Irreducible components of the moduli stack of torsion free sheaves of K3 surfaces and their dimensions

Yuki Mizuno

Abstract

In this paper, we study the irreducible components of the moduli stack of torsion free sheaves of rank 2 with fixed Mukai vector on a K3 surface of Picard number 1 and their dimensions.

Contents

0 Introduction **1**

1 Preliminaries **5**

1.1 K3 surfaces and Mukai vector 5

1.2 The moduli stacks of torsion free sheaves 5

1.3 The topological spaces associating to algebraic stacks and dimension of algebraic stacks 6

1.4 Stability, Harder-Narasimhan filtration 7

2 Proof of Theorem 0.1 and Corollary 0.3 **8**

2.1 Irreducibility of substacks of the moduli stacks of torsion free sheaves 8

2.2 Determination of the irreducible components of $\mathscr{M}^{\mathrm{tf}}(v)$ 10

2.3 Proof of Theorem 0.1 and Corollary 0.3 15

Reference **15**

0 Introduction

It is well known that we can realize the moduli spaces of the line bundles on algebraic varieties as Picard varieties (for example, [6]). However, although the moduli spaces of vector bundles of rank 2 or more are known not to be realized as schemes because there are too many objects to handle, it is also known that the moduli spaces are realized as projective varieties by introducing stability (for example, [10]). Although we can similarly construct the moduli schemes of vector bundles on higher dimensional varieties by introducing stability, in this case, it is known that the moduli schemes does not generally become projective schemes but quasi-projective schemes (ibid.). At this time, by replacing a vector bundle by a torsion free sheaf which is a kind of generalization a vector bundle, it is known that we can realized the moduli schemes of semi stable torsion free sheaves as projective schemes (for example, [26]). The reason that a torsion free sheaf is a kind of generalization is a toraion free sheaf is a vector bundle except for a part whose codimension 2 or more. In particular, a torsion free sheaf on a surface is a vector bundle except for finite points (in detail, [19]).

Studying the structure of the moduli schemes of stable sheaves is interesting in itself. However, they are not sufficient in that they can not parametrize all vector bundles or all torsion free sheaves because we must make assumptions about the sheaves we treat when constructing the moduli schemes of (semi)stable sheaves. But, it is known that when thinking in the category of algebraic stacks, all vector bundles or torsion free sheaves with fixed rank and Chern class are realized as an algebraic stack in the sense of Artin stack (for example, [14]). By the way, the moduli stacks of (semi)stable sheaves are often treated, but it seems that little has been done to study the moduli stacks of vector bundles or torsion free sheaves. In particular, some results for the moduli stacks of torsion free sheaves can be seen in detail in Strømme ([22]) or Walter ([23]). Additionally, in Walter ([23]), irreducible decomposition of the moduli stacks of the torsion free sheaves on ruled surfaces and calculations of the dimensions of points in them.

By the way, for K3 surafces, by Mukai ([16], [17]), the dimensions of the moduli schemes of stable sheaves can be written uniformly by using Mukai vector (in detail, definition 1.2), and the dimensions are dependent only on the length of Mukai vector. That is,

Theorem 0.0 ([16],[17])**.** Let X be a K3 surface, and v be an element of $\mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, H be an ample divisor on X. Then, the moduli scheme $M_H(v)$ of stable sheaves for H with mukai vector is nonsingular and for any sheaf $E \in M_H(v)$

$$\dim_E M_H(v) = \langle v, v \rangle + 2$$

It also seems that irreducible decomposition of the moduli stacks of torsion free sheaves on K3 surfaces have not been done. Furthermore, we have a question that how we write the dimensions of the points by using Mukai vector from the above result.This time, we did irreducible decomposition of the moduli stack of the torsion free sheaves on K3 surfaces of Picard number 1 with fixed Mukai vector and computation of calculations of the dimensions of points on it. Although the irreducible components are not finite, it turned out that they can be divided into three types. The first is a component whose general members are Gieseker-semistable sheaves, the second is a component whose general members are not Gieseker-semistable but μ-semistable sheaves and the third is a component whose general members are not μ-semistable sheaves. The last two kinds of components are given closures of the stacks of Harder-Narasimhan filtrations. Let $\mathscr{M}^{\mathrm{tf}}(v)$ be the moduli stack of torsion free sheaves on K3 surface X with Mukai vector v and $\mathscr{M}^{\mathrm{ss}}(v)$ be the moduli stack of Gieseker-semistable sheaves (in detail, definition 1.6). The stacks of Harder-Narasimhan filtrations are given as follows (in detail. definition 2.2).

$$\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) := \left\{ E \in \mathscr{M}^{\mathrm{tf}}(v) \;\middle|\; \begin{array}{c} {}^{\exists}(0 \subset E_1 \subset E) : \text{Harder-Narasimahn filtration} \\ \text{s.t. } v(E_1) = v_1, v(E/E_1) = v_2 \end{array} \right\}$$

where, $v := ([v]_0, [v]_1, [v]_2), v_1 := ([v_1]_0, [v_1]_1, [v_1]_2), v_2 := ([v_2]_0, [v_2]_1, [v_2]_2) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$.

In the following theorem, when we write v_1, v_2, they are always assumed to be elements of $\mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$. And we set

$$\begin{aligned} I &:= \{(v_1, v_2) \mid v_1 + v_2 = v, [v_1]_0 = [v_2]_0 = 1\} \\ J &:= \{(v_1, v_2) \mid \langle v_1, v_2 \rangle < 1\} \\ K &:= \{(v_1, v_2) \mid 2[v_1]_1 = 2[v_2]_1 = [v]_1\} \end{aligned}$$

The main theorem is the following.

Theorem 0.1. Let X be a K3 surface of $\rho(X) = 1$, and v_0 : primitive $\in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, $m \in \mathbb{Z}$ and set $v := mv_0$.

