# ある種のフロイデンタール多様体の射影幾何について
# Projective geometry of Freudenthal varieties of certain type

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## 0. Introduction

H. Freudenthal constructed, in a series of his papers (see [10] and its references), the exceptional Lie algebras of type $E_8$, $E_7$, $E_6$ and $F_4$, with defining various projective varieties. The purpose of our work is to study projective geometry for his varieties of certain type, which are called *varieties of planes* in the symplectic geometry of Freudenthal (see [10, 4.11], [24, 2.3]).

Let $\mathfrak{g}$ be a graded, simple, finite-dimensional Lie algebra over the complex number field $\mathbb{C}$ with grades between $-2$ and 2, $\dim \mathfrak{g}_2 = 1$ and $\mathfrak{g}_1 \neq 0$, namely a *graded Lie algebra of contact type*: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (see §1). We set

$$\mathcal{V} := \{x \in \mathfrak{g}_1 \setminus \{0\} | (\operatorname{ad} x)^2 \mathfrak{g}_{-2} = 0\},$$

and define an algebraic set $V$ in $\mathbb{P}(\mathfrak{g}_1)$ to be the projectivization of $\mathcal{V}$:

$$V := \pi(\mathcal{V}),$$

where $\pi : \mathfrak{g}_1 \setminus \{0\} \to \mathbb{P}(\mathfrak{g}_1)$ is the natural projection. Then we call $V \subseteq \mathbb{P}(\mathfrak{g}_1)$ (with the reduced structure) the *Freudenthal variety* associated to the graded Lie algebra $\mathfrak{g}$ of contact type, which is a natural generalization of Freudenthal's varieties mentioned above: Note that $V$ is not necessarily connected in this general setting. We here consider moreover the projectivization of a closed set $\{x \in \mathfrak{g}_1 | (\operatorname{ad} x)^{k+1} \mathfrak{g}_{-2} = 0\}$, and denote it by $V_k$: we have

$$\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \mathbb{P},$$

where we set $\mathbb{P} := \mathbb{P}(\mathfrak{g}_1)$ for short. Clearly, $V_3$ is a quartic hypersurface, $V_2$ is an intersection of cubics and $V_1 = V$ is an intersection of quadrics, with a few exceptions.

In the literature, several results have been known about the structure of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-space, case-by-case for each exceptional Lie algebra of types $E_8$, $E_7$, $E_6$ and $F_4$, from the view-point of the invariant theory of prehomogeneous vector spaces (see [13], [15], [20], [23]). By virtue of those results, it can be shown, for example, that the stratification of $\mathbb{P}$ given by the differences of $V_k$'s exactly corresponds to the orbit decomposition of the $\mathfrak{g}_0$-space $\mathfrak{g}_1$ for those exceptional Lie algebras, and also that Freudenthal varieties $V$ associated to the algebras of type $E_8$, $E_7$, $E_6$ and $F_4$ are respectively projectively equivalent to the 27-dimensional $E_7$-variety arising from the 56-dimensional irreducible representation, the orthogonal Grassmann variety of isotropic 6-planes in $\mathbb{C}^{12}$ (namely, the 15-dimensional spinor variety), the Grassmann variety of 3-planes in $\mathbb{C}^6$ and the symplectic Grassmann variety of isotropic 3-planes in $\mathbb{C}^6$, with $\dim \mathbb{P} = 55, 31, 19$ and 13, respectively (see Appendix 1): for those homogeneous projective varieties, we refer to [12, §23.3].

In this article we study the Freudenthal varieties $V$ with the filtration $\{V_k\}$ of the ambient space $\mathbb{P}$, from the view-point of projective geometry, not individually but systematically in terms

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of abstract Lie algebras, without depending on the classification of simple Lie algebras as well as on the known results for each case of types $E_8$, $E_7$, $E_6$ and $F_4$.

Before stating the main result, we note that the Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2 \simeq \mathbb{C}$ defines a non-degenerate skew-symmetric form on $\mathfrak{g}_1$, so that this form allows us to identify $\mathfrak{g}_1$ with its dual space, hence $\mathbb{P}$ with its dual space, and $\mathfrak{g}_1$ is even-dimensional. Moreover, the quartic form on $\mathfrak{g}_1$ defining $V_3$ has a differential which via the symplectic form defines a vector field on $\mathfrak{g}_1$, and this vector field defines a 1-dimensional distribution on $\mathbb{P}$ away from the singular locus of $V_3$ (see Proposition A1). We denote by $L_P$ the (closure of the) integral curve of this distribution passing through $P \in \mathbb{P} \setminus \operatorname{Sing} V_3$. On the other hand, we have a rational map $\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$ defined by $x \mapsto (\operatorname{ad} x)^3 \mathfrak{g}_{-2}$ with base locus $V_2$, which turns out to be a Cremona transformation of $\mathbb{P}$: It is deduced that $\gamma^{-1}(V) = V_3 \setminus V_2$, $\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$, $\gamma^2 = 1$ on $\mathbb{P} \setminus V_3$, and $\gamma$ is explicitly given by the partial differentials of $q$ (see Proposition A2). Note that our $\gamma$ is a special case of the Cremona transformations in [7, Theorem 2.8 (ii)].

Our main results are summarized as follows (see Theorems A, B, C, D, E, Corollaries A2, B1, B3 and C):

**Theorem.** *Assume that $V$ is irreducible. Then we have:*

(1) *$V$ is a Legendrian subvariety of $\mathbb{P}$, that is, the projectivization of a Lagrangian subvariety of $\mathfrak{g}_1$, with $\dim V = n-1$, spans $\mathbb{P}$, and is an orbit of the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_0$, hence smooth, where $\dim \mathfrak{g}_1 = 2n$. In particular, the projective dual $V^*$ of $V$ is equal to the union of tangents to $V$ via the symplectic form.*

