GAUSS MAPS OF RANK ZERO

HAJIME KAJI

0. Introduction

Let X be a projective variety of dimension n in $\mathbb{P}^N$ defined over an algebraically closed field K of characteristic $p \geq 0$. The *Gauss map* of $X \subseteq \mathbb{P}^N$, denoted by γ, is by definition the rational map from X to the Grassmann variety $\mathbb{G}(n, \mathbb{P}^N)$ which sends each smooth point x of X to the embedded tangent space T_xX to X at x in $\mathbb{P}^N$ ([13, §1, (e)], [32, I, §2]). To avoid trivial exceptions we treat γ only for a non-linear $X \subseteq \mathbb{P}^N$. According to a theorem of F. L. Zak [32, I, 2.8. Corollary], γ is finite for a smooth X, and it is well known that a general fibre of γ is linear if $p = 0$ ([13, (2.10)], [32, I, 2.3. Theorem]); hence γ is birational for a smooth X in $p = 0$.

In this article we introduce an intrinsic property of a projective variety X as follows:

(GMRZ) *there exists an embedding ι of X into some $\mathbb{P}^M$ such that the Gauss map γ is of rank zero.*

Here the *rank* of a rational map is defined to be the rank of its differential at a general point, and the differential of a rational map is by definition the induced K-linear map between Zariski tangent spaces. Note that a variety X satisfies (GMRZ) only if $p > 0$, since the rank of a rational map is equal to the dimension of its image if $p = 0$.

The theory of rational curves on projective varieties was initiated by an epoch-making work [27] of S. Mori about 30 years ago, settling the Hartshorne conjecture on characterisation of projective spaces in the affirmative. A central and significant notion there has been a "minimal free rational curve." Here, a rational curve (or a morphism) $f : \mathbb{P}^1 \to X$ is said to be *free* if the pull-back f^*T_X of the tangent bundle T_X on X is generated by its global sections ([9, p. 85], [25, II.3.1]), and a free f *minimal* if f^*T_X is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{d-2} \oplus \mathcal{O}_{\mathbb{P}^1}^{n-d+1}$ with $d = \deg(-f^*K_X)$ ([9, p. 93], [25, IV.2.8]; it is addressed as *standard* in [17]). In fact, a family of minimal free rational curves has been employed essentially in various situations, for instance, characterisations of projective spaces and quadric hypersurfaces ([1], [4], [7], [8], [26]), studies of Fano varieties ([2], [3], [28]), theories of varieties of minimal rational tangents ([17], [18], [19], [20], [23]), and so on. Those beautiful works are all established on the existence of a family of minimal free rational curves.

One of the most basic results in characteristic zero case to guarantee that existence is

Theorem A ([25, IV.2.10])**.** *Let X be a smooth projective variety in $p = 0$. If there exists a free rational curve on X, then there exists a minimal free rational curve on X.*

Note that for a smooth X in arbitrary characteristic $p \geq 0$, the existence of free rational curves is equivalent to the separable uniruledness ([25, IV.1.9]).

2000 *Mathematics Subject Classification.* 14N05 (14C05, 14J45, 14J70).

Key words and phrases. Gauss map, inseparable, minimal free rational curve, normal bundle, hypersurface.

E-mail address: kaji@waseda.jp.

Workshop "2010 Algebra and Geometry of Subvarieties of Projective Space (in honor of Professor Fyodor Zak on his 60th birthday)," KAIST (2010/01/28, Daejeon, KOREA)..

In positive characteristic case, however, the conclusion of Theorem A turns out to fail, as we will see below. The property (GMRZ) imposes strong restrictions on rational curves on algebraic varieties: In fact, first of all we have

Theorem 0.1. *Let X be a projective variety, and assume that X satisfies* (GMRZ) *with an embedding $\iota : X \hookrightarrow \mathbb{P}^M$. Let $f : \mathbb{P}^1 \to X$ be a minimal free rational curve such that X is smooth along $f(\mathbb{P}^1)$, and set $a := \deg f^*\iota^*\mathcal{O}_{\mathbb{P}^M}(1)$. Then one of the following holds:*

1. $\deg(-f^*K_X) = n+1$, $a > p$ *and* $p \mid a-1$.
2. $\deg(-f^*K_X) = p = 2$ *and* $2 \mid a$.