We assume $[v]_0 = 2$ and v satisfies one of the following disjoint conditions
$(a) : \langle v, v \rangle > 0$
$(b) : \langle v, v \rangle < -2$, $\langle v_0, v_0 \rangle \neq -2$
$(c) : \langle v, v \rangle = 0, -2$, v : primitive
then,

$$\mathscr{M}^{\mathrm{tf}}(v) = \overline{\mathscr{M}^{\mathrm{ss}}(v)} \cup \bigcup_{(v_1,v_2) \in S \cup S_{\mathrm{even}}} \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$$

where,

$$S := \begin{cases} (I \cap J) \setminus K & \text{if } (a) \text{ or } (c) \\ I \setminus K & \text{if } (b) \end{cases}$$

$$S_{\mathrm{even}} := \begin{cases} I \cap J \cap K & \text{if } (a) \text{ or } (c), \text{ and } 2|[v]_1 \\ I \cap K & \text{if } (b), \text{ and } 2|[v]_1 \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover, if $(v_1, v_2) \in S$, then the sheaves in $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ are not μ-semistable. If $(v_1, v_2) \in S_{\mathrm{even}}$, then the sheaves in $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ are not Gieseker-semistable but μ-semistable.

Remark 0.2. In the theorem 0.1, if v satisfies (b), then $\mathscr{M}^{\mathrm{ss}}(v)$ is an empty category.

Although the dimensions of the moduli stacks of semistable sheaves $\mathscr{M}^{\mathrm{ss}}(v)$ or Harder-Narasimhan filtrations $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ are determined by Yoshioka ([12], [13]), all of these are not necessarily irreducible components in $\mathscr{M}^{\mathrm{tf}}(v)$, so this alone can not determine the dimensions of points in $\mathscr{M}^{\mathrm{tf}}(v)$. However, this time, we completely determined the dimensions of the points in $\mathscr{M}^{\mathrm{tf}}(v)$ by the main results.

Corollary 0.3. Under the notation of the theorem 0.1, for all $E \in \mathscr{M}^{\mathrm{tf}}(v)$, the dimension $\dim_E \mathscr{M}^{\mathrm{tf}}(v)$ of $\mathscr{M}^{\mathrm{tf}}(v)$ at E is the following.

In the cases of (a) or (c),

$$\dim_E \mathscr{M}^{\mathrm{tf}}(v) = \begin{cases} \langle v, v \rangle & E \in \mathscr{M}^{\mathrm{ss}}(v) \text{ or } E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \text{ of } \langle v_1, v_2 \rangle \geq 1 \\ \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_2 \rangle + 2 & E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \text{ of } \langle v_1, v_2 \rangle < 1 \end{cases}$$

In the case of (b),

$$\dim_E \mathscr{M}^{\mathrm{tf}}(v) = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_2 \rangle + 2$$

This result is similar to the result of Walter ([23]) for ruled surfaces. However, because the canonical sheaves are obvious in the case of K3 surfaces, therefore if E is a coherent sheaf, $\mathrm{Ext}^2(E, E) \simeq \mathrm{Hom}(W, E) \neq 0$ from Serre duality. This makes it difficult to calculate the dimensions. In addtion, unlike ruled surfaces, K3 surfaces of Picard number 1 are not fibered surfaces. From this point, the approach of the proof is different.

In the future, Brill-Noether theory is expected as an application of the main result. Brill-Noether theory was originally conceived to study the detailed properties of algebraic curves which can not be known just from Riemann-Roch Theorem. In detail, we define Brill-Noether locus of an algebraic curve C.

$$W^r_d(C) := \{L \in \mathrm{Pic}^d(C) \mid h^0(C, L) \geq r + 1\}$$

where, $\mathrm{Pic}^d(C)$ is a connected component of $\mathrm{Pic}(C)$ which parametrizes the line bundles whose degree are d. Then, for example, it is known that C is hyper elliptic is equivalent to $W^1_2(C) \neq \emptyset$. In addtion, That C is a plane curve of degree d is equivalent to $W^2_d \neq \emptyset$ (for example, [15]).

We can generalize the concept of Brill-Noehter locus. Under the notation of 0.0, We set

$$W^r_H(v) := \{E \in M_H(v) \mid h^0(X, E) \geq r + 1\}$$

The above Brill-Noether locus for the moduli schemes of stable sheaves has been found to be related to the birational geometry of the moduli schemes of stable sheaves (for example, [3], [4]). In Walter ([23]), it is stated that the irreducible components of Brill-Noether locus of Hilbert schemes of points on smooth projective surfaces correspond to irreducible components satisfying a certain condition of some moduli stacks of torsion free sheaves on them. By Göttsche and Huybrechts ([7]) or Yoshioka ([24]), it is also known that for K3 surfaces birational maps between Hilbert schemes of points and some kinds of moduli schemes of stable sheaves can be construct or that they can be deformation equivalent under certain conditions. Combining these facts, it is expected that it will be useful for further analysis of the structure of $W^r_H(v)$.

Acknowledgements

The auther sincerely thanks his supervisor Professor Hajime Kaji for many helphul advices and encouragenments. He is also grateful to Professor Ryo Okawa for useful discussions and comments, and the members of algebraic geometry laboratories.

1 Preliminaries

In this paper, a surface means 2 dimensional algebraic variety over $\mathbb{C}$, an algebraic stack means an Artin stack over $\mathbb{C}$. In addtion, an open (resp. closed, resp. locally closed) substack means a strictly substack whose inclusion map is open (resp. closed, resp. locally closed) immersion (in detail, [21]).