(2) *$V_2$ is the singular locus of $V_3$, and for any $P \in \mathbb{P} \setminus V_2$, $L_P$ is the line in $\mathbb{P}$ joining $P$ and $\gamma(P)$. Moreover, we have:*
   - (a) *If $P \in \mathbb{P} \setminus V_3$, then $L_P$ is a unique secant line of $V$ passing through $P$, there is no tangent line to $V$ passing through $P$, $L_P \cap V$ consists of harmonic conjugates with respect to $P$ and $\gamma(P)$, and $L_P \setminus V \subseteq \mathbb{P} \setminus V_3$. Moreover, $\gamma$ preserves $L_P$, and the automorphism of $L_P$ induced from $\gamma$ leaves each point in $L_P \cap V$ invariant and permutes $P$ and $\gamma(P)$.*
   - (b) *If $P \in V_3 \setminus V_2$, then there is no secant line of $V$ passing through $P$, $L_P$ is a unique tangent line to $V$ passing through $P$, $L_P \cap V = \gamma(P)$, and $L_P \setminus V \subseteq V_3 \setminus V_2$. Moreover, $L_P$ is contracted by $\gamma$ to the contact point $\gamma(P)$, and conversely the fibre of $\gamma$ on $Q \in V$ consists of the points $P \in V_3 \setminus V_2$ such that $Q \in L_P$, or equivalently, $P$ lies on some tangent to $V$ at $Q$.*

   *In particular, $V$ is a variety with one apparent double point, and $V_3$ is the union of tangents to $V$.*

(3) *For any $P \in V_2 \setminus V$, the family of secants of $V$ passing through $P$ is of dimension at least 1, and all of those secants are isotropic with respect to the symplectic form: In particular, $V_2 \setminus V$ is covered by isotropic secants of $V$.*

(4) *For any $Q, R \in V$, the secant line joining $Q$ and $R$ is isotropic if and only if the tangents to $V$ at $Q$ and at $R$ are disjoint.*

(5) *For any $P \in V_3 \setminus V_2$ and $Q \in V$, if the secant line joining $Q$ and the contact point $\gamma(P)$ of $L_P$ is not isotropic, then there is a twisted cubic curve contained in $V$ to which $L_P$ and $L_R$ are tangent at $\gamma(P)$ and at $Q$, respectively, where $R$ is a point on some tangent to $V$ at $Q$ away from $V_2$, determined by $P$ and $Q$.*

(6) *If $V_2 \neq V$, then $V$ is ruled, that is, covered by lines contained in $V$.*

(7) *For any $P \in V$, the double projection from $P$ gives a birational map from $V$ onto $\mathbb{P}^{n-1}$, and by the inverse $V$ is written as the closure of the image of a cubic Veronese embedding of a certain affine space $\mathbb{A}^{n-1}$ under some projection to $\mathbb{P}$.*

We show also that *the three conditions, $V = \emptyset$, $V_3 = \mathbb{P}$ and $V_2 = \mathbb{P}$ are equivalent to each other* (see Corollary A1), and that *if $V$ is neither empty nor irreducible, then $\mathfrak{g}_1$ decomposes* *naturally into two irreducible $\mathfrak{g}_0$-submodules of dimension $n$ and $V$ is the (disjoint) union of the projectivizations of those summands* (see Corollary B2).

Finally we should mention that S. Mukai announced a theorem [22, (5.8)] on cubic Veronese varieties without proofs. Our work was originated by looking for proofs of the corresponding statements for Freudenthal varieties (Corollaries A2, B1, C and Theorem D): In fact, we see from his list [22, (5.10)] of cubic Veronese varieties (and the list in Appendix 1) that the notion of our Freudenthal varieties coincides with that of his cubic Veronese varieties. Our result gives a partial explanation for this coincidence (see Theorem D).

This is a joint work with Osami Yasukura. For proofs of the results here, see [19].

## 1. Preliminaries

For a finite-dimensional, simple Lie algebra $\mathfrak{g}$ of rank $\geq 2$, a graded decomposition of contact type is obtained as follows: Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a basis $\Delta$ of the root system $R$ with respect to $\mathfrak{h}$, and fix an order on $R$ defined by $\Delta$. Denote by $\rho$ the highest root of $\mathfrak{g}$, let $E_+$ and $E_-$ be highest and lowest weight vectors, respectively, and set $H := [E_+, E_-]$. By multiplying suitable scalars, one may assume that $(E_+, H, E_-)$ form an $\mathfrak{sl}_2$-triple, that is, those vectors have the following standard relations: $[H, E_+] = 2E_+$, $[H, E_-] = -2E_-$ and $[E_+, E_-] = H$. Then, the eigenspace decomposition of $\mathfrak{g}$ with respect to $\operatorname{ad} H$ gives $\mathfrak{g}$ a graded decomposition of contact type: In other words, if we set $\mathfrak{g}_\lambda := \{x \in \mathfrak{g} | [H, x] = \lambda x\}$ for $\lambda \in \mathbb{C}$, then it follows that $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\dim \mathfrak{g}_2 = 1$ and $\mathfrak{g}_1 \neq 0$: In fact, $\mathfrak{g}_1 = 0$ if and only if $\mathfrak{g} = \mathfrak{sl}_2$. In terms of root spaces of $\mathfrak{g}$, we have

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \setminus (R_\rho \cup \{\rho\})} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \quad \mathfrak{g}_{\pm 1} = \bigoplus_{\alpha \in R_\rho} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm\rho} = \mathbb{C}E_\pm,$$

where $R^+$ is the set of positive roots and $R_\rho := \{\alpha \in R^+ | \rho - \alpha \in R\}$: Indeed, let $\mathfrak{s}_\rho$ be the subalgebra of $\mathfrak{g}$ spanned by $E_+$, $H$ and $E_-$, which is isomorphic to $\mathfrak{sl}_2$. Then the irreducible decomposition of $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module gives the decomposition above (see, for full details, [27]). Conversely, for a graded decomposition $\mathfrak{g} = \sum \mathfrak{g}_i$ of contact type, taking suitable bases $E_+$ for $\mathfrak{g}_2$ and $E_-$ for $\mathfrak{g}_{-2}$ with $H := [E_+, E_-]$, one may assume that $(E_+, H, E_-)$ form an $\mathfrak{sl}_2$-triple, as before. Then, we see that $E_+$ and $E_-$ are some highest and lowest weight vectors, respectively, and each $\mathfrak{g}_i$ is recovered as an $(\operatorname{ad} H)$-eigenspace. Therefore, the graded decompositions of contact type are unique up to automorphism of $\mathfrak{g}$, so that the Freudenthal variety $V$ is essentially unique and determined by $\mathfrak{g}$ itself (see Appendix 1).