In particular, we have $a > 1$.

Using Theorem 0.1, one can give a counter-example for Theorem A, that is, a projective variety which admits a free rational curve, but no minimal free rational curve, in each characteristic $p > 0$ (Theorem 2.2; Cf. [25, IV.2.10.1]).

Theorem 0.1 is derived basically from the following:

Theorem 0.2. *Let X be a projective variety, let $f : \mathbb{P}^1 \to X$ be an unramified morphism, and denote by N_f the dual of the kernel of the natural homomorphism $f^* : f^*\Omega^1_X \to \Omega^1_{\mathbb{P}^1}$. Assume that X is smooth along $f(\mathbb{P}^1)$ and $N_f^\vee \simeq \bigoplus_{i \geq -1} \mathcal{O}_{\mathbb{P}^1}(i)^{r_i}$ for some non-negative integers r_i $(i \geq -1)$. Then we have:*

1. *If X satisfies* (GMRZ)*, then $r_{i-1}r_i = 0$ for any $i \geq 0$.*
2. *Moreover if $r_{-1} > 0$, then $p \mid \deg f^*\iota^*\mathcal{O}_{\mathbb{P}^M}(1) - 1$ for any embedding $\iota : X \hookrightarrow \mathbb{P}^M$ with Gauss map of rank zero, and if $r_i > 0$ for some $i \geq 0$, then $p = 2$ or $p \mid i+1$.*

Theorem 0.2 is proved by investigating bundles of principal parts (§1). As a consequence of Theorem 0.2, we have

Theorem 0.3.
1. *Let X be a projective variety with a non-constant morphism π to a variety Y, and assume that there exists a smooth point y of Y such that the fibre $X_y := \pi^{-1}(y)$ is isomorphic to a projective space $\mathbb{P}^l$ and π is smooth along X_y. Then X satisfies* (GMRZ) *only if $p = 2$ and $l = 1$. Moreover, a product $\prod_{1 \leq i \leq r} \mathbb{P}^{n_i}$ of two or more projective spaces $(r \geq 2, n_i \geq 1)$ satisfies* (GMRZ) *if and only if $p = 2$ and $n_i = 1$ for any i.*
2. *A Grassmann variety $\mathbb{G}(l, l+m)$ of l-dimensional subspaces of an $(l+m)$-dimensional vector space $(l, m \geq 1)$ satisfies* (GMRZ) *if and only if $l = 1$ or $m = 1$.*
3. *A smooth quadric hypersurface Q in $\mathbb{P}^N$ $(N \geq 3)$ satisfies* (GMRZ) *if and only if $p = 2$ and $N = 3$.*
4. *A smooth cubic hypersurface X in $\mathbb{P}^N$ $(N \geq 3)$ satisfies* (GMRZ) *only if $p = 2$.*

For a higher dimensional cubic hypersurface, we moreover have

Theorem 0.4. *A smooth cubic hypersurface X in $\mathbb{P}^N$ with $N \geq 5$ satisfies* (GMRZ) *if and only if $p = 2$ and X is projectively equivalent to a Fermat hypersurface.*

We will also consider a general hypersurface of low degree with (GMRZ):

Theorem 0.5. *A general hypersurface X in $\mathbb{P}^N$ of degree d with $3 \leq d \leq 2N-3$ satisfies* (GMRZ) *only if $p = 2$ and $d = 2N-3$.*

To obtain Theorems 0.4 and 0.5 above, we need in addition detailed studies on projective geometry on cubic hypersurfaces with Gauss map of rank zero (§4) and on the normal bundles of conics in a hypersurface (§5), respectively: In fact, we establish a characterisation theorem of a Fermat cubic X in $\mathbb{P}^N$, in terms of the Gauss map γ_0 induced from the original embedding $X \subseteq \mathbb{P}^N$ (Theorem 3.2), and we show that the splitting type of $N_{C/X}$ has 'good' bounds for a general hypersurface X and for a general conic C in X (Corollary 4.3).

This paper is a brief summary of a joint work [10] with Satoru Fukasawa and Katsuhisa Furukawa.