1.1 K3 surfaces and Mukai vector[1]

Definition 1.1. Let X be a smooth projective surface over $\mathbb{C}$. Then,
X is K3 surface if $K_X = 0$ and $H^1(X, \mathscr{O}_X) = 0$

Definition 1.2. Let X be a K3 surface and E be a coherent sheaf on X. Then,
$v(E) := (\mathrm{rank}(E),\, c_1(E),\, \frac{c_1(E)^2}{2} - c_2(E) + \mathrm{rank}(E)) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$

Definition 1.3. Let X be a K3 surface and
$v := ([v]_0, [v]_1, [v]_2), v' := ([v']_0, [v']_1, [v']_2) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$. Then,
$\langle v, v' \rangle := -[v]_0[v']_2 + [v]_1[v']_1 - [v]_2[v']_0 \in \mathbb{Z}$

Definition 1.4. $v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$ is primitive
if $[v' \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z},\ m \in \mathbb{Z},\ v = mv'] \Rightarrow m = 1$ or -1

Remark 1.5.

- X is a K3 surface $\Rightarrow \mathrm{Pic}(X) = \mathrm{NS}(X)$
- For all $v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, $\langle v, v \rangle \in 2\mathbb{Z}$.
- $E, E' \in \mathrm{Coh}(X)$, $v(E) = v(E') \Rightarrow (\mathrm{rank}(E), c_1(E), c_2(E)) = (\mathrm{rank}(E'), c_1(E'), c_2(E'))$
- $\forall v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, $\exists E$: $\mathrm{Coh}(X)$ s.t. $v(E) = v$

1.2 The moduli stacks of torsion free sheaves

Definition 1.6 (Moduli stacks of torsion free sheaves)**.** Let X be a smooth projective surface over $\mathbb{C}$ and $(2, D, c_2) \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$

We define the moduli stack $\mathscr{M}^{\mathrm{tf}}(2, D, c_2)$ of torsion-free sheaves with rank r and Chern polynomial $1 + Dt + c_2t^2$ to be the following category

1. Objects : $(U,\, E)$, where
   - U : scheme over $\mathbb{C}$

[1] For further information about K3 surfaces, see [9]

- E : quasi-coherent sheaf of finite presentation on $X \times_{\mathbb{C}} U (=: Z)$, flat over U s.t. E_t : torsion free sheaves on $Z_t = X_{k(t)}$ s.t. $\mathrm{rank}(E_t) = 2$, $c_1(E_t) = D \mid_{X_{k(t)}}$, $c_2(E_t) = c_2$

2. Morphisms : we define the maps from (U, E) to (U', E') as $(\varphi : U \to U', \alpha : \varphi^* E \to E'$: isomorphism$)$

Remark 1.7. $\mathscr{M}^{\mathrm{tf}}(2, D, c_2)$ is an algebraic stack.And, we can define the moduli stack of coherent sheaves in the same way. We denote it by $\mathscr{M}(2, D, c_2)$.

1.3 The topological spaces associating to algebraic stacks and dimension of algebraic stacks

Definition 1.8 ([5], [14])**.** Let $\mathscr{X}$ be an algebraic stack over $\mathbb{C}$

$$|\mathscr{X}| := \coprod_{K/\mathbb{C}:\text{extension}} \mathscr{X}(\mathrm{Spec}(K)) / \sim$$

where, $E \sim E' \overset{\text{def}}{\Longleftrightarrow} \exists K''$: extension of K, K' such that $E \mid_{X_{\mathrm{Spec}(K'')}} \simeq E' \mid_{X_{\mathrm{Spec}(K'')}}$

$$\begin{pmatrix} E \in \mathscr{X}(\mathrm{Spec}(K)), E' \in \mathscr{X}(\mathrm{Spec}(K')) \\ K, K'/\mathbb{C} : \text{extension of } \mathbb{C} \end{pmatrix}$$

Definition 1.9 ([5], [14])**.** Let $\mathscr{X}$ be an algebraic stack over $\mathbb{C}$.

A set $\{U \subseteq |\mathscr{X}| \mid \exists \mathscr{U}$: open substack of $\mathscr{X}$ such that $|\mathscr{U}| = U\}$ is a family of subsets of $\mathscr{X}$ satisfying the axiom of open set, by this, we can think of $|\mathscr{X}|$ as a topological set.

Remark 1.10. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks, this induces a continuous map $|f| : |\mathscr{X}| \to |\mathscr{Y}|$

Definition 1.11 ([5], [14])**.** Let $P : U \to \mathscr{X}$ be a smooth morphism from a scheme and $u \in U$ such that $u \mapsto x$ then, we define $\dim_u(P)$ as follows.

$$\begin{array}{ccc} U \times_{\mathscr{X}} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & \square & \downarrow x \\ U & \xrightarrow{P} & \mathscr{X} \end{array}$$

then,

$$\dim_u(P) := \dim_x(U \times_{\mathscr{X}} \mathrm{Spec}(k))$$

Definition 1.12 ([5], [14])**.** Let $\mathscr{X}$ be an algebraic stack over $\mathbb{C}$ and $x \in \mathscr{X}(\mathrm{Spec}(K))(K/\mathbb{C}$: extension$)$, $P : U \to \mathscr{X}$ be a smooth morphism from a scheme , $u \in U$ such that $u \mapsto x$. Then,

$$\dim_x(\mathscr{X}) := \dim_u(U) - \dim_u(P)$$

Remark 1.13. In general, $-\chi(E, E) + \mathrm{ext}^2(E, E) \geq \dim_{[E]} \mathscr{M}^{\mathrm{tf}}(2, D, c_2) \geq -\chi(E, E)$

