in our proof is the following propositions:

Proposition 4 (decomposition formula): *For $Y \in \mathfrak{g}_{-1}, P \in \mathfrak{g}_1$, we have*

$$[Y, P] = Y^+ \times P - \langle Y^+, P\rangle H,$$

where we set $Y^+ := [X_\lambda, Y]$.

Proof: Dividing into two, applying the Jacobi identity to the latter term below, we have

$$\begin{aligned}
[Y, P] = [[X_{-\lambda}, Y^+]P] &= \frac{1}{2}[[X_{-\lambda}, Y^+]P] + \frac{1}{2}[[X_{-\lambda}, Y^+]P] \\
&= \frac{1}{2}[[X_{-\lambda}, Y^+]P] + \frac{1}{2}\left(-[[Y^+, P]X_{-\lambda}] - [[P, X_{-\lambda}]Y^+]\right) \\
&= \frac{1}{2}[[X_{-\lambda}, Y^+]P] + \frac{1}{2}[X_{-\lambda}[Y^+, P]] + \frac{1}{2}[[X_{-\lambda}, P]Y^+] \\
&= \frac{1}{2}\left([[X_{-\lambda}, Y^+]P] + [[X_{-\lambda}, P]Y^+]\right) + [X_{-\lambda}, \langle Y^+, P\rangle X_\lambda] \\
&= Y^+ \times P - \langle Y^+, P\rangle H. \quad \square
\end{aligned}$$

Proposition 5: *For $P, Q \in \mathfrak{g}_1$, we have*

$$P \times Q = 0, P \in M \Rightarrow \langle P, Q\rangle = 0.$$

Proof: In (STS2), set $R := P$. Then we have

$$[PQP] - [PPQ] = \langle P, P\rangle Q - \langle P, Q\rangle P + 2\langle Q, P\rangle P.$$

Since it follows from the assumption that $[PQP] = [P, P \times Q] = [P, 0] = 0$ and $[PPQ] = [Q, P \times P] = [Q, 0] = 0$, we obtain

$$\langle P, Q\rangle P = 0.$$

Therefore it follows $\langle P, Q\rangle = 0$ whether $P = 0$ or not. $\square$

Proof of Theorem 4: It suffices to show that if $h \in T_{gx_+}X$ with $g \in \operatorname{Int}\mathfrak{g}$, then $gx_+ \in \{x_+, x_-\}$. Since $T_{gx_+}X(\mathfrak{g}) = \mathbb{P}_*([\mathfrak{g}, gX_\lambda])$, in terms of Lie algebra this is equivalent to showing that for $g \in \operatorname{Int}\mathfrak{g}, Y \in \mathfrak{g}$, we have

$$H = [Y, gX_\lambda] \Rightarrow gX_\lambda \in \mathfrak{g}_2 \cup \mathfrak{g}_{-2}.$$

Here we have

$$gX_\lambda \in \mathfrak{g}_i$$

for some i with $-2 \le i \le 2$: Indeed, since $gX_\lambda \in W$, it follows

$$[H, gX_\lambda] = [[Y, gX_\lambda], gX_\lambda] = (\operatorname{ad} gX_\lambda)^2 Y = -2D(Y, gX_\lambda)gX_\lambda,$$

hence gX_λ is an eigenvector of $\operatorname{ad} H$. If we write $Y = \sum_{j=-2}^{2} Y_j$ $(Y_j \in \mathfrak{g}_j)$, then we have

$$H = [Y, gX_\lambda] = \sum_{j=-2}^{2} [Y_j, gX_\lambda].$$

Since $H \in \mathfrak{g}_0$ and $[Y_j, gX_\lambda] \in \mathfrak{g}_{i+j}$, taking the component of degree 0, we obtain

$$H = [Y_{-i}, gX_\lambda].$$

Thus setting $Y := Y_{-i}$, we may assume $Y \in \mathfrak{g}_{-i}$.

Now we first have $i \neq 0$. Suppose $i = 0$: it follows from Lemma 4 that

$$H = [Y, gX_\lambda] \in [\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{D}_0,$$

that is, $H \in \mathfrak{D}_0$. This contradicts to $[H, X_\lambda] = 2X_\lambda \neq 0$. Thus we have $i \neq 0$.