1. Bundles of principal parts

For a line bundle $\mathcal{L}$ on a projective variety X, we denote by $\mathcal{P}^1_X(\mathcal{L})$ the bundle of principal parts of $\mathcal{L}$ of first order ([14, §16], [30, §2]), which is equipped with a natural exact sequence,

$$0 \to \Omega^1_X \otimes \mathcal{L} \to \mathcal{P}^1_X(\mathcal{L}) \to \mathcal{L} \to 0 \qquad (\xi).$$

A generically surjective homomorphism $a^1 : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \otimes \mathcal{O}_X \to \mathcal{P}^1_X(\mathcal{O}_X(1))$ is associated to a projective variety X in $\mathbb{P}^N$. The Gauss map γ of X is formally defined to be the rational map $X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$ associated with a^1 by the universality of $\mathbb{G}(n, \mathbb{P}^N)$, where $n := \dim X$.

If a vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a_1)^{r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{r_m}$, then $[a_1^{r_1}, \dots, a_m^{r_m}]$ is called the *splitting type* of $\mathcal{E}$. Note that, according to a theorem of A. Grothendieck ([15, V, Exercise 2.6]), every vector bundle on $\mathbb{P}^1$ splits into a direct sum of line bundles, as above. By abuse of notation, a vector bundle of splitting type $[a_1^{r_1}, \dots, a_m^{r_m}]$ is denoted by the same symbol, for simplicity.

Proposition 1.1. *Let X be a projective variety, let $f : \mathbb{P}^1 \to X$ be an unramified morphism, and denote by N_f the dual of the kernel of the natural homomorphism $f^* : f^*\Omega^1_X \to \Omega^1_{\mathbb{P}^1}$. Assume that X is smooth along $f(\mathbb{P}^1)$, and $N_f^\vee = [-1^{r_{-1}}, 0^{r_0}, \dots, i^{r_i}, \dots]$. Then for an embedding $\iota : X \hookrightarrow \mathbb{P}^M$, we have*

$$f^*\mathcal{P}^1_X(\iota^*\mathcal{O}_{\mathbb{P}^M}(1)) = \begin{cases} [a-2, a-1^{r_{-1}}, a^{r_0+1}, a+1^{r_1}, a+2^{r_2}, \dots, a+i^{r_i}, \dots], & \text{if } p|a, \\ [a-1^{r_{-1}+2}, a^{r_0}, a+1^{r_1}, a+2^{r_2}, \dots, a+i^{r_i}, \dots], & \text{otherwise,} \end{cases}$$

where $a := \deg f^*\iota^*\mathcal{O}_{\mathbb{P}^M}(1)$.

Proposition 1.2. *Let X be a projective variety, let $f : \mathbb{P}^1 \to X$ be a morphism, and assume that X is smooth along $f(\mathbb{P}^1)$. If X satisfies* (GMRZ)*, then the splitting type of $f^*\mathcal{P}^1_X(\iota^*\mathcal{O}_{\mathbb{P}^M}(1))$ is divisible by p.*

Proof of Theorem 0.2. According to Proposition 1.1, if both r_{-1} and r_0 were positive, then $a-1$ and a would be divisible by p by Proposition 1.2. If both r_0 and r_1 were positive, then a and $a+1$ would be divisible by p. Similarly for any $i \geq 2$, if both r_{i-1} and r_i were positive, then $a+i-1$ and $a+i$ would be divisible by p. Anyway this is a contradiction. Moreover, using Propositions 1.1 and 1.2, we see that if $r_{-1} > 0$, then $p|a-1$. If $r_0 > 0$, then $p|a-2$ and $p|a$; hence $p = 2$. Furthermore we see that $r_i > 0$ implies $p|i+1$ for any odd $i \geq 1$, and that $r_i > 0$ implies $p = 2$ or $p|i+1$ for any even $i \geq 2$. This completes the proof. □

Example 1.3. Let X be an n-fold product $(\mathbb{P}^1)^n$ of $\mathbb{P}^1$ in $p = 2$, set

$$I_k := \{(a_1, \dots, a_n) \in \{0, 1, 2\}^n | \#\{j | a_j = 1\} = k\},$$

and let $\iota : X \dashrightarrow \mathbb{P}^M$ be a rational map defined by

$$(1 : y_1) \times \cdots \times (1 : y_n) \mapsto (y_1^{a_1} \cdots y_n^{a_n})_{(a_1, \dots, a_n) \in I_0 \cup I_1},$$

where $M + 1 = 2^{n-1}(n+2)$. Then by a direct computation as in [11, Proof of Proposition] one can verify that ι gives an embedding of X with Gauss map of rank zero; hence $(\mathbb{P}^1)^n$ in $p = 2$ satisfies (GMRZ).

