Irreducible components of the moduli stack of torsion free sheaves of K3 surfaces and their dimensions

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Introduction I

Question

Study the structure of moduli of sheaves on algebraic varieties (irreducibility, smoothness, birational property etc.)

- To construct moduli schemes of sheaves
  $\rightarrow$ we must restrict sheaves to (semi) stable sheaves, which are a kind of torsion-free sheaves. (in detail, [HL10])
- To treat all torsion-free sheaves
  $\rightarrow$ we need the notion of moduli stacks. (in detail, [LMB00])

Introduction II

Theorem ([Mukai84(a)],[Mukai84(b)])

X : K3 surface, $v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$
H : ample divisor
$M^s_H(v)$: the moduli scheme of stable sheaves for H with Mukai vector v.
Then, $M^s_H(v)$ is smooth and

$$\dim_E M^s_H(v) = \langle v, v \rangle + 2 \quad ({}^\forall E \in M^s_H(v))$$

- We can uniformly write the dimensions of the moduli schemes of stable sheaves on K3 surfaces by using Mukai vector ($\overset{1:1}{\longleftrightarrow}$ rank & Chern class)

$\rightsquigarrow$ How about moduli stacks?

Introduction III

Purpose of the study

$\rightsquigarrow$ Irreducible decomposition of the moduli stack of torsion -free sheaves of rank 2 on K3 surfaces of Picard number 1 and computation of dimensions at the points.

- Cases of ruled surfaces and $\mathbb{P}^2$
  $\rightarrow$ by C. Walter ([Walter95]) etc.
- Expected application
  $\rightarrow$ study of Brill-Noehter theory. (Study of special loci of the moduli schemes of stable sheaves) (cf. [GH96], [Walter95], [Yoshioka99])

K3 surfaces , Mukai vector I

Definition (K3 surface)

X : smooth projective surface / $\mathbb{C}$

X : K3 surface $\overset{\text{def}}{\Longleftrightarrow}$ $K_X = 0$ and $H^1(X, \mathscr{O}_X) = 0$

Examples

- Smooth quartic hypersurfaces in $\mathbb{P}^3$
- Smooth complete intersections of a quadric and a cubic hypersurfaces in $\mathbb{P}^4$
- smooth complete intersections of three quadric hypersurfaces in $\mathbb{P}^5$
- Kummer surfaces
- A double cover of $\mathbb{P}^2$ branched along a smooth sextic curve

K3 surfaces , Mukai vector II

Definition (Mukai vector)

X : K3 surface
E : coherent sheaf on X
$v(E) := (\mathrm{rank}(E), c_1(E), \frac{c_1(E)^2}{2} - c_2(E) + \mathrm{rank}(E)) \quad \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$

Definition (Mukai pairing)

X : K3 surface
$v := ([v]_0, [v]_1, [v]_2), v' := ([v']_0, [v']_1, [v']_2) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$
$\langle v, v' \rangle := -[v]_0[v']_2 + [v]_1[v']_1 - [v]_2[v']_0 \in \mathbb{Z}$

Definition

$v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$: primitive
$\overset{\mathrm{def}}{\Longleftrightarrow} v' \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$ and $m \in \mathbb{Z}$, $v = mv' \Rightarrow m = 1$ or -1

K3 surfaces , Mukai vector III

Remark

- ${}^{\forall}v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, $\langle v, v \rangle \in 2\mathbb{Z}$
- $E, E' \in \mathrm{Coh}(X)$, $v(E) = v(E')$
  $\Rightarrow (\mathrm{rank}(E), c_1(E), c_2(E)) = (\mathrm{rank}(E'), c_1(E'), c_2(E'))$
- ${}^{\forall}v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, ${}^{\exists}E \in \mathrm{Coh}(X)$ s.t. $v(E) = v$

Moduli stacks of torsion-free sheaves I

Definition (Moduli stacks of torsion-free sheaves)

X : K3 surface, $v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$

then, the moduli stack of torsion-free sheaves on X whose Mukai vectors are v is the following category.(denoted by $\mathscr{M}^{\mathrm{tf}}(v)$)

- Objects : (U, E), where
  - U : $\mathrm{Sch}/\mathbb{C}$
  - E : quasi-coherent sheaf of finite presentation on $X \times_{\mathbb{C}} U$ and flat$/U$ s.t. E_t : torsion-free sheave on $X_{k(t)}$ with $v(E_t) = v$ ($^{\forall} t \in U$)
- Morphisms : from (U, E) to (U', E') $\rightsquigarrow (\varphi : U \to U', \alpha : \varphi^* E \to E'$: isomorphism$)$

Moduli stacks of torsion-free sheaves II

Definition (stacks of Harder-Narasimhan filtration)

X : K3 surface
$v, v_1, v_2 \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$
then,

$$\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) := \left\{ E \in \mathscr{M}^{\mathrm{tf}}(v) \;\middle|\; \begin{array}{l} {}^{\exists}(0 \subset E_1 \subset E) : \text{HN-filtaration} \\ \text{s.t. } v(E_1) = v_1, v(E/E_1) = v_2 \end{array} \right\}$$

$$\mathscr{M}^{\mathrm{ss}}(v) := \{ E \in \mathscr{M}^{\mathrm{tf}}(v) \mid E : \text{semistable} \}$$

- ${}^{\forall}E \in \mathscr{M}^{\mathrm{tf}}(v)$, HN-filtration is uniquely determined.