Next we have $i \neq \pm 1$. Suppose $i = 1$: we have $Y \in \mathfrak{g}_{-1}, gX_\lambda \in \mathfrak{g}_1$, and it follows from Proposition 4 that

$$H = [Y, gX_\lambda] = Y^+ \times gX_\lambda - \langle Y^+, gX_\lambda \rangle H.$$

Taking account of the decomposition $\mathfrak{g}_0 = \mathfrak{D}_0 \oplus \mathbb{C} \cdot H$, comparing both sides above, one gets

$$Y^+ \times gX_\lambda = 0 \quad \text{and} \quad \langle Y^+, gX_\lambda \rangle = -1.$$

Since it follows from Lemma 3 that $gX_\lambda \in \mathfrak{g}_1 \cap W \subseteq M$, by Proposition 5 one obtains from the former that $\langle Y^+, gX_\lambda \rangle = 0$. But this contradicts to the latter. Thus, $i \neq 1$. Similarly we obtain $i \neq -1$.

Therefore $i = 2$ or $i = -2$. $\square$

Appendix A: Homogeneous spaces M

It can be shown that $\pi(M \setminus \{0\}) \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ is a projective variety that is a homogeneous space of (an algebraic group with) Lie algebra $\mathfrak{D}_0$ [Y2]. On the other hand, it is known [A, M2, YA] that the adjoint varieties $X(\mathfrak{g})$ correspond to the meta-symplectic geometry while the Severi varieties [FR, LV, Z] correspond to the projective geometry, in the magic square of H. Freudenthal [F] as follows:

elliptic geometry	B_1	A_2	C_3	F_4
projective geometry	A_2	$A_2 + A_2$	A_5	E_6
symplectic geometry	C_3	A_5	D_6	E_7
metasymplectic geometry	F_4	E_6	E_7	E_8

In this context $\pi(M \setminus \{0\}))$ correspond to the symplectic geometry, and are called *cubic Veronese varieties* in [M2].

Now our result is

Theorem 5 (homogeneous spaces M of $\mathfrak{D}_0$):

$$X(\mathfrak{g}) \cap \mathbb{P}_*(\mathfrak{g}_1) = \pi(M \setminus \{0\}).$$

Note that $\mathbb{P}_*(\mathfrak{g}_1)$ is a linear subspace of $T_{x_+}X$ of codimension 2: In fact, $T_{x_+}X$ is spanned by the point x_+ of contact, the point h corresponding to the characteristic element, and this linear space $\mathbb{P}_*(\mathfrak{g}_1)$.

Proof of Theorem 5: We here show the inclusion $\supseteq$: the converse is just Lemma 3. By virtue of Theorem 1 this is equivalent to showing that if $Y \in M$, then

$$(\operatorname{ad} Y)^2 Z + 2D(Y, Z)Y = 0$$

for all $Z \in \mathfrak{g}$. Since this equation is linear on Z, it suffices to show that the equation holds for $Z \in \mathfrak{g}_i$ with $-2 \leq i \leq 2$.

We first consider the case $i \neq -1$. We then have $D(Y, Z) = 0$ since $Y \in \mathfrak{g}_1$. Therefore it suffices to show that

$$(\operatorname{ad} Y)^2 Z = 0.$$

If $i = 1, 2$, then the claim follows since $(\operatorname{ad} Y)^2 Z \in \mathfrak{g}_{i+2} = 0$ with $i + 2 > 2$. If $i = -2$, then we have $Z = cX_{-\lambda}$ for some $c \in \mathbb{C}$, and $(\operatorname{ad} Y)^2 Z = c(\operatorname{ad} Y)^2 X_{-\lambda} = 0$ since $Y \in M$.

In case of $i = 0$, set $Z_1 := (\operatorname{ad} Y)Z \in \mathfrak{g}_1$. Then the claim above is written as $[Y, Z_1] = 0$, that is, $\langle Y, Z_1 \rangle = 0$. By virtue of Proposition 5 this is reduced to showing the next

Claim: $Y \in M, Z \in \mathfrak{g}_0 \Rightarrow Y \times Z_1 = 0$, where $Z_1 := (\operatorname{ad} Y)Z$.