Proof of Theorem 0.3. Each only-if-part of (1–4) follows from the splitting type of the normal bundles of a projective line L in X and Theorem 0.2, where we note that every X in question contains a projective line L. The if-parts of (1) and (3) follow from Example 1.3, and that of (2) follows from [12, Example 3.1]. $\square$

2. Absence of minimal free rational curves

Proof of Theorem 0.1. It follows from [25, IV, 2.11] that a minimal free f is unramified. Theorem 0.2 implies $N_f^\vee = [-1^{n-1}]$ or $[0^{n-1}]$.

Suppose $N_f^\vee = [-1^{n-1}]$. Then we have $\deg(-f^*K_X) = n+1$, and it follows from Theorem 0.2 that $p \mid a-1$. We show $a \neq 1$ as follows: Assume $a = 1$, and identify X with $\iota(X) \subseteq \mathbb{P}^M$. Then $L := f(\mathbb{P}^1)$ is a line in $\mathbb{P}^M$. We fix a point $x = f(o) \in L$ with $o \in \mathbb{P}^1$, where x is a smooth point of X. Since $h^1((f^*T_X)(-1)) = 0$, it follows from [25, II, 1.7] that $\operatorname{Hom}(\mathbb{P}^1, X; o \mapsto x)$ is smooth at f. For an irreducible component $V \subseteq \operatorname{Hom}(\mathbb{P}^1, X; o \mapsto x)$ containing f, we consider the evaluation morphism $F : \mathbb{P}^1 \times V \to X$. Since $f^*T_X = [2, 1^{n-1}]$, it follows from [25, II, 3.10] that $\operatorname{rk} d_{(o,f)}F = n$; hence F is dominant. On the other hand, setting $E := F^*\mathcal{O}_{\mathbb{P}^M}(1)$, we see from [25, II. 3.9.2] that the image of a morphism $g \in V$ is a line in X passing through x, which implies that X is a cone with vertex x. Since X is non-linear by our convention, X is singular at x. Thus we reach a contradiction.

If $N_f^\vee = [0^{n-1}]$, then we have $\deg(-f^*K_X) = 2$; hence it follows from Proposition 1.1 that $p = 2$ and $p \mid a$. $\square$

Remark 2.1. Both cases (1–2) in Theorem 0.1 actually occur:

(1) According to [12, Example 3.1], $\mathbb{P}^n$ satisfies (GMRZ), and we have $T_{\mathbb{P}^n}|_L = [1^{n-1}, 2]$ for each line $L \subset \mathbb{P}^n$.

(2) Let $X = (\mathbb{P}^1)^n$ with $p = 2$, which satisfies (GMRZ) by Example 1.3. Let us consider an embedding $f : \mathbb{P}^1 \to X$ such that $f(\mathbb{P}^1)$ is a product of $\mathbb{P}^1$ and a point in $(\mathbb{P}^1)^{n-1}$. Then f is minimal free with $f^*T_X = [0^{n-1}, 2]$.

Theorem 2.2. *Assume $p > 0$, and let X be a Fermat hypersurface of degree $ep+1$ in $\mathbb{P}^N$ with $e \in \mathbb{N}$. Then X satisfies* (GMRZ), *and we have:*

(1) *X has no minimal free line, or equivalently, no free line.*

(2) *If $N > e(p+1)$, then X has no minimal free rational curve.*

(3) *If $N \geq 2ep+1$, then X has a free $f : \mathbb{P}^1 \to X$ with $\deg f^*\mathcal{O}_X(1) = ep$.*

Thus a Fermat hypersurface $X \subseteq \mathbb{P}^N$ of degree $ep+1$ with $N \geq 2ep+1$ gives a counter-example for Theorem A in each characteristic $p > 0$.

Remark 2.3. Let X be a Fermat hypersurface of degree $p^r + 1$ in $\mathbb{P}^N$. It follows from [9, pp. 50–51] that $N_{L/X} = [1-p^r, 1^{N-3}]$ for each line $L \subseteq X$, from which one can deduce that Theorem 2.2 (1) holds for this X.