1.4 Stability, Harder-Narasimhan filtration

Definition 1.14. Let X be a smooth projective surface over $\mathbb{C}$, H be an ample divisor on X, E is a torsion free sheaf on X. Then,

$$\mu(E) := \frac{c_1(E).H}{\mathrm{rank}(E)} \qquad P(E,m) := \chi(E(mH)) = \sum_{i=0}^{\dim(E)} \frac{\alpha_i}{i!} m^i \qquad p(E,m) := \frac{P(m)}{\alpha_{\dim(E)}}$$

and,

$$E : \mu\text{-(semi)stable if } \mu(F) \underset{(=)}{<} \mu(E)$$

$(0 \neq \forall F \subset E, \mathrm{rank}(F) < \mathrm{rank}(E))$

$$E : \text{Gieseker-(semi)stable if } p(F,m) \underset{(=)}{<} p(E,m)$$

$(0 \neq \forall F \subset E)$

where, $p(F,m) \underset{(=)}{<} p(E,m)$ if $p(F,m) \underset{(=)}{<} p(E,m)(m \gg 0)$

Theorem 1.15 (Harder-Narasimhan(HN) filtration)**.** Let X be a smooth projective surface over $\mathbb{C}$ H be an ample divisor on X, E be a torsion free sheaf on X.

Then, for E, there exists a unique filtration (we call this the Harder-Narasimhan(HN) filtration of E for μ-semistable)

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$$

, s.t. E_i/E_{i-1} : μ-semistable for H, $i = 1, \cdots s$, and $\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_{s-1}/E_{s-2}) > \mu(E_s/E_{s-1})$

In the same way, there exists a unique filtration (we call this the Harder-Narasimhan(HN) filtration of E for Gieseker-semistable)

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$$

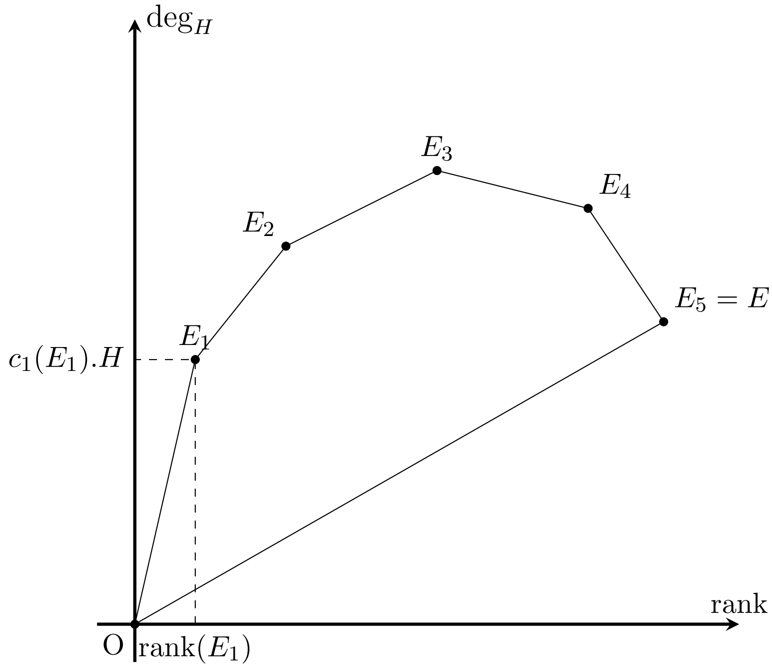
, s.t. E_i/E_{i-1} : Gieseker-semistable for H, $i = 1, \cdots s$, and $p(E_1/E_0, m) > p(E_2/E_1, m) > \cdots > p(E_{s-1}/E_{s-2}, m) > p(E_s/E_{s-1}, m)$

Definition 1.16. Let X be a smooth projective surface over $\mathbb{C}$ and H be an ample divisor on X, E be a torsion free sheaf on X and Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$$

be the HN-filtration of E for H. Then, we can define the polygon which has as the vertexes $(0,0)$, $(\mathrm{rank}(E_1), \deg_H(E_1))$, $(\mathrm{rank}(E_1) + \mathrm{rank}(E_2), \deg_H(E_2))$, $\cdots$, $(\mathrm{rank}(E_1) + \cdots + \mathrm{rank}(E_{s-1}), \deg_H(E_{s-1}))$, $(\mathrm{rank}(E), \deg_H(E))$. We call this HN-polygon of E for H.We denote it by HNP(E).

Here, as an example, we draw the polygon in case $s = 5$ as follows.



HN-polygon of E for H

2 Proof of Theorem 0.1 and Corollary 0.3

In this section, we always assume X is a K3 surface of $\rho(X) = 1$ and H is an ample divisor generating $\mathrm{Pic}(X)$.

And we often denote $\mathscr{M}(2, D, c_2)(resp.\mathscr{M}^{\mathrm{tf}}(2, D, c_2)$ by $\mathscr{M}(v)(resp.\mathscr{M}^{\mathrm{tf}}(v))$ where, $v := (2, dH, \frac{d^2H^2}{2} - c_2 + 2)$ (we always assume v satisfies the assumption of the theorem 0.1.). And, We denote Gieseker-semistable part and μ-semistable part $\mathscr{M}^{\mathrm{tf}}(v)$ by $\mathscr{M}^{\mathrm{ss}}(v)$ or $\mathscr{M}^{\mu\mathrm{ss}}(v)$(they become open substacks). p, p' are points of a topological spaces, we denote $p \rightsquigarrow p'$ by $\overline{\{p\}} \ni p'$.