Proof of Claim: By the definition we have $Y \times Z_1 = \frac{1}{2}\{[Y[Z_1, X_{-\lambda}]] + [Z_1[Y, X_{-\lambda}]]\}$. For each term of the right hand, we have

$$\begin{aligned}
[Y[Z_1, X_{-\lambda}]] &= [Y[[Y, Z]X_{-\lambda}]] \\
&= -[Y[[Z, X_{-\lambda}]Y]] - [Y[[X_{-\lambda}, Y]Z]] \\
&= 0 - [Y[[X_{-\lambda}, Y]Z]] \quad (\because [Z, X_{-\lambda}] \in \mathfrak{g}_{-2}, Y \in M) \\
&= -[Y[[X_{-\lambda}, Y]Z]],
\end{aligned}$$

and

$$\begin{aligned}[Z_1[Y, X_{-\lambda}]] &= [[Y, Z], [Y, X_{-\lambda}]] \\ &= -[[Z, [Y, X_{-\lambda}]]Y] - [[[Y, X_{-\lambda}]Y]Z] \\ &= -[[Z, [Y, X_{-\lambda}]]Y] - [0, Z] \quad (\because Y \in M) \\ &= -[[Z, [Y, X_{-\lambda}]]Y] \\ &= [Y[[X_{-\lambda}, Y]Z]].\end{aligned}$$

Thus we obtain $Y \times Z_1 = 0$. $\square$

This completes the proof when $i \neq -1$, and we next consider the case $i = -1$. In this case we show that

$$(\operatorname{ad} Y)^2 Z = 4\langle Y, Z^+\rangle Y = -2D(Y, Z)Y,$$

where we set $Z^+ := [X_\lambda, Z]$ for $Z \in \mathfrak{g}_{-1}$: note that one has $Z = [X_{-\lambda}, Z^+]$.

We have $(\operatorname{ad} Y)^2 Z = 4\langle Y, Z^+\rangle Y$: Indeed, applying the Jacobi identity twice, we have

$$\begin{aligned}(\operatorname{ad} Y)^2 Z &= [Y[Y[X_{-\lambda}, Z^+]]] \\ &= -[Y[X_{-\lambda}[Z^+, Y]]] - [Y[Z^+[Y, X_{-\lambda}]]] \\ &= -2\langle Z^+, Y\rangle[Y[X_{-\lambda}, X_\lambda]] + \{[Z^+[[Y, X_{-\lambda}]Y]] + [[Y, X_{-\lambda}], [Y, Z^+]]\} \\ &= -2\langle Z^+, Y\rangle[Y, -H_\lambda] + [Z^+, 0] + 2\langle Y, Z^+\rangle[[Y, X_{-\lambda}]X_\lambda] \quad (\because Y \in M) \\ &= 2\langle Y, Z^+\rangle Y + 0 + 2\langle Y, Z^+\rangle Y \\ &= 4\langle Y, Z^+\rangle Y.\end{aligned}$$

On the other hand, we have $D(Y, Z) = -2\langle Y, Z^+\rangle$: Indeed, we have $D(X_\lambda, X_{-\lambda}) = 1$ and

$$\begin{aligned}D(Y, Z) &= D(Y, [X_{-\lambda}, Z^+]) \\ &= -D(Y, [Z^+, X_{-\lambda}]) \\ &= -D([Y, Z^+], X_{-\lambda}) \\ &= -2\langle Y, Z^+\rangle D(X_\lambda, X_{-\lambda}) \\ &= -2\langle Y, Z^+\rangle.\end{aligned}$$

This completes the proof of Theorem 5. $\square$

Appendix B: Higher secant varieties

For a complex projective variety $X \subseteq \mathbb{P}^N_{\mathbb{C}}$, set

$$S^{(k)}X := \overline{\bigcup_{x_0,\dots,x_k \in X, \dim\langle x_0,\dots,x_k\rangle = k} \langle x_0, \dots, x_k\rangle},$$

where $\langle x_0, \dots, x_k\rangle$ is the linear subspace of $\mathbb{P}^N_{\mathbb{C}}$ spanned by the points $x_0, \dots, x_k$. We call $S^{(k)}X$ the *variety of k-secants* of $X \subseteq \mathbb{P}^N_{\mathbb{C}}$. Of course $S^{(0)}X = X$ and $S^{(1)}X = \operatorname{Sec} X$.