3. Characterisation of a cubic hypersurface with (GMRZ)

Let X be a smooth cubic hypersurface in $\mathbb{P}^N$ with $N \geq 3$. We denote the Gauss map of $X \subseteq \mathbb{P}^N$ by $\gamma_0 : X \to \mathbb{G}(N-1, \mathbb{P}^N) = \check{\mathbb{P}}^N$.

Proposition 3.1. *We assume that $N \geq 5$, $p = 2$ and $N_{L/X}^\vee = [-1^{N-3}, 1]$ for any projective line $L \subseteq X$. Then, the Gauss map γ_0 of X in $\mathbb{P}^N$ is of rank zero.*

Theorem 3.2. *Let X be a smooth cubic hypersurface in $\mathbb{P}^N$ with $N \geq 3$ in $p = 2$. Then, the Gauss map γ_0 of $X \subseteq \mathbb{P}^N$ is of rank zero if and only if X is projectively equivalent to the Fermat cubic hypersurface.*

Proof of Theorem 0.4. Denote by γ_0 the Gauss map of the embedding of X in $\mathbb{P}^N$ as a cubic hypersurface, as before. For the if-part, it is easily verified by a direct computation that γ_0 is of rank zero; hence X satisfies (GMRZ). For the only-if-part, it follows from Theorem 0.2 that $N^\vee_{L/X} \simeq [-1^{N-3}, 1]$ for any projective line $L \subseteq X$. Then, γ_0 is of rank zero by Proposition 3.1; hence X is projectively equivalent to a Fermat by Theorem 3.2. □

4. General conics on general hypersurfaces

First we have the following:

Proposition 4.1. *A general hypersurface X in $\mathbb{P}^N$ of degree d with $3 \le d \le 2N-3$ satisfies* (GMRZ) *only if $p = 2$ and either $d = 2N-3$ or $d = N-1$.*

To complete the proof of Theorem 0.5, we study normal bundles of general conics on X. Let $\mathcal{R}$ be the set of (irreducible reduced) conics in $\mathbb{P}^N$. Here $\mathcal{R}$ is an open subvariety of $\mathrm{Hilb}^{2t+1}(\mathbb{P}^N/K)$, the Hilbert scheme attached to the Hilbert polynomial $2t+1$. For an integer $d \ge 1$, we set $\mathcal{H} := |\mathcal{O}_{\mathbb{P}^N}(d)|$, and

$$I := \{\, (X, C) \in \mathcal{H} \times \mathcal{R} \mid C \subseteq X \,\},$$

which is a projective space bundle over $\mathcal{R}$, with projections $p_{\mathcal{H}} : I \to \mathcal{H}$ and $p_{\mathcal{R}} : I \to \mathcal{R}$. We moreover set $I^0 := \{\, (X, C) \in I \mid X \text{ is smooth along } C \,\}$, and

$$\mu_\xi := 3N - 2d - 2 + (N-2)\xi,$$

where we note that $\mu_\xi = \chi(f^* N_{C/X} \otimes \mathcal{O}_{\mathbb{P}^1}(\xi))$ for any $(X, C) \in I^0$.

Fix a conic C, and take an embedding $f : \mathbb{P}^1 \to \mathbb{P}^N$ with $f(\mathbb{P}^1) = C$. From the exact sequence, $0 \to \mathcal{I}_C^2 \to \mathcal{I}_C \to N^\vee_{C/\mathbb{P}^N} \to 0$ on $\mathbb{P}^N$, we obtain the following K-linear map,

$$\delta_C : H^0(\mathbb{P}^N, \mathcal{I}_C(d)) \to \mathsf{D} := \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(f^* N_{C/\mathbb{P}^N}, f^*\mathcal{O}_{\mathbb{P}^N}(d)),$$

which gives each $X \in p_{\mathcal{H}}(p_{\mathcal{R}}^{-1}(C) \cap I^0)$ a natural homomorphism of normal bundles,

$$\delta_C(X) : f^* N_{C/\mathbb{P}^N} \to f^* N_{X/\mathbb{P}^N} \simeq f^*\mathcal{O}_{\mathbb{P}^N}(d).$$