2.1 Irreducibility of substacks of the moduli stacks of torsion free sheaves

Definition 2.1. Let H_0 be an ample divisor on X, and $v \in \mathbb{Z} \bigoplus \mathrm{Pic(X)} \bigoplus \mathbb{Z}$. For $v := (r, dH, a)$,

$$\deg_{H_0}(v) := dH.H_0$$

Definition 2.2. As a full substack of $\mathscr{M}^{\mathrm{tf}}(2, D, c_2)$, we define $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ to be the category having the following objects , where $v_i = (r_i, d_iH, a_i) \in \mathbb{Z} \bigoplus \mathrm{Pic(X)} \bigoplus \mathbb{Z}$, $i = 1, 2$ s.t. $\deg_H(v_1) > \deg_H(v_2)$ or $\deg_H(v_1) = \deg_H(v_2)$ and $a_1 > a_2$

- Objects : $(U, E) \in \mathrm{ob}\,\mathscr{M}^{\mathrm{tf}}(v)$ s.t. $\forall t \in U$, $^\exists(0 \subset E_1 \subset E_t)$: HN-filtration of E_t s.t. $v(E_1) = (r_1, d_1H|_{X_{k(t)}}, a_1)$, $v(E_t/E_1) = (r_2, d_2H|_{X_{k(t)}}, a_2)$

Lemma 2.3 ([25])**.** For $\langle v, v\rangle > 0$, $\mathscr{M}^{\mu\mathrm{ss}}(v)$ is an irreducible algebraic stack. □

Lemma 2.4. Let v_1, v_2 be elements of $\mathbb{Z}\bigoplus \mathrm{Pic}(X)\bigoplus \mathbb{Z}$ and we assume we can write them as 2.2. Then, $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ is a locally closed substack of $\mathscr{M}^{\mathrm{tf}}(v)$.

Proof . For X, the elements of $\mathbb{Z}\bigoplus \mathrm{Pic}(X)\bigoplus \mathbb{Z}$ and the Hilbert polynomials for H is 1 to 1 correspondence. Actually, for E : coherent sheaf of rank $= 2$, let $c_t(E) = 1+dHt+c_2t^2$ be the Chern polynomial of E. Then, we have $\mathrm{td}(X) = (1, 0, 2)$, $c_t(E) = 1+(d+2m)t+(m^2+dm+c_2)t^2$, $\mathrm{ch}(E(m)) = (\mathrm{rank}(E), (d+2m)H, (m^2H^2+mdH^2+\frac{d^2}{2}-c_2))$, so by Hirzebrugh-Riemann-Roch thoerem, $\chi(E, m) = H^2m^2+dH^2m+(\frac{d^2}{2}-c_2)H^2+2$. Therefore, when we give a polynomial, d, c_2 are uniquely determined. Then, from [8, Theorem 1.4], $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ is a locally closed substack. □

For the next lemma, we prepare some notation. $v \in \mathbb{Z}\bigoplus \mathrm{Pic}(X)\bigoplus \mathbb{Z}$, $\mathrm{Quot}_X(F, v) = \{F \twoheadrightarrow E \mid E : X\text{coherent sheaf s.t.} v(E) = v\}$, $R^{N,m}(v) = \{\varphi : \mathscr{O}_X(-m)^{\oplus N} \twoheadrightarrow E \in \mathrm{Quot}_X(\mathscr{O}_X(-m)^{\oplus N}, v) \mid H^0(\varphi(m)$ is an isomorphism, $H^p(X, E(m)) = 0 (\forall p > 0)\}$. In this case, $R^{N,m}(v) \subseteq \mathrm{Quot}_X(\mathscr{O}_X(-m)^{\oplus N}, v)$ is an open subscheme.

Lemma 2.5. $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ is an irreducible algebraic stack.

Proof . For the proof of this lemma, we consider the following stacks.

$$\mathscr{F}(v_1, v_2) = \{0 \subset E_1 \subset E \mid E \in \mathscr{M}(v), E_1, E/E_1 : \mu\text{-semistable}, v(E_1) = v_1, v(E/E_1) = v_2\}$$

Then, when $\deg_H(v_1) > \deg_H(v_2)$ or $\deg_H(v_1) = \deg_H(v_2)$ and $a_1 > a_2$, $|\mathscr{F}(v_1, v_2)| \cap |\mathscr{M}^{\mathrm{tf}}(v)| = |\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|$. Therefore, $|\mathscr{F}(v_1, v_2)| \cap |\mathscr{M}^{\mathrm{tf}}(v)| \subseteq |\mathscr{F}(v_1, v_2)|$: open subset, so it is enough to prove $|\mathscr{F}(v_1, v_2)|$ is irreducible. The following fact is useful to prove this.

Lemma 1 ([13])**.** Let $v_1, v_2 \in \mathbb{Z}\bigoplus \mathrm{Pic}(X)\bigoplus \mathbb{Z}$,

$$\mathscr{F}^n(v_1, v_2) := \left\{ 0 \subset E_1 \subset E \in \mathscr{F}(v_1, v_2) \;\middle|\; \begin{array}{c} E \in \mathscr{M}(v), E_1, E/E_1 : \mu\text{-semistable} \\ v(E_1) = v_1, v(E/E_1) = v_2, \hom(E_1, E/E_1) = n \end{array} \right\}$$

$$R^n(v_1, v_2) := \{(E_1, E_2) \in R^{N',m'}(v_1) \times R^{N'',m''}(v_2) \mid E_1, E_2 : \mu\text{-semistable}, \hom(E_1, E_2) = n\}$$

, where N', N'', m', m'' are non negative integers s.t. $R^{N',m'}(v_1) \cap \mathscr{M}^{\mu\mathrm{ss}}(v) \to \mathscr{M}^{\mu\mathrm{ss}}(v_1)$, $R^{N'',m''}(v_2) \cap \mathscr{M}^{\mu\mathrm{ss}}(v_2) \to \mathscr{M}^{\mu\mathrm{ss}}(v'')$: surjective. Then, there exists a vector bundle Y^n on $R^n(v_1, v_2)$ and an algebraic group G^n acting on this s.t.