Now, we define a symmetric product $\times : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ by the formula:

$$-2a \times b = [b, [a, E_-]] + [a, [b, E_-]],$$

which induces a symmetric map $L : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \operatorname{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$ and a ternary product $[\,,\,] : \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ by

$$[a, b, c] = L(a, b)c = [a \times b, c].$$

Note that the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful since $\mathfrak{g}$ is simple (see [27, Lemma 3.2 (1)]): we may assume $\mathfrak{g}_0 \subseteq \operatorname{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$, so that we identify $L(a, b)$ with $a \times b$. We think of $\mathfrak{g}_1$ as an $\mathfrak{g}_0$-module via the adjoint action: For example, we often write $Dx$ instead of $(\operatorname{ad} D)x$ and $[D, x]$ for $D \in \mathfrak{g}_0$ and $x \in \mathfrak{g}_1$. As the skew-symmetric form $\langle , \rangle : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{C}$ and the quartic form on $\mathfrak{g}_1$ defining $V_3$ mentioned in Introduction, we use the ones determined by

$$2\langle a, b \rangle E_+ = [a, b], \quad 2q(x)E_+ = (\operatorname{ad} x)^4 E_-.$$

Note that the skew-symmetric form $\langle , \rangle$ is *non-degenerate* since $\mathfrak{g}$ is simple (see [27, Lemma 3.2 (2)]).

With the notation above, it follows that

$$\begin{aligned} V = V_1 &= \pi\left(\{x \in \mathfrak{g}_1 \setminus \{0\} | x \times x = 0\}\right), \\ V_2 &= \pi\left(\{x \in \mathfrak{g}_1 \setminus \{0\} | [xxx] = 0\}\right), \\ V_3 &= \pi\left(\{x \in \mathfrak{g}_1 \setminus \{0\} | \langle x, [xxx]\rangle = 0\}\right), \end{aligned}$$

and $q(x) = \langle x, [xxx]\rangle$. Note that $V_0 = \emptyset$ since $[[x, E_-]E_+] = x$ for any $x \in \mathfrak{g}_1$: Indeed, it follows from the Jacobi identity that $[[x, E_-]E_+] = [[x, E_+], E_-] + [x[E_-, E_+]] = [x, -H] = x$ since $[x, E_+] \in \mathfrak{g}_3 = 0$. On the other hand, it follows from Lemma 1 below that $V \neq \mathbb{P}$.

**Lemma 1.** *Let $\mathfrak{g}_{00}$ be the subalgebra of $\mathfrak{g}_0$ defined by $\mathfrak{g}_{00} := \mathrm{Ker}(\mathrm{ad}\, E_+|\mathfrak{g}_0) = \mathrm{Ker}(\mathrm{ad}\, E_-|\mathfrak{g}_0)$. Then we have $\mathfrak{g}_0 = \mathfrak{g}_{00} \oplus \mathbb{C}H$, and $\mathfrak{g}_{00}$ is linearly spanned by the elements in $\mathfrak{g}_0$ of the form $a \times b$ with $a, b \in \mathfrak{g}_1$. In particular, $\mathfrak{g}_{00} \neq 0$, and $x \times x \neq 0$ for some $x \in \mathfrak{g}_1$.*

**Lemma 2** (Asano [3])**.** *For any $a, b, c \in \mathfrak{g}_1$ and $D \in \mathfrak{g}_{00}$, we have*

(1) $\langle Da, b\rangle + \langle a, Db\rangle = 0$.
(2) $D(a \times b) = Da \times b + a \times Db$.
(3) $D[abc] = [(Da)bc] + [a(Db)c] + [ab(Dc)]$.

If we denote by $G_{00}$ the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_{00}$, then Lemma 2 tells that the symplectic form $\langle, \rangle$, the symmetric product $\times$ and the ternary product $[, , ]$ are equivariant with respect to the action of $G_{00}$, so that each $V_i$ is stable under the action of $G_{00}$, that is, a union of some orbits of $G_{00}$. We should mention that the above proofs of (2) and (3) in Lemma 2 are due to the referee, much simpler than the ones in [3].

**Lemma 3** (Asano [3])**.** *We have $[abc] - [acb] = \langle a, c\rangle b - \langle a, b\rangle c + 2\langle b, c\rangle a$ for any $a, b, c \in \mathfrak{g}_1$.*

## 2. Basic Results

**Proposition 1.** *If $x \in \mathcal{V}$, then $[axx] = 3\langle a, x\rangle x$ for any $a \in \mathfrak{g}_1$* (Asano [2])*. In particular, $\mathbb{C}x \subseteq \mathfrak{g}_{00}x$, and if $a \times x = 0$, then $\langle a, x\rangle = 0$.*

**Proposition 2.** *We have $\langle [abc], d\rangle = \langle [cda], b\rangle$ for any $a, b, c, d \in \mathfrak{g}_1$.*

**Proposition 3.** *If $x \in \mathcal{V}$ and $D, E \in \mathfrak{g}_{00}$, then $Dx \times x = 0$* (Asano [2])*, $\langle Dx, x\rangle = 0$, $\langle Dx, Ex\rangle = 0$ and $[(Dx)(Ex)x] = 0$.*

**Proposition 4.** *For any $a \in \mathfrak{g}_1$, we have:*

(1) $[aaa] \times a = 0$.
(2) $[aa[aaa]] = 3q(a)a$.
(3) $[aaa] \times [aaa] = -3q(a)a \times a$.
(4) $[[aaa][aaa][aaa]] = -9q(a)^2 a$.
(5) $q([aaa]) = 9q(a)^3$.

**Proposition 5.** *If $b = a + x$ with $a \in \mathfrak{g}_1$ and $x \in \mathcal{V}$, then we have:*

(1) $b \times b = a \times a + 2a \times x$.
(2) $[bbb] = [aaa] + 3[aax] + 6\langle x, a\rangle(a - x)$.
(3) $q(b) = q(a) + 4\langle x, [aaa]\rangle + 12\langle x, a\rangle^2$.