In addition, we have a decomposition, $f^* N_{C/X} = \bigoplus_{i=1}^{N-2} \mathcal{O}_{\mathbb{P}^1}(b_i(C/X))$ for some integers $b_i(C/X)$ determined by $(X, C) \in I^0$. Then, we set

$$I_{[\ge\xi]} := \{\, (X, C) \in I^0 \mid \min\{\, b_i(C/X) \,\} \ge \xi \,\},$$
$$I_{[\le\xi]} := \{\, (X, C) \in I^0 \mid \max\{\, b_i(C/X) \,\} \le \xi \,\},$$

where we note that $I_{[\ge\xi]}$ (resp. $I_{[\le\xi]}$) is an open subset of I by virtue of the upper semi continuity of $-\min\{\, b_i(C/X) \,\}$ (resp. $\max\{\, b_i(C/X) \,\}$) for (X, C) ([25, II, (3.9.2)]).

Proposition 4.2. (1) *$I_{[\ge\xi]}$ is not empty if $\mu_{-\xi-1} \ge 0$ and $\xi \le 2$.*
(2) *$I_{[\le\xi]}$ is not empty if $\mu_{-\xi-1} \le 0$ and $\xi \le 2d$.*

Corollary 4.3. *Assume $\mu_0 = 3N - 2d - 2 \ge 0$. Then for a general hypersurface X in $\mathbb{P}^N$ of degree d, there exists a conic C lying in X. Moreover for a general conic $C \subseteq X$, we have:*

(1) $\max\{\, b_i(C/X) \,\} \le 1$ *if* $\mu_{-2} = \mu_0 - 2(N-2) \le 0$,
(2) $\min\{\, b_i(C/X) \,\} \ge 0$ *if* $\mu_{-1} = \mu_0 - (N-2) \ge 0$.

Hence if $N - 2 \le \mu_0 \le 2(N-2)$ (i.e., $-N/2 + N + 1 \le d \le N$), then $f^* N^\vee_{C/X} = [-1^{2(N-d)}, 0^{N-2-2(N-d)}]$.

Proof of Theorem 0.5. Let $X \subseteq \mathbb{P}^N$ be a general hypersurface of degree $d \geq 3$ such that X satisfies (GMRZ). From Proposition 4.1, it is sufficient to show that the case of $p = 2$ and $d = N - 1$ does not occur.

Assume $p = 2$ and $N = d + 1 \geq 4$. It follows from Corollary 4.3 that $f^* N_{C/X}^{\vee} = [-1^2, 0^{N-4}]$ for a general conic $C \subseteq X$. Hence Theorem 0.2 implies $N = 4$ and $2 | a - 1$, where we set $a := \deg f^* \iota^* \mathcal{O}_{\mathbb{P}^M}(1)$ for an embedding $\iota : X \hookrightarrow \mathbb{P}^M$ with Gauss map of rank zero. On the other hand, from the Lefschetz theorem [15, III, Exercise 11.6 (c)], it follows $\operatorname{Pic} X = \operatorname{Pic} \mathbb{P}^4$ for $X \subseteq \mathbb{P}^4$; hence a is divisible by $2 = \deg f^*(\mathcal{O}_{\mathbb{P}^4}(1)|_X)$. This is a contradiction. $\square$

References

[1] M. Andreatta and J. A. Wiśniewski, On manifolds whose tangent bundle contains an ample subbundle, Invent. Math. **146** (2001), 209–217.

[2] M. Andreatta, E. Chierici, and G. Occhetta, Generalized Mukai conjecture for special Fano varieties, Cent. Eur. J. Math. **2** (2004), 272–293

[3] C. Araujo, Rational curves of minimal degree and characterizations of projective spaces, Math. Ann. **335** (2006), 937–951.

[4] C. Araujo, S. Druel, and S. Kovács, Cohomological characterizations of projective spaces and hyperquadrics, Invent. Math. **174** (2008), 233–253.

[5] A. Beauville, Sur les hypersurfaces dont les sections hyperplanes sont à module constant, Progr. Math. 86, The Grothendieck Festschrift, Vol. I, Birkhauser Boston, Boston, MA, 1990, pp. 121–133.

[6] N. Bourbaki, Éléments de Mathématique, Algèbre, Chapitre 9, Hermann, Paris, 1959.

[7] K. Cho, Y. Miyaoka, and N. I. Shepherd-Barron, Characterizations of projective space and applications to complex symplectic manifolds, Higher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math., **35**, Math. Soc. Japan, Tokyo, 2002, pp. 1–88.