$$\mathscr{F}^n(v_1, v_2) \simeq [Y^n/G^n]$$

□

Now, if $\deg_H(v_1) > \deg_H(v_2)$ or $\deg_H(v_1) = \deg_H(v_2)$ and $a_1 > a_2$, then $\hom(E_1, E/E_1) = 0$. For, if $\hom(E_1, E/E_1) \neq 0$, we have $0 \neq \exists\phi : E_1 \to E/E_1$. And E/E_1 : torsion free, so $0 \neq \mathrm{Im}(\phi)$ implies $\mathrm{rank}(\mathrm{Im}(\phi)) = 1$. Therefore, $\mathrm{rank}(\mathrm{Ker}(\phi) = 0$. In the same way, because E_1 : torsion free, so we get $\mathrm{Ker}(\phi) = 0$ and E_1 is a subsheaf of E/E_1. Then, from the assumption we have E/E_1 is stable, $p(E_1) > p(E/E_1)$ contradicts this. Therefore, $\hom(E_1, E/E_1) = 0$. Moreover, in this case, $\mathscr{F}(v_1, v_2) \simeq \mathscr{F}^0(v_1, v_2)$. So, if we prove Y^0 is irreducible, from the above lemma, we can get $\mathscr{F}^0(v_1, v_2) \simeq [Y^0/G^0]$ and Y^0 is an atlas of $[Y^0/G^0]$. In particular, from the fact $Y^0 \to [Y^0/G^0]$ is surjective, we get $[Y^0/G^0]$ is irreducible.

Next, we prove that $R^{N',m'}(v_1)^{\mu ss}, R^{N'',m''}(v_2)^{\mu ss}$ are irreducible. In particular, it is enough to see $R^{N',m'}(v_1)^{\mu ss}$. Now, let $v_1 := (r_1, d_1 H, a_1)$, the following morphism is isomorphism.

$$\begin{array}{ccccc} \otimes \mathscr{O}_X(-d_1) & : & R^{N',m'}(v_1)^{\mu ss} & \longrightarrow & R^{N',m'+d_1}((1,0,-a_1 - \frac{d_1^2 H^2}{2}))^{\mu ss} \\ & & \cup & & \cup \\ & & (\mathscr{O}_X(-m)^{\oplus N} \twoheadrightarrow E) & \longmapsto & (\mathscr{O}_X(-m-d_1)^{\oplus N} \twoheadrightarrow E(-d_1)) \end{array}$$

Then, the moduli scheme $M(1, \mathscr{O}_X, a_1 + \frac{d_1^2 H^2}{2})$ is a quotient of $R^{N',m'+d_1}((1,0,-a_1 - \frac{d_1^2 H^2}{2}))^{\mu ss}$ by an action of $\mathrm{PGL}(N)$. In addition, $\pi : R^{N',m'+d_1}((1,0,-a_1 - \frac{d_1^2 H^2}{2}))^{\mu ss} \to M(1, \mathscr{O}_X, a_1 + \frac{d_1^2 H^2}{2})$ is a principal $\mathrm{PGL}(N)$-bundle and open map(because this is a quotient map). And, this moduli scheme is isomorphic to a Hilbert scherme of points on X. i.e., $M(1, \mathscr{O}_X, a_1 + \frac{d_1^2 H^2}{2}) \simeq \mathrm{Hilb}^{a_1 + \frac{d_1^2 H^2}{2}}(X)$. At last, from [10, Theorem 6.A.1], we have $\mathrm{Hilb}^{a_1 + \frac{d_1^2 H^2}{2}}(X)$ is irreducible and, we can apply the following lemma.

Lemma 2 ([1])**.** $f : X \to Y$ is open and surjective morphism and any fiber is irreducible at any closed point. Then,

$$Y : \text{irreducible} \Rightarrow X : \text{irreducible}$$

Therefore, we have $R^{N',m'+d_1}((1,0,-a_1 - \frac{d_1^2 H^2}{2}))$ is irreducible, so $R^{N',m'}(v_1)$ is irreducible. Therefore, $R^{N',m'}(v_1) \times R^{N'',m''}(v_2)$ is irreducible, and Y^0 is a vector bundle on this, so this is also irreducible. □

At the end of this subsection, we mention to irreducibility of $\mathscr{M}^{ss}(v)$ not in the case $\langle v, v \rangle > 0$. This is necessary to the proof of Theorem 0.1.

Lemma 2.6. Let v : primitive and $\langle v, v \rangle = 0, -2$. Then, $\mathscr{M}^{ss}(v)$ is an irreducible algebraic stack.

Proof . At first, v is primitive, so all semistable sheaves are stable. Let $M(v)$ be the moduli scheme of stable sheaves whose mukai vector is v. In the same way, $\exists N, m \geq 0$ s.t. $R^{N,m}(v)^{ss} \to M(v)$ is a principal $\mathrm{PGL}(N)$ bundle and an open map, and from [10] and [26] $M(v)$ is not empty and irreducible. So from 2, $R^{N,m}(v)^{ss}$ is also irreducible. Surjectivity of $R^{N,m}(v)^{ss} \to \mathscr{M}^{ss}(v)$ implies irreducibility of $\mathscr{M}^{ss}(v)$. □

2.2 Determination of the irreducible components of $\mathscr{M}^{tf}(v)$

Lemma 2.7. (i) We define $R^{N,m}(v)^{tf} \subseteq R^{N,m}(v)$ to be $R^{N,m}(v) \times_{\mathscr{M}(v)} R^{N,m}(v)^{tf}$. Then, $[R^{N,m}(v)^{tf}/\,\mathrm{GL}(N)]$ is an open immersion of $\mathscr{M}(v)^{tf}$.