**Proposition 6.** *For any $a \in \mathfrak{g}_1$, we have:*

(1) $3[aa[aab]] = 8\langle b, [aaa]\rangle a + 8\langle a, b\rangle[aaa] + \langle a, [aaa]\rangle b$ *for any $b \in \mathfrak{g}_1$.*
(2) *If $q(a) \neq 0$, then the linear map $L(a, a)$ has full rank.*

**Proposition 7.** *For any $a \in \mathfrak{g}_1$ and $x \in \mathcal{V}$, we have*

(1) $[aaa] \times x + 3[aax] \times a + 6\langle x, a\rangle a \times a = 0$.
(2) $3[aax] \times [aax] + 8\langle x, [aaa]\rangle a \times x - 8\langle x, a\rangle[aaa] \times x = 0$. *In particular, if $[aaa] = 0$, then $[aax] \times [aax] = 0$, and moreover, $\mathbb{C}x + \mathbb{C}[aax] \subseteq \mathcal{V} \cup \{0\}$.*

## 3. A Line Field and a Cremona Transformation

**Proposition A1.** *The quartic form $q$ on $\mathfrak{g}_1$ has a differential at $a \in \mathfrak{g}_1$ as follows:*

$$dq(a) : t_a\mathfrak{g}_1 \to \mathbb{C}; b \mapsto 4\langle b, [aaa]\rangle,$$

*where $t_a\mathfrak{g}_1$ is the Zariski tangent space to $\mathfrak{g}_1$ at $a$, naturally identified with $\mathfrak{g}_1$. In particular, the singular locus of $V_3$ is equal to $V_2$. The vector field on $\mathfrak{g}_1$ corresponding to $dq$ via $\langle , \rangle$ induces a 1-dimensional distribution $\mathcal{D}$ on $\mathbb{P}$ away from $\operatorname{Sing} V_3 = V_2$, which is given by*

$$\mathcal{D} : \pi(a) \mapsto (\mathbb{C}a + \mathbb{C}[aaa])/\mathbb{C}a,$$

*where $\pi(a) \in \mathbb{P} \setminus V_2$ and we naturally identify the Zariski tangent space $t_{\pi a}\mathbb{P}$ with the quotient space $\mathfrak{g}_1/\mathbb{C}a$.*

**Proposition A2.** *Let $\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$ be a rational map induced from the cubic, $a \mapsto [aaa]$. Then*:

1. *$\gamma^{-1}(V) = V_3 \setminus V_2$.*
2. *$\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$.*
3. *$\gamma^2 = 1$ on $\mathbb{P} \setminus V_3$, hence $\gamma$ gives an automorphism of $\mathbb{P} \setminus V_3$.*
4. *$\gamma$ is explicitly given by the partial differentials of $q$.*

*In particular, $\gamma$ is a Cremona transformation of $\mathbb{P}(\mathfrak{g}_1)$ with order 2 if $V_2 \neq \mathbb{P}$.*

A *secant line* of $V$ is by definition a line in $\mathbb{P}$ which passes through at least two distinct points of $V$ and is not contained in $V$. We note that for a line $L$ in $\mathbb{P}$ if the scheme-theoretic intersection $L \cap V$ has length more than 2, then $L \subseteq V$: Indeed, $V$ is an intersection of quadric hypersurfaces.

**Theorem A.** *Let $L_P$ be the closure of the integral curve of $\mathcal{D}$ through $P \in \mathbb{P} \setminus V_2$, where $\mathcal{D}$ is the 1-dimensional distribution on $\mathbb{P} \setminus V_2$ induced from the quartic form $q$. Then we have:*

1. *For any $P \in \mathbb{P} \setminus V_2$, $L_P$ is the line in $\mathbb{P}$ joining $P$ and $\gamma(P)$.*
2. *If $P \in \mathbb{P} \setminus V_3$, then we have:*
   - (a) *$L_P$ is a secant line of $V$, and $L_P \cap V$ consists of harmonic conjugates with respect to $P$ and $\gamma(P)$.*
   - (b) *$L_P \setminus V \subseteq \mathbb{P} \setminus V_3$.*
   - (c) *$L_P$ is a unique secant line of $V$ passing through $P$.*
   - (d) *There is no tangent line to $V$ passing through $P$.*
   - (e) *$\gamma(L_P \setminus V) = L_P \setminus V$, and the automorphism of $L_P$ induced from $\gamma$ leaves each point in $L_P \cap V$ invariant and permutes $P$ and $\gamma(P)$.*
3. *If $P \in V_3 \setminus V_2$, then we have:*
   - (a) *$L_P$ is a tangent line to $V$, and $L_P \cap V = \{\gamma(P)\}$.*
   - (b) *$L_P \setminus V \subseteq V_3 \setminus V_2$.*
   - (c) *There is no secant line of $V$ passing through $P$.*
   - (d) *$L_P$ is a unique tangent line to $V$ passing through $P$.*
   - (e) *$\gamma(L_P \setminus V) = \gamma(P)$, and $\gamma^{-1}(Q) = \{P \in V_3 \setminus V_2 | Q \in L_P\} = T_Q V \setminus V_2$ for any $Q \in V$, where $T_Q V$ is the embedded tangent space to $V$ at $Q$.*

**Corollary A1.** *The three conditions, $V = \emptyset$, $V_3 = \mathbb{P}$ and $V_2 = \mathbb{P}$ are equivalent to each other.*

*Remark A.* It can be shown that $V = \emptyset$ if and only if the Lie algebra $\mathfrak{g}$ is of type $C$ (see Appendix): In fact, using a theorem of Asano [30, 1.6.Theorem], [4], one can show that if $q \equiv 0$, then $\mathfrak{g} \simeq \mathfrak{sp}_{2n+2}$, where $\dim \mathfrak{g}_1 = 2n$; The converse is checked by an explicit computation.

Recall that a projective variety $V \subseteq \mathbb{P}$ is called a *variety with one apparent double point* if for a general point $P \in \mathbb{P}$ there exists a unique secant line of $V$ passing through $P$ (see [25, IX]).

**Corollary A2.** *If $V \neq \emptyset$, then $V$ is a variety with one apparent double point. In particular, $V$ is non-degenerate in $\mathbb{P}$.*

## 4. The Homogeneity

**Theorem B.** *Let $G_{00}$ be the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_{00}$, where $\mathfrak{g}_{00}$ is the subalgebra of $\mathfrak{g}_0$ defined by $\mathfrak{g}_{00} := \mathrm{Ker}(\mathrm{ad}\, E_\pm|\mathfrak{g}_0)$. Then we have:*

(1) *$G_{00}$ acts transitively on each of irreducible components of $\mathcal{V}$. In particular, we have $t_x\mathcal{V} = \mathfrak{g}_{00}x$ for any $x \in \mathcal{V}$, where $t_x\mathcal{V}$ is the Zariski tangent space to $\mathcal{V}$ at $x$.*

(2) *$\mathfrak{g}_{00}x = (\mathfrak{g}_{00}x)^\perp$ with $2\dim \mathfrak{g}_{00}x = \dim \mathfrak{g}_1$ for any $x \in \mathcal{V}$, and $\mathfrak{g}_1 = \mathfrak{g}_{00}x \oplus \mathfrak{g}_{00}y$ for any $x, y \in \mathcal{V}$ with $\langle x, y\rangle \neq 0$.*

Recall that the *tangent variety* of $V$, denoted by $\mathrm{Tan}\, V$, is the union of embedded tangent spaces to $V$, and the *projective dual* of $V$, denoted by $V^*$, is the set of hyperplanes tangent to $V$ (see, for example, [11, §3]).