[8] K. Cho and E. Sato, Smooth projective varieties with the ample vector bundle $\bigwedge^2 T_X$ in any characteristic, J. Math. Kyoto Univ. **35** (1995), 1–33.

[9] O. Debarre, Higher-dimensional algebraic geometry, Universitext, Springer, 2001.

[10] S. Fukasawa, K. Furukawa and H. Kaji, Projective varieties admitting an embedding with Gauss maps of rank zero, to appear in Adv. Math.

[11] S. Fukasawa and H. Kaji, The separability of the Gauss map and the reflexivity for a projective surface, Math. Z. **256** (2007), 699–703.

[12] S. Fukasawa and H. Kaji, Existence of a non-reflexive embedding with birational Gauss map for a projective variety, Math. Nachr. **281** (2008), 1412–1417.

[13] P. Griffiths and J. Harris, Algebraic geometry and local differential geometry, Ann. Sci. École Norm. Sup. (4) **12** (1979), 355–432.

[14] A. Grothendieck and J. A. Dieudonné, Éléments de géométrie algébrique, IV, Étude locale des schémas et des morphismes de schémas, Inst. Hautes Études Sci. Publ. Math., **32** (1967).

[15] R. Hartshorne, Algebraic Geometry, Graduate Texts in Math. **52**, Springer, New York, 1977.

[16] A. Hefez, Duality for projective varieties, Ph. D. Thesis, M.I.T., 1984.

[17] J.-M. Hwang, Geometry of minimal rational curves on Fano manifolds, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), ICTP Lect. Notes, **6**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, 335–393.

[18] J.-M. Hwang and S. Kebekus, Geometry of chains of minimal rational curves, J. Reine Angew. Math. **584** (2005), 173–194.

[19] J.-M. Hwang and N. Mok, Uniruled projective manifolds with irreducible reductive G-structures, J. Reine Angew. Math. **490** (1997), 55-64.

[20] J.-M. Hwang and N. Mok, Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation, Invent. Math. **131** (1998), 393-418.

[21] H. Kaji, On the normal bundles of rational space curves, Math. Ann. **273** (1985), 163–176.

[22] N. Katz, *Pinceaux de Lefschetz: Théorème d'existence*, Exposé XVII, "Groupes de Monodromie en Géométrie Algébrique (SGA7), Part II," Lecture Notes in Math. **340**, Springer, Berlin, 1973, pp. 212–253.

[23] S. Kebekus and L. Solá Conde, Existence of rational curves on algebraic varieties, minimal rational tangents, and applications, Global aspects of complex geometry, Springer, Berlin, 2006, pp. 359–416.

[24] S. L. Kleiman, Tangency and duality, In: Proceedings of the 1984 Vancouver conference in algebraic geometry, CMS Conference Proceedings **6**, Amer. Math. Soc., Providence, 1986, pp. 163–226.

[25] J. Kollár, Rational curves on algebraic varieties, Ergebnisse Der Mathematik Und Ihrer Grenzgebiete, 3. Folge, Bd. **32**, Springer, Berlin, 1996.

[26] Y. Miyaoka, Numerical characterisations of hyperquadrics, Complex analysis in several variables—Memorial Conference of Kiyoshi Oka's Centennial Birthday, Adv. Stud. Pure Math., **42**, Math. Soc. Japan, Tokyo, 2004, pp. 209–235

[27] S. Mori, Projective manifolds with ample tangent bundles, Ann. Math. (2) **110** (1979), 593-606

[28] G. Occhetta, A note on the classification of Fano manifolds of middle index, Manuscripta Math. **117** (2005), 43–49.

[29] R. Pardini, Some remarks on plane curves over fields of finite characteristic, Compos. Math. **60** (1986), 3–17.

[30] R. Piene, Numerical characters of a curve in projective n-space, In: Real and Complex Singularities, Oslo 1976, pp. 475–495.

[31] A. H. Wallace, Tangency and duality over arbitrary fields, Proc. London Math. Soc. (3) **6** (1956), 321–342.

[32] F. L. Zak, Tangents and secants of algebraic varieties, Transl. Math. Monographs **127**, Amer. Math. Soc., Providence, 1993.

Ohkubo 3-4-1, Shinjuku, Tokyo 169–8555, JAPAN
E-mail address: `kaji@waseda.jp`