(ii) In the same way, we define $R^{N,m}_{(v_1,v_2)}(v) \subseteq R^{N,m}(v)$ to be $R^{N,m}(v) \times_{\mathscr{M}(v)} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \simeq R^{N,m}(v) \times_{\mathscr{M}^{\mathrm{tf}}(v)} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$. Then, $[R^{\mathrm{N,m}}_{(v_1,v_2)}(v)/\operatorname{GL}(N)]$ is an open immersion of $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v))$. Moreover, $R^{N,m}_{(v_1,v_2)}(v) \subset R^{N,m}(v)^{\mathrm{tf}}$ is the set torsion free sheaf whose HN-type is (v_1, v_2) as a set.

(iii) There exist the following fiber products.

$$\begin{array}{ccccc} R^{N,m}(v)^{\mathrm{tf}} & \longrightarrow & [R^{N,m}(v)^{\mathrm{tf}}/\operatorname{GL}(N)] & \longrightarrow & \mathscr{M}^{\mathrm{tf}}(v) \\ \downarrow & \square & \downarrow & \square & \downarrow \\ R^{N,m}(v) & \longrightarrow & [R^{N,m}(v)/\operatorname{GL}(N)] & \longrightarrow & \mathscr{M}(v) \end{array}$$

$$\begin{array}{ccccc} R^{N,m}_{(v_1,v_2)}(v) & \longrightarrow & [R^{N,m}_{(v_1,v_2)}(v)/\operatorname{GL}(N)] & \longrightarrow & \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \\ \downarrow & \square & \downarrow & \square & \downarrow \\ R^{N,m}(v)^{\mathrm{tf}} & \longrightarrow & [R^{N,m}(v)^{\mathrm{tf}}/\operatorname{GL}(N)] & \longrightarrow & \mathscr{M}(v) \end{array}$$

Proof . For (i),

Lemma 3 ([11])**.** $[R^{\mathrm{N,m}}(v)/\operatorname{GL}(N)] \to \mathscr{M}(v)$ is an open immersion.

Therefore, it is sufficient to prove the existence of the first fiber product of (iii). Actually, for (iii), from the property of fiber products it is sufficient to show the existence of

$$\begin{array}{ccccc} R^{N,m}(v)^{\mathrm{tf}} & \longrightarrow & [R^{N,m}(v)/\operatorname{GL}(N)] \times_{\mathscr{M}(v)} \mathscr{M}^{\mathrm{tf}}(v) & \longrightarrow & \mathscr{M}^{\mathrm{tf}}(v) \\ \downarrow & \square & \downarrow & \square & \downarrow \\ R^{N,m}(v) & \longrightarrow & [R^{N,m}(v)/\operatorname{GL}(N)] & \longrightarrow & \mathscr{M}(v) \end{array}$$

Then, all vertical arrows are open immersions. Especially, from the middle arrow and the property of quotients stacks, we can show $[R^{N,m}(v)/\operatorname{GL}(N)] \times_{\mathscr{M}(v)} \mathscr{M}(v)^{\mathrm{tf}} \simeq [S/\operatorname{GL}(N)]$, ($\exists S \subset R^{N,m}(v)$: $\operatorname{GL}(N)$-invariant open subscheme), In general, we have the following bijective correspondence. ([2],[21])

$$\begin{array}{ccc} (\text{locally closed substacks of } [R^{N,m}(v)/\operatorname{GL}(N)]) & \longrightarrow & (\operatorname{GL}(N)\text{-invariant locally closed subschemes of } R^{N,m}) \\ \Cup & & \Cup \\ \mathscr{Y} & \longmapsto & \mathscr{Y} \times_{[R^{N,m}(v)/\operatorname{GL}(N)]} R^{N,m} \\ [S/\operatorname{GL}(N)] & \longleftarrow\!\!\shortmid & S \end{array}$$

Lemma 4 ([14])**.** The following is a cartesian product of algebraic stacks.

$$\begin{array}{ccc} \mathscr{Y}' & \longrightarrow & \mathscr{X}' \\ \downarrow & \square & \downarrow \\ \mathscr{Y} & \longrightarrow & \mathscr{X} \end{array}$$

then, the natural morphism $|\mathscr{Y}'| \to |\mathscr{Y}| \times_{|\mathscr{X}|} |\mathscr{X}'|$ is surjective.

Definition 2.8 ([21])**.** Let $\mathscr{X}$ be an algebraic stack, $T \subseteq |\mathscr{X}|$ be a closed subset. Then, there exists a unique closed substack $\mathscr{Z} \subseteq \mathscr{X}$ s.t. $|\mathscr{Z}| = T$ and $\mathscr{Z}$ is a reduced stacks. Then, we denote $\mathscr{Z}$ by T_{red}.

Lemma 2.9. (i) Let $(\overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|})_{\mathrm{red}} \subseteq \mathscr{M}(v)^{\mathrm{tf}}$ be the reduced indeced closed substack of $|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|$ in $|\mathscr{M}^{\mathrm{tf}}(v)|$. In the same way, we denote $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)_{\mathrm{red}}$ the reduced induced closed substack of $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$. Then,

$$\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)_{\mathrm{red}} \to (\overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|})_{\mathrm{red}} : \text{open immersion}$$

(ii) $\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)_{\mathrm{red}} = \dim(\overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|})_{\mathrm{red}}$

Proof . (i) We consider the following diagram.

$$\begin{array}{ccc}
\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) & \xrightarrow{\text{locally closed}} & \mathscr{M}(v)^{\mathrm{tf}} \\
\text{closed}\uparrow & & \uparrow\text{closed} \\
\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)_{\mathrm{red}} & \xrightarrow{\exists} & (\overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|})_{\mathrm{red}} \\
 & & \uparrow\textit{open} \\
 & & \mathscr{U}
\end{array}$$

where, $|\mathscr{U}| = |\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|$. Moreover, we get the second row morphism from [21, Lem97.10.2]. Now, we pullback respectively $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)_{\mathrm{red}}$ and $\mathscr{U}$, by $\coprod_{N,m\geq 0} R^{N,m}(v)^{\mathrm{tf}} \to$ and, by $\mathscr{M}(v)^{\mathrm{tf}}$, to $R^{N,m}(v)$. Then, $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)_{\mathrm{red}} \times_{\mathscr{M}^{\mathrm{tf}}(v)} R^{N,m}(v)$ and $\mathscr{U} \times_{\mathscr{M}^{\mathrm{tf}}(v)} R^{N,m}(v)$ are the same reduced locally closed subschemes in $R^{N,m}$, so $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)_{\mathrm{red}}$ and $\mathscr{U}$ correspond. Therefore, we get (i).