**Corollary B1.** *Assume that $V \neq \emptyset$. Then we have:*

(1) *$G_{00}$ acts transitively on each of irreducible components of $V$, and $V$ is smooth, equidimensional of dimension $n-1$, where $\dim \mathfrak{g}_1 = 2n$.*

(2) *Denote by $L^*$ the set of hyperplanes containing a linear subspace $L \subseteq \mathbb{P}$. Then we have $(T_QV)^* = T_QV$ for any $Q \in V$, hence*
$$\mathrm{Tan}\, V = V^*,$$
*where we identify $\mathbb{P}$ with its dual space $\mathbb{P}^\vee := \mathbb{P}(\mathfrak{g}_1^*)$ via the symplectic form $\langle , \rangle$.*

**Corollary B2.** *If $V$ is neither empty nor irreducible, then there are irreducible $\mathfrak{g}_{00}$-modules $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of dimension $n$ such that $\mathfrak{g}_1 = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, and we have*
$$V = \mathbb{P}(\mathfrak{s}_1) \sqcup \mathbb{P}(\mathfrak{s}_2),$$
*where $\dim \mathfrak{g}_1 = 2n$.*

*Remark B1.* It is known that $V$ is irreducible unless $\mathfrak{g}$ is of type $A$ or $C$ (see Appendix 1): In fact, if $\mathfrak{g} = \mathfrak{so}_m$, then $V$ is a Segre embedding of $\mathbb{P}^1 \times Q$ in $\mathbb{P}^{2m-9}$, where $Q$ is a quadric hypersurface in $\mathbb{P}^{m-5}$; if $\mathfrak{g}$ is of type $G_2$, then $V$ is a cubic Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$; for other exceptional Lie algebras $\mathfrak{g}$, see Introduction. Conversely, it follows from a direct computation that we are in the case above if $\mathfrak{g} = \mathfrak{sl}_{n+2}$ with $n \geq 1$.

**Corollary B3.** *If $V \neq \emptyset$ and $V_2 \neq V$, then $V$ is ruled, that is, covered by lines contained in $V$.*

*Remark B2.* It can be shown that $V = V_2$ if and only if $\mathfrak{g}$ is of type $G_2$.

## 5. Isotropic Secants

**Proposition C.** *For $P = \pi(u) \in \mathbb{P}$, let $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$ be a rational map induced from $L(u, u)$ with base locus $B_P = \mathbb{P}(\mathrm{Ker}\, L(u, u))$. If $V$ is irreducible and $P \in V_2 \setminus V$, then $\dim \Phi_P(V \setminus B_P) \geq 1$, hence $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ and $\mathrm{codim}\, B_P \geq 2$.*

*Remark C1.* The irreducibility condition for $V$ is essential in Proposition C: In fact, there is an example of $u$ satisfying the assumption above such that $\mathrm{rk}\, L(u, u) = 1$ in case of $\mathfrak{g} = \mathfrak{sl}_m$, where $V$ is not irreducible (see Remark B3).

*Remark C2.* It follows easily from Proposition 6 that $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ if $P \notin V_2$, and $\mathrm{codim}\, \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ if $P \in V_3$, though we do not use these facts in this article.

Recall that the *secant locus* $\Sigma_P$ as well as the *tangent locus* $\Theta_P$ of $V$ with respect to a given point $P \in \mathbb{P}$ are defined by
$$\Sigma_P^\circ := \{Q \in V | \exists R \in V \setminus \{Q\}, P \in Q * R\}, \quad \Sigma_P := \overline{\Sigma_P^\circ},$$
$$\Theta_P := \{Q \in V | P \in T_QV\},$$
where we denote by $Q * R$ the line in $\mathbb{P}$ joining $Q$ and $R$, and by $T_QV$ the embedded tangent space to $V$ at $Q$ in $\mathbb{P}$ (see, for example, [11]).

**Theorem C.** *Assume that $V$ is irreducible. Then we have:*

(1) *For any $x, y \in \mathcal{V}$, $\langle x, y\rangle = 0$ if and only if $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y \neq 0$. In particular, a secant line joining $Q, R \in V$ is isotropic with respect to the symplectic form if and only if $T_QV \cap T_RV \neq \emptyset$.*
(2) *$V_2 \setminus V$ is covered by isotropic secants of $V$. More precisely, for any $u \in \mathfrak{g}_1$, we have that $[uuu] = 0$ and $u \times u \neq 0$ if and only if $u = x + y$ for some $x, y \in \mathcal{V}$ such that $\langle x, y\rangle = 0$ and $x \times y \neq 0$.*
(3) *If $P \in V_2 \setminus V$, then $\Phi_P(V \setminus B_P) \subseteq \Sigma_P$ and $\Phi_P(V \cap P^\perp \setminus B_P) \subseteq \Theta_P$, where $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$ is the rational map induced from $L(u, u)$ with base locus $B_P = \mathbb{P}(\operatorname{Ker} L(u, u))$ and $P^\perp = \mathbb{P}(u^\perp)$ with $P = \pi(u)$.*
(4) *We have $\dim \Sigma_P \geq 1$ for any $P \in V_2 \setminus V$.*

*Remark C3.* The irreducibility condition for $V$ is essential in (1) above: In fact, it is easily seen that the conclusion does not hold in case of $\mathfrak{g} = \mathfrak{sl}_m$.

