Irreducible components of the moduli stack of torsion-free sheaves of K3 surfaces and their dimensions

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1 Introduction

A moduli space is a space consisting of points that represent a certain type of object. When we consider the moduli space of sheaves, in the category of scheme, it only parametrizes the (semi) stable sheaves, which is a kind of torsion-free sheaves. If trying to realize a moduli space for all torsion-free sheaves, the concept of a stack is needed. Stack is , roughly speaking, an extension of scheme. In this study, we performed the irreducible decomposition of torsion-free sheaves on K3 surfaces and calculated the dimensions of them at each point.

2 Preliminaries

Definition 2.1 (K3 surface)**.** X : smooth projective surface / $\mathbb{C}$
X : K3 surface $\overset{\text{def}}{\iff} K_X = 0$ and $H^1(X, \mathscr{O}_X) = 0$

Definition 2.2 (Mukai vector)**.** X : K3 surface, E : coherent sheaf on X
$v(E) := (\mathrm{rank}(E),\, c_1(E),\, \frac{c_1(E)^2}{2} - c_2(E) + \mathrm{rank}(E)) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$

Definition 2.3 (Mukai pairing)**.** X : K3 surface
$v := ([v]_0, [v]_1, [v]_2), v' := ([v']_0, [v']_1, [v']_2) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$
$\langle v, v' \rangle := -[v]_0[v']_2 + [v]_1[v']_1 - [v]_2[v']_0 \in \mathbb{Z}$

Definition 2.4. $v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$: primitive
$\overset{\text{def}}{\iff} v' \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$ and $m \in \mathbb{Z}$, $v = mv' \Rightarrow m = 1$ or -1

Remark 2.5.

- $^\forall v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, $\langle v, v \rangle \in 2\mathbb{Z}$
- $E, E' \in \mathrm{Coh}(X)$, $v(E) = v(E') \Rightarrow (\mathrm{rank}(E), c_1(E), c_2(E)) = (\mathrm{rank}(E'), c_1(E'), c_2(E'))$
- $^\forall v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, $^\exists E \in \mathrm{Coh}(X)$ s.t. $v(E) = v$

Definition 2.6 (Moduli stacks of torsion-free sheaves)**.** X : K3 surface , $v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$
Then, we define the moduli stack of torsion-free sheaves $\mathscr{M}^{\mathrm{tf}}(v)$ as the following category.

1. Objects : (U, E), where
   - U : Sch/$\mathbb{C}$,
   - E : quasi-coherent sheaf of finite presentation on $X \times_{\mathbb{C}} U (=: \mathscr{X})$, flat/$U$
     s.t. E_t : torsion-free sheaves on $\mathscr{X}_t = X_{k(t)}$ with $v(E_t) = v$ ($^\forall t \in U$)
2. Morphisms : from (U, E) to $(U', E') \rightsquigarrow (\varphi : U \to U', \alpha : \varphi^* E \to E'$: isomorphism)

Definition 2.7 (Stacks of Harder-Narasimhan filtration)**.** $v, v_1, v_2 \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$

$$\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) := \left\{ E \in \mathscr{M}^{\mathrm{tf}}(v) \;\middle|\; \begin{array}{c} ^\exists(0 \subset E_1 \subset E) : \text{Harder-Narasimhan filtration} \\ \text{with } v(E_1) = v_1, v(E/E_1) = v_2 \end{array} \right\}$$

$$\mathscr{M}^{\mathrm{ss}}(v) := \{ E \in \mathscr{M}^{\mathrm{tf}}(v) \mid E : \text{semistable} \}$$

3 Main Theorem

Theorem 3.1. Let X be a K3 surface of $\rho(X)=1$, $v_0 \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$: primitive, $m \in \mathbb{Z}$.
$v := mv_0$. We assume $[v]_0 = 2$ and v satisfies one of the following disjoint conditions.
$(a) : \langle v, v \rangle > 0$
$(b) : \langle v, v \rangle < -2$, $\langle v_0, v_0 \rangle \neq -2$
$(c) : \langle v, v \rangle = 0, -2$, v : primitive
then, we have the irreducible decomposition of $\mathscr{M}^{\mathrm{tf}}(v)$ as follows.

$$\mathscr{M}^{\mathrm{tf}}(v) = \overline{\mathscr{M}^{\mathrm{ss}}(v)} \cup \bigcup_{(v_1,v_2) \in S \cup S_{\mathrm{even}}} \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$$

, where (we always assume $v_1, v_2 \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$)

$$I := \{(v_1, v_2) \mid v_1 + v_2 = v, \mathrm{rank}(v_1) = \mathrm{rank}(v_2) = 1\}$$
$$J := \{(v_1, v_2) \mid \langle v_1, v_2 \rangle < 1\} \quad K := \{(v_1, v_2) \mid 2[v_1]_1 = 2[v_2]_1 = [v]_1\}$$

$$S := \begin{cases} (I \cap J) \setminus K & \text{if } (a) \text{ or } (c) \\ I \setminus K & \text{if } (b) \end{cases} \qquad S_{\mathrm{even}} := \begin{cases} I \cap J \cap K & \text{if } (a) \text{ or } (c), \text{ and } 2 \mid [v]_1 \\ I \cap K & \text{if } (b), \text{ and } 2 \mid [v]_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover, we can classify the irreducible components into 3 types according to the general members of them.

$$^{\forall}E \in \mathscr{M}^{\mathrm{ss}}(v) \text{ is semistable}$$
$$(v_1, v_2) \in S \Rightarrow {}^{\forall}E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \text{ is not } \mu\text{-semistable}$$
$$(v_1, v_2) \in S_{\mathrm{even}} \Rightarrow {}^{\forall}E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \text{ is not semistable but } \mu\text{-semistable}$$

Remark 3.2. v satisfies $(b) \Rightarrow \mathscr{M}^{\mathrm{ss}}(v)$: empty category

Corollary 3.3. The dimension of $\mathscr{M}^{\mathrm{tf}}(v)$ at ${}^{\forall}E \in \mathscr{M}^{\mathrm{tf}}(v)$ is the following.

$$v \text{ satisfies } (a) \text{ or } (c) \Rightarrow \dim_E \mathscr{M}^{\mathrm{tf}}(v) = \begin{cases} \langle v, v \rangle + 1 & (E \in \mathscr{M}^{\mathrm{ss}}(v) \cup \bigcup_{\langle v_1, v_2 \rangle \geq 1} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)) \\ \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_2 \rangle + 2 & (E \in \bigcup_{\langle v_1, v_2 \rangle < 1} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)) \end{cases}$$
$$v \text{ satisfies } (b) \Rightarrow \dim_E \mathscr{M}^{\mathrm{tf}}(v) = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_2 \rangle + 2$$

Remark 3.4. By Yoshioka([Kimura-Yoshioka11], [Kurihara-Yoshioka08]), it is known that

$$\dim \mathscr{M}^{\mathrm{ss}}(v) := \sup_{E \in \mathscr{M}^{\mathrm{ss}}(v)} (\dim_E \mathscr{M}^{\mathrm{ss}}(v)) = \langle v, v \rangle + 1$$
$$\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) := \sup_{E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)} (\dim_E \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)) = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1 + v_2 \rangle + 2$$

But $\dim_E \mathscr{M}^{\mathrm{tf}}(v)$ is NOT necessarily equal to $\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ for $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$.

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