(ii) It is sufficient to prove the following claim.

Claim . Let $\mathscr{X}$ be a reduced irreducible algebraic stack, $\mathscr{U} \subseteq \mathscr{X}$ be an open substack(then, $\overline{|\mathscr{U}|} = \mathscr{X}$ holds.). And, we assume $\dim_p(\mathscr{U}) = \text{constant}(\forall p \in \mathscr{U})$ and we ca choose a locally noetherian scheme as an atlas $\mathscr{X}$. Then,

$$\dim(\mathscr{U}) = \dim(\mathscr{X})$$

Proof of Claim . We suppose $q \in |\mathscr{X}| - |\mathscr{U}|$. Then, from the assumption $\exists f : X \to \mathscr{X}$ s.t. f is smooth, $q' \in X \simeq |X|$ with $|f|(q') = q$.We consider the following cartesian product.

$$\begin{array}{ccc}
U := \mathscr{U} \times_{\mathscr{X}} X & \xrightarrow{\text{open im}} & X \\
\text{sm}\downarrow f' & \square & f\downarrow sm \\
\mathscr{U} & \xrightarrow[\text{open im}]{} & \mathscr{X}
\end{array}$$

Moreover, let $p \in |\mathscr{U}|$ with $\overline{\{p\}} = |\mathscr{X}|$. Then, $\exists p' \rightsquigarrow q'$ s.t. $p' \mapsto p$, p' is the generic point of an irreducible component of X. Then, we have $\dim_{p'}(f) = \dim_{q'}(f)$ (because relative dimension of the morphism is locally constant.).

$$\begin{aligned}\dim_q(\mathscr{X}) &= \dim_{q'}(X) - \dim_{q'}(f) \\ &= \dim_{q'}(X) - \dim_{p'}(f)\end{aligned}$$

Then, let r be the generic point of an irreducible component which contains q' and whose dimension is the larger than that of any other irreducible component containing q', then $q = |f| = (q') \in |f|(\overline{\{r'\}}) \subseteq \overline{(|f|(r'))}$. Moreover, we suppose $|f|(r') := r$, then we have $p \rightsquigarrow r$. So we get $r = p$ because of maximality of r' for inclusion relationship. And,

$$\begin{aligned}\dim_p(\mathscr{U}) &= \dim_p(U) - \dim_p(f') \\ &= (\dim_p(X) - \dim_p(U \hookrightarrow X)) - \dim_p(f) \\ &= \dim_p(X) - \dim_p(f) \\ &= \dim_p(\mathscr{X})\end{aligned}$$

so, $\dim_p(\mathscr{U}) = opdim_p(\mathscr{X})$, we get

$$\begin{aligned}\dim_p(\mathscr{U}) = \dim_p(\mathscr{X}) &= \dim_{r'}(X) - \dim_{r'}(f) \\ &= \dim_{q'}(X) - \dim_{q'}(f) \\ &= \dim_q(\mathscr{X})\end{aligned}$$

Therefore, $\dim(\mathscr{U}) = \dim(\mathscr{X})$ □

We mention to the fact which is about $\dim(\mathscr{M}^{\mathrm{ss}}(v))$ and $\dim(\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v))$ and necessary to the proof of the next lemma.

Lemma 5 ([12],[13])**.** (i) $\dim(\mathscr{M}^{\mathrm{ss}}(v))= \langle v, v\rangle + 1$

(ii) $\dim(\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)) = \langle v_1, v_1\rangle + \langle v_2, v_2\rangle + \langle v_1, v_2\rangle + 2$

Remark 2.10. To be accurate, in the above lemma, H must be general for v, v_1, v_2([12],[13]) , but in this case, the Picard number is 1, so the condition holds regardless of v, v_1, v_2. Moreover, in general, (i) of the above lemma holds in $\langle v, v\rangle > 0$ or v : primitive. In the condition of the Theorem 0.1, this necessarily holds. Note that because the ranks of v_1, v_2 are 1, v_1, v_2 are always primitive.

Lemma 2.11. $v_1, v_2, v_1', v_2' \in \mathbb{Z} \bigoplus \mathrm{Pic(X)} \bigoplus \mathbb{Z}$ such that $v_1 \neq v_1'$ or $v_2 \neq v_2'$. Then, $|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)| \nsubseteq |\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|$

Proof . Suppose $\overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|} \subseteq \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$. By (2.5), Let p, p' be respectively the generic points of $\overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$ and $\overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$, $\exists N, m \in \mathbb{Z}_{\geq 0}$ s.t. $|R^{N,m}_{(v_1,v_2)}(v)| \to \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$: dense, i.e., $R^{N,m}_{(v_1,v_2)}(v) \ni \exists q \mapsto p \in\to \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$. Then, we can think of q as the generic point of $R^{N,m}_{(v_1,v_2)}(v)$ (otherwise, we take a maximal point of general points of it.). Then, let $\overline{\{q\}} =: V \subseteq R^{N,m}_{(v_1,v_2)}(v)$,

Claim . $\dim(V) = \dim(\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)) + N^2$

Proof of Claim . From (2.7), $[R^{N,m}_{(v_1,v_2)}(v)/\operatorname{GL}(N)