Our result is

Theorem 6 (defining equations of higher secant varieties):

$$S^{(k)}X(A_l) = \pi(K_{k+1} \setminus \{0\})$$

for $0 \le k \le l+1$.

As in the proof of Lemma 1, by virtue of [E, Theorem 2.1] it suffices to show that the equality holds set-theoretically.

From the definition of higher secant varieties and Proposition 1 we obtain the inclusion $\subseteq$ (see the proof of Proposition 1). The converse $\supseteq$ follows from

Proposition 6 (decompoisiton of traceless matrices): *For $A \in M_n\mathbb{C}$, if* $\operatorname{tr} A = 0$ *and* $\operatorname{rk} A = r \ge 1$, *then there exist* $A_1, \dots, A_r \in M_n\mathbb{C}$ *such that*

$$A = \sum_{i=1}^{r} A_i, \quad \operatorname{tr} A_i = 0, \quad \operatorname{rk} A_i = 1.$$

Proof: We show the assertion by induction on the rank r of A. Assume $r \ge 2$: the assertion is trivial in case of $r = 1$. It suffices to show

Claim: *There exist* $A_1, A_0 \in M_n\mathbb{C}$ *such that*

$$A = A_1 + A_0, \quad \operatorname{tr} A_i = 0, \quad \operatorname{rk} A_1 = 1, \quad \operatorname{rk} A_0 = r-1.$$

To show this, since the ground field $\mathbb{C}$ is algebraically closed, one may assume that A is a Jordan normal form as follows:

$$A = J(n_1, a_1) \oplus J(n_2, a_2) \oplus \cdots \oplus J(n_k, a_k), \quad (n_i \in \mathbb{N}, a_i \in \mathbb{C}),$$

where $J(m, b) \in M_m\mathbb{C}$ denotes a Jordan cell with eigenvalue $b \in \mathbb{C}$:

$$J(m,b) = \begin{bmatrix} b & 1 & & \\ & b & \ddots & \\ & & \ddots & 1 \\ & & & b \end{bmatrix} \in M_m\mathbb{C}.$$

We first consider the case when *there exist* i, j *such that* $(a_i - a_j)a_ia_j \neq 0$. To simplify the notation, write $b := a_i, c := a_j, l := n_i, m := n_j$, assume $i = 1, j = 2$, and set

$$R := \bigoplus_{i=3}^{k} J(n_i, a_i) \in M_{n-l-m}\mathbb{C}.$$

Then we have

$$A = J(l, b) \oplus J(m, c) \oplus R$$

with $(b - c)bc \neq 0$. Now set

$$\begin{aligned} S &:= O_{l-1} \oplus \frac{1}{b-c}\begin{bmatrix} -bc & bc \\ -bc & bc \end{bmatrix} \oplus O_{m-1}, \\ T &:= J(l, b) \oplus J(m, c) - S, \end{aligned}$$

where $O_m \in M_m\mathbb{C}$ denotes zero matrix. Then we have

$$T = \left[\begin{array}{c|c|c} J(l-1,b) & 1 & \mathbf{0} \\ \hline & U & 1 \\ \hline \mathbf{0} & & J(m-1,c) \end{array}\right],$$

where we set

$$U := \frac{1}{b-c}\begin{bmatrix} b^2 & -bc \\ bc & -c^2 \end{bmatrix}.$$

We set

$$A_1 := S \oplus O_{n-l-m}, \quad A_0 := T \oplus R.$$

These matrices have the required properties. Indeed, we have

$$\begin{aligned} A_1 + A_0 =& (S + T) \oplus R = J(l, b) \oplus J(m, c) \oplus R = A, \\ \operatorname{tr} A_1 =& \operatorname{tr} S = 0, \\ \operatorname{tr} A_0 =& \operatorname{tr} T + \operatorname{tr} R = (\operatorname{tr} J(l, b) \oplus J(m, c) - \operatorname{tr} S) + \operatorname{tr} R = \operatorname{tr} A = 0, \\ \operatorname{rk} A_1 =& \operatorname{rk} S = 1, \end{aligned}$$

where one needs $bc \neq 0$. Moreover, we have $\operatorname{rk} T < \operatorname{rk} J(l, b) \oplus J(m, c)$: indeed, $\det T = 0$ since $\det U = 0$, while $\det J(l, b) \oplus J(m, c) = b^l c^m \neq 0$. Therefore

$$\operatorname{rk} A_0 = \operatorname{rk} T + \operatorname{rk} R \leq (\operatorname{rk} J(l, b) \oplus J(m, c) - 1) + \operatorname{rk} R = \operatorname{rk} A - 1 = r - 1.$$

But the converse inequality is trivial since $A = A_1 + A_0$ with $\operatorname{rk} A_1 = 1$. Thus we have

$$\operatorname{rk} A_0 = r - 1.$$

This completes the proof when there exist i, j such that $(a_i - a_j)a_ia_j \neq 0$.