Main theorem I

Main theorem

Let X be a K3 surface of $\rho(X) = 1$, $v_0 \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$: primitive, $m \in \mathbb{Z}$, $v := mv_0$.
And we assume $[v]_0 = 2$ and v satisfies one of the following disjoint conditions
$(a) : \langle v, v \rangle > 0$
$(b) : \langle v, v \rangle < -2$, $\langle v_0, v_0 \rangle \neq -2$
$(c) : \langle v, v \rangle = 0, -2$, v : primitive
then, we have the irreducible decomposition of $\mathscr{M}^{\mathrm{tf}}(v)$ as follows.

$$\mathscr{M}^{\mathrm{tf}}(v) = \overline{\mathscr{M}^{\mathrm{ss}}(v)} \cup \bigcup_{(v_1,v_2) \in S \cup S_{\mathrm{even}}} \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$$

where, (we always assume $v_1, v_2 \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$)

$$I := \{(v_1, v_2) \mid v_1 + v_2 = v, [v_1]_0 = [v_2]_0 = 1\}$$

$$J := \{(v_1, v_2) \mid \langle v_1, v_2 \rangle < 1\}, \quad K := \{(v_1, v_2) \mid 2[v_1]_1 = 2[v_2]_1 = [v]_1\}$$

$$S := \begin{cases} (I \cap J) \setminus K & \text{if } (a) \text{ or } (c) \\ I \setminus K & \text{if } (b) \end{cases} \quad S_{\mathrm{even}} := \begin{cases} I \cap J \cap K & \text{if } (a) \text{ or } (c), \text{ and } 2 \mid [v]_1 \\ I \cap K & \text{if } (b), \text{ and } 2 \mid [v]_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Main theorem II

(Continuition of Main theorem)
Moreover, we can classify the irreducible components into 3 types according to the general members of them,

$$
\begin{aligned}
&{}^{\forall}E \in \mathscr{M}^{\mathrm{ss}}(v) \text{ is semistable} \\
(v_1, v_2) \in S \Rightarrow &{}^{\forall}E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \text{ is not } \mu\text{-semistable} \\
(v_1, v_2) \in S_{\mathrm{even}} \Rightarrow &{}^{\forall}E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \text{ is not semistable but } \mu\text{-semistable}
\end{aligned}
$$

Remark

- v satisfies $(b) \Rightarrow \mathscr{M}^{\mathrm{ss}}(v)$: empty category

Main theorem III

Corollary

The dimension of $\mathscr{M}^{\mathrm{tf}}(v)$ at ${}^{\forall}E \in \mathscr{M}^{\mathrm{tf}}(v)$ is the following.

In the cases of (a) or (c)

$$\dim_E \mathscr{M}^{\mathrm{tf}}(v) = \begin{cases} \langle v, v \rangle + 1 & (E \in \mathscr{M}^{\mathrm{ss}}(v) \cup \bigcup_{\langle v_1, v_2 \rangle \geq 1} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)) \\ \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_2 \rangle + 2 & (E \in \bigcup_{\langle v_1, v_2 \rangle < 1} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)) \end{cases}$$

In the case of (b)

$$\dim_E \mathscr{M}^{\mathrm{tf}}(v) = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_2 \rangle + 2$$

Remark

By Yoshioka([Kimura-Yoshioka11], [Kurihara-Yoshioka08]), it is known that

$$\dim \mathscr{M}^{\mathrm{ss}}(v) = \langle v, v \rangle + 1$$

$$\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_2 \rangle + 2$$

But $\dim_E \mathscr{M}^{\mathrm{tf}}(v)$ is NOT necessarily equal to $\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ for $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$.

Further studies

1. Expected application of the results (Brill-Noether loci of the moduli schemes of stable sheaves or Hilbert schemes etc.)
2. Remaining cases (v : nonprimitive and $\langle v_0, v_0 \rangle = 0, -2$)
3. Extension of the result to the case of $\rho(X) \geq 2$
4. Relationship between the moduli of stable sheaves and the moduli of unstable sheaves

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