**Corollary C.** *If $V$ is irreducible, then $V_3 = \operatorname{Tan} V$.*

## 6. Double Projections

**Proposition D.** *For any $x, y \in \mathcal{V}$, let $\Psi_{xy} : \mathfrak{g}_1 \to \mathfrak{g}_1$ be a linear map defined by $\Psi_{xy}(a) := [axy] + \langle a, x\rangle y$. Then:*

(1) *If $\langle x, y\rangle \neq 0$, then $\operatorname{Ker} \Psi_{xy} = \mathfrak{g}_{00}x$ and $\Psi_{xy}(\mathfrak{g}_1) = \mathfrak{g}_{00}y$. In particular, a rational map $\Psi_{PQ} : \mathbb{P} \dashrightarrow \mathbb{P}$ induced from $\Psi_{xy}$ is a double projection from $P$ with image $T_QV$, that is, a projection with center $T_PV$ onto $T_QV$, hence defines a morphism*
$$\Psi_{PQ} : \mathbb{P} \setminus T_PV \to T_QV,$$
*where $T_PV$ is the embedded tangent space to $V$ at $P$ with $P = \pi(x)$ and $Q = \pi(y)$.*
(2) *Moreover for any $R \in V$, the four points $R$, $[PQR]$, $\Psi_{PR}(Q)$ and $\Psi_{QR}(P)$ are collinear, and $[PQR]$ is the harmonic conjugate of $R$ with respect to $\Psi_{PR}(Q)$ and $\Psi_{QR}(P)$, where we set $[PRR] := \pi([xyz])$ with $R = \pi(z)$. In particular, this holds for general $P, Q, R \in V$ and gives a geometric meaning of our ternary product.*

*Remark D1.* In terms of the Lie bracket, we have $\Psi_{ab}(c) = [b[a[c, E_-]]]$.

**Theorem D.** *For any $P, Q \in V$, if the secant line joining $P$ and $Q$ is not isotropic, that is, $T_PV \cap T_QV = \emptyset$, then we have:*

(1) *$V \setminus P^\perp = (\Psi_{PQ}|_{V \setminus T_PV})^{-1}(T_QV \setminus P^\perp)$.*
(2) *The double projection $\Psi_{PQ}$ gives an isomorphism $V \setminus P^\perp \to T_QV \setminus P^\perp$. In fact, a rational map $\Gamma_{QP} : T_QV \dashrightarrow V$ induced from a map $\Gamma_{yx} : \mathfrak{g}_{00}y \to \mathcal{V} \cup \{0\}$ defined by*
$$\Gamma_{yx}(t) := \langle x, [ttt]\rangle x + 3\langle x, t\rangle [ttx] + 12\langle x, t\rangle^2 t$$
*gives the inverse of $\Psi_{PQ}|_{V \setminus P^\perp}$, where $P = \pi(x)$ and $Q = \pi(y)$.*
(3) *The base locus of $\Gamma_{QP}$ is $T_QV \cap P^\perp \cap V_2$.*

*In particular, if $V$ is irreducible, then $\Psi_{PQ}$ gives a birational map from $V$ to $T_QV$, and $V$ is the closure of the image of a composition of a cubic Veronese embedding of the affine space $T_QV \setminus P^\perp$ with some projection to $\mathbb{P}$.*

*Remark D2.* The morphism $\Psi_{PQ} : V \setminus T_PV \to T_QV$ is not necessarily surjective: In fact, if $\mathfrak{g}$ is of type $G_2$, then for any $P \in V$, $P^\perp$ is the osculating plane to the twisted cubic $V \subseteq \mathbb{P}^3$ at $P$, $V \cap P^\perp = \{P\}$, and $\Psi_{PQ}(V \setminus T_PV) = T_QV \setminus P^\perp$ for any $Q \in V$ with $P \neq Q$.

*Remark D3.* We have proved in the above that $\Psi_{xy} : \mathcal{V} \setminus x^\perp \to \mathfrak{g}_{00}y \setminus x^\perp$ is an isomorphism.

*Remark D4.* We here give another expression of the inverse map of the double projection $\Psi_{PQ}$. We first note that there is an isomorphism of affine spaces,

$$\iota : \mathfrak{g}_{00}y \cap x^{\perp} \to T_QV \setminus P^{\perp}$$

defined by $\iota(a) := \pi(a+y)$. Indeed, the inverse is given by $\iota^{-1}(\pi(t)) := \frac{\langle x,y\rangle}{\langle x,t\rangle}t - y$ for $\pi(t) \in T_QV \setminus P^{\perp}$, where $T_QV = \mathbb{P}(\mathfrak{g}_{00}y)$ and $P^{\perp} = \mathbb{P}(x^{\perp})$. Now let $\rho : \mathfrak{g}_{00}y \cap x^{\perp} \to V$ be the composition of $\iota$ with the rational map $\Gamma_{QP} : T_QV \dashrightarrow V$ in Theorem D (2). Then $\rho$ is the inverse of $\Psi_{PQ}$ via $\iota$, and it follows from part (1) and (4) of Proposition 3 that

$$\rho(a) = \pi\left(\frac{\langle x, [aaa]\rangle}{12\langle x,y\rangle^2}x + \frac{1}{4\langle x,y\rangle}[aax] + a + y\right).$$

In particular, the Freudenthal variety $V$ is equal to the closure of the image of the affine space $\mathfrak{g}_{00}y \cap x^{\perp}$ under the cubic Veronese embedding $\rho$.

## 7. Twisted Cubic Curves

**Proposition E.** *For any $P \in V_3 \setminus V_2$ and $Q \in V$, if the secant line joining $Q$ and the contact point $\gamma(P)$ of $L_P$ is not isotropic, then we have:*

(1) *$Q \in L_{\Phi_P(Q)}$ and $\Phi_P^3(Q) = \gamma(P) \in L_P = L_{\Phi_P^2(Q)}$ with $\Phi_P(Q), \Phi_P^2(Q) \in V_3 \setminus V_2$.*

(2) *$L_P \cap L_{\Phi_P(Q)} = \emptyset$, hence $Q, \Phi_P(Q), \Phi_P^2(Q)$ and $\Phi_P^3(Q)$ are linearly independent in $\mathbb{P}$.*

**Theorem E.** *For any $P \in V_3 \setminus V_2$ and $Q \in V$ such that the secant line joining $Q$ and the contact point $\gamma(P)$ of $L_P$ is not isotropic, that is, $T_QV \cap T_{\gamma(P)}V = \emptyset$, let $\mathbb{P}_{PQ}$ be the linear subspace of dimension 3 in $\mathbb{P}$ spanned by $Q$, $\Phi_P(Q)$, $\Phi_P^2(Q)$ (or equivalently $P$) and $\Phi_P^3(Q) = \gamma(P)$, that is, spanned by $L_P$ and $L_{\Phi_P(Q)}$, the unique tangent lines to $V$ passing through $P$ and $\Phi_P(Q)$. Then we have:*