We next consider the contrary case. We see that the eigenvalues a_i take at most 2 values, that is,

$$\{a_1, \dots, a_k\} = \{0, a\}, \quad (a \in \mathbb{C})$$

as sets. Moreover we find that $a = 0$: Because if $a \neq 0$, then

$$0 = \operatorname{tr} A = qa,$$

where $q > 0$ is the multiplicity of the eigenvalue a, and this is a contradiction since the characteristic of $\mathbb{C}$ is zero. Thus A is nilpotent and the assertion now easily follows. $\square$

For several projective varieties and their secant varieties, determinantal realzations are known, and it might be interesting to compare Theorem 6 with the following:

$X \subseteq \mathbb{P}$	matrices M	matrices for $S^{(k)}X$
$\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$	$M_{m,n}\mathbb{C}$	M_{k+1}
$v_2(\mathbb{P}^{m-1})$	$Symm_m\mathbb{C}$	M_{k+1}
$\mathbb{G}(2, m)$	$Alt_m\mathbb{C}$	M_{2k+2}
$v_{m+n-2}(\mathbb{P}^1)$	$Cat_{m,n}\mathbb{C}$	M_{k+1}

Notation. We denote by $Symm_m\mathbb{C}$ and by $Alt_m\mathbb{C}$ the spaces of symmetric and alternating matrices of degree m, respectively. We denote by $Cat_{m,n}\mathbb{C}$ the catalecticant space of size $m \times n$, that is,

$$Cat_{m,n}\mathbb{C} := \{[a_{ij}] \in M_{m,n}\mathbb{C} | a_{ij} = a_{i+1j-1} \ (\forall i, j)\}.$$

For a linear space M of matrices we denote by M_r the locus of matrices in M with rank at most r, as before.

The table above reads as follows: The second row, for example, means that we have

$$S^{(k)}v_2(\mathbb{P}^{m-1}) = \pi((Symm_m\mathbb{C})_{k+1} \setminus \{0\}),$$

for $k \geq 0$, where $\pi : Symm_m\mathbb{C} \setminus \{0\} \to \mathbb{P}_*(Symm_m\mathbb{C})$ is the canonical projection; note that to obtain the equality in the scheme-theoretic sense for $Alt_m\mathbb{C}$ one should take Pfaffians insead of minors. For proofs, see [H].

One might expect some formulas similar to Theorem 6 would hold for other adjoint varieties. But this is not straightforward: In fact, for exmple, the adjoint variety associated to the simple Lie algebra $Alt_m\mathbb{C}$, which is of type B or D, is not a Grassmann Variety $\mathbb{G}(2, m)$ but a Fano variety of projective lines in a quadric hypersurface in $\mathbb{P}^{m-1}_{\mathbb{C}}$, which is a subvariety of codimension 3 in $\mathbb{G}(2, m)$. Thus to obtain defining equations for those adjoint varieties, one needs more polynomials other than Pfaffians.

It is observed that for those varieties X in the table above including $X(A_l)$, the varieties of k-secants, $S^{(k)}X$, have defining equations of degree $k+2$: In fact, the defining equations of $S^{(k)}X$ for Grassmann varieties come from Pfaffians, and for others from ordinary minors. On the other hand, according to W. Lichtenstein [L], any homogeneous projective variety has defining equation of degree 2, where a homogeneous projective variety is by definition a (unique) closed orbit in $\mathbb{P}_*(V)$ of the action of an algebraic group G induced by a rational representation $G \curvearrowright V$. So it would be natural to pose

Conjecture. *The variety of k-secants, $S^{(k)}X$, has defining equations of degree $k+2$ if X is an adjoint variety, or more generally, if X is a homogeneous projective variety.*

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