(1) *The intersection $V \cap \mathbb{P}_{PQ}$ is a twisted cubic curve in $\mathbb{P}_{PQ} \simeq \mathbb{P}^3$ given explicitly by the image of $L_P$ under the cubic map $\Gamma_{\gamma(P)Q}$:*

$$V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P).$$

(2) *The twisted cubic curve in $\mathbb{P}_{PQ}$ above has the following properties:*
   (a) *$L_P$ and $L_{\Phi_P(Q)}$ are respectively the tangent lines at $\gamma(P)$ and at $Q$, and*
   (b) *$\gamma(P)^{\perp} \cap \mathbb{P}_{PQ}$ and $Q^{\perp} \cap \mathbb{P}_{PQ}$ are respectively the osculating planes at $\gamma(P)$ and at $Q$, which are spanned by $L_P$ and $\Phi_P(Q)$ and by $L_{\Phi_P(Q)}$ and $\Phi_P^2(Q)$, respectively.*

*Remark E1.* The morphism $\Gamma_{\gamma(P)Q} : L_P \to \mathbb{P}_{PQ}$ is given by $(\lambda : \mu) \mapsto (2\lambda^3 : 6\lambda^2\mu : 9\lambda\mu^2 : 9\mu^3)$ in terms of homogeneous coordinate with respect to the basis $\{D^2x, D^3x\}$ for $L_P$ and $\{x, Dx, D^2x, D^3x\}$ for $\mathbb{P}_{PQ}$.

*Remark E2.* Set $E := L(Dx, Dx)$, $F := [D, E]$ with $D := L(t,t)$ as in the above, and denote by $\mathfrak{g}_{00PQ}$ the subalgebra of $\mathfrak{g}_{00}$ generated by $D$, $E$ and $F$. Then it follows that

$$[F, D] = \frac{4}{3}\langle D^3x, x\rangle D, \quad [F, E] = -\frac{4}{3}\langle D^3x, x\rangle E,$$

so that $\mathfrak{g}_{00PQ}$ is isomorphic to the Lie algebra $\mathfrak{sl}_2$. If we denote by $\mathfrak{g}_{1PQ}$ the subspace of $\mathfrak{g}_1$ spanned by $x$, $Dx$, $D^2x$ and $D^3x$, then we see that $\mathfrak{g}_{1PQ}$ is an irreducible $\mathfrak{g}_{00PQ}$-module of dimension 4 with

$$F(D^kx) = (2k-3)\frac{2}{3}\langle D^3x, x\rangle D^kx,$$

and the twisted cubic curve $V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P)$ is a unique closed orbit in $\mathbb{P}_{PQ} = \mathbb{P}(\mathfrak{g}_{1PQ})$ under the natural action of the group of inner automorphisms of $\mathfrak{g}_{00}$ with Lie algebra $\mathfrak{g}_{00PQ}$.

Thus, for any $P \in V_3 \setminus V_2$ and $Q \in V$ with $T_{\gamma(P)}V \cap T_QV = \emptyset$, a subalgebra $\mathfrak{g}_{00PQ}$ of $\mathfrak{g}_{00}$ isomorphic to $\mathfrak{sl}_2$ and an irreducible $\mathfrak{g}_{00PQ}$-submodule $\mathfrak{g}_{1PQ}$ of $\mathfrak{g}_1$ with dimension 4 are associated to $P$ and $Q$. If $\mathfrak{g}$ is of type $G_2$, then $\mathfrak{g}_{00PQ}$ and $\mathfrak{g}_{1PQ}$ are respectively equal to $\mathfrak{g}_{00}$ and $\mathfrak{g}_1$ themselves.

## Appendix 1. A Classification of Freudenthal Varieties

We here give a classification of Freudenthal varieties $V$ in terms of the root data of $\mathfrak{g}$. It would be interesting to compare $V$ with the adjoint variety associated to $\mathfrak{g}$ since those varieties are closely related to each other: In fact, for a simple graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_i$ of contact type, denote by $V$ the Freudenthal variety associated to $\mathfrak{g}$, as before, and denote by $X$ the orbit of the inner automorphism group of $\mathfrak{g}$ through $\pi(E_+)$ in $\mathbb{P}(\mathfrak{g})$, which is the minimal closed orbit in $\mathbb{P}(\mathfrak{g})$, called the *adjoint variety* associated to $\mathfrak{g}$ (see [16]). Then, according to [17, Theorem B], we have $V = X \cap \mathbb{P}(\mathfrak{g}_1)$.

### Adjoint Varieties and Freudenthal Varieties

| $\mathfrak{g}$ | $X \subseteq \mathbb{P}(\mathfrak{g})$ | $\mathfrak{g}_{00}$ | $V \subseteq \mathbb{P}(\mathfrak{g}_1)$ |
|---|---|---|---|
| $\mathfrak{sl}_m$ | $(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}) \cap (1) \subseteq \mathbb{P}^{m^2-2}$ | $\mathfrak{gl}_1 \oplus \mathfrak{sl}_{m-2}$ | $\mathbb{P}^{m-3} \sqcup \mathbb{P}^{m-3} \subseteq \mathbb{P}^{2m-5}$ |
| $\mathfrak{so}_m$ | $\mathbb{G}_{\text{orthog.}}(2, m) \subseteq \mathbb{P}^{\binom{m}{2}-1}$ | $\mathfrak{sl}_2 \oplus \mathfrak{so}_{m-4}$ | $\mathbb{P}^1 \times Q^{m-6} \subseteq \mathbb{P}^{2m-9}$ |
| $\mathfrak{sp}_{2m}$ | $v_2\mathbb{P}^{2m-1} \subseteq \mathbb{P}^{\binom{2m+1}{2}-1}$ | $\mathfrak{sp}_{2m-2}$ | $\emptyset \subseteq \mathbb{P}^{2m-3}$ |
| $\mathfrak{e}_6$ | $E_6(\omega_2)^{21} \subseteq \mathbb{P}^{77}$ | $\mathfrak{sl}_6$ | $\mathbb{G}(3, 6) \subseteq \mathbb{P}^{19}$ |
| $\mathfrak{e}_7$ | $E_7(\omega_1)^{33} \subseteq \mathbb{P}^{132}$ | $\mathfrak{so}_{12}$ | $S_5 = \mathbb{G}_{\text{orthog.}}(6, 12) \subseteq \mathbb{P}^{2^5-1}$ |
| $\mathfrak{e}_8$ | $E_8(\omega_8)^{57} \subseteq \mathbb{P}^{247}$ | $\mathfrak{e}_7$ | $E_7(\omega_6) \subseteq \mathbb{P}^{55}$ |
| $\mathfrak{f}_4$ | $F_4(\omega_1)^{15} \subseteq \mathbb{P}^{51}$ | $\mathfrak{sp}_6$ | $\mathbb{G}_{\text{sympl.}}(3, 6) \subseteq \mathbb{P}^{13}$ |
| $\mathfrak{g}_2$ | $G_2(\omega_2)^5 \subseteq \mathbb{P}^{13}$ | $\mathfrak{sl}_2$ | $v_3\mathbb{P}^1 \subseteq \mathbb{P}^3$ |

*Notation*: We denote by $\cap(1)$ cutting by a general hyperplane, and by $v_d$ the Veronese embedding of degree $d$. We denote by $\mathbb{G}(r, m)$ a Grassmann variety of $r$-planes in $\mathbb{C}^m$, and denote by $\mathbb{G}_{\text{orthog.}}(r, m)$ and by $\mathbb{G}_{\text{symp.}}(r, m)$ respectively an orthogonal and a symplectic Grassmann varieties of isotropic $r$-planes in $\mathbb{C}^m$. A simple exceptional Lie algebra of Dynkin type $G$ is denoted by the lowercase of $G$ in the German character, as in [12], a simple algebraic group of type $G$ is denoted by just $G$, and for a dominant integral weight $\omega$ of $G$, the minimal closed orbit of $G$ in $\mathbb{P}(V_\omega)$ is denoted by $G(\omega)$, where $V_\omega$ is the irreducible representation space of $G$ with highest weight $\omega$: For example, $\mathfrak{g}_2$ in the list is the simple Lie algebra of type $G_2$, and $G_2(\omega_2)$ is the minimal closed orbit of an algebraic group of type $G_2$ in $\mathbb{P}(V_{\omega_2})$, where $\omega_2$ is the second fundamental dominant weight with the standard notation of Bourbaki [6].

## Appendix 2. The Filtration of The Ambient Space

- $\mathfrak{e}_{6,7,8}, \mathfrak{f}_4$
  - $\mathbb{P}$
  - |
  - $V_3 = \operatorname{Tan} V$
  - |
  - $V_2 = \operatorname{Sing} V_3$
  - |
  - $V$
  - |
  - $\emptyset$
- $\mathfrak{g}_2$
  - $\mathbb{P}^3$
  - |
  - $V_3 = \operatorname{Tan} V$
  - |
  - $(V_2)_{\text{red}} = \operatorname{Sing} V_3 = V = v_3\mathbb{P}^1$
  - |
  - $\emptyset$

- $\mathfrak{sl}_{m\geq 3}$

$$\begin{array}{ll}
\circ & \mathbb{P}^{2m-5} \\
| & \\
\circ & (V_3)_{\mathrm{red}} = V_2 = Z(\sum x_i y_i) \\
| & \\
\circ & V = \mathbb{P}^{m-3} \sqcup \mathbb{P}^{m-3} = Z(x_0, \dots, x_{m-3}) \sqcup Z(y_0, \dots, y_{m-3}) \\
| & \\
\circ & \emptyset
\end{array}$$

- $\mathfrak{sp}_{2m}$

$$\begin{array}{ll}
\circ & \mathbb{P}^{2m-3} = V_3 = V_2 \\
| & \\
\circ & V = \emptyset
\end{array}$$

- $\mathfrak{so}_8$

$$\mathfrak{g}_{00} = \mathfrak{so}_4 \oplus \mathfrak{sl}_2 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$$
$$\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^4 \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = M_{2,2,2}$$

$$\begin{array}{ccccl}
 & & \circ & & \mathbb{P}^7 \\
 & & | & & \\
 & & \circ & & V_3 = \operatorname{Tan} V = Z(\text{hyper-determinant for } M_{2,2,2}) \\
 & / & | & \backslash & \\
\big(\mathbb{P}^3 \times \mathbb{P}^1 \ \circ & & \circ & & \circ \ \mathbb{P}^1 \times \mathbb{P}^3\big) \cdots\cdots V_2 = \operatorname{Sing} V_3 \\
 & \backslash & | & / & \\
 & & \circ & & V = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\
 & & | & & \\
 & & \circ & & \emptyset
\end{array}$$

- $\mathfrak{so}_{m\geq 9}$

$$\begin{array}{ll}
\mathfrak{g}_{00} & = \mathfrak{sl}_2 \oplus \mathfrak{so}_{m-4} \\
\mathfrak{g}_1 & = \mathbb{C}^2 \otimes \mathbb{C}^{m-4} = M_{m-4,2} \\
V_{20} & = \pi(\{X \in M_{m-4,2} |\operatorname{rk} X = 2, {}^{\tau}XX = 0\}) \\
 & = \{[a \otimes c + b \otimes d] | [a] * [b] = \mathbb{P}^1, [c] * [d] \subseteq Q\} \\
 & : \dim 2m - 12
\end{array}$$

$$\begin{array}{ccccl}
 & & \circ & & \mathbb{P}^{2m-9} \\
 & & | & & \\
 & & \circ & & V_3 = \operatorname{Tan} V \\
 & / & & \backslash & \\
(V_{20} \ \circ & & & & \circ \ V_{11}) \cdots V_2 = \operatorname{Sing} V_3 \\
 & \backslash & & / & \\
 & & \circ & & V = \mathbb{P}^1 \times Q^{m-6} \\
 & & | & & \\
 & & \circ & & \emptyset
\end{array}$$

$$\begin{array}{ll}
V_{11} & = \pi(\{X \in M_{m-4,2} |\operatorname{rk} X = 1, \operatorname{rk} {}^{\tau}XX = 1\}) \\
 & = \mathbb{P}^1 \times \mathbb{P}^{m-5}
\end{array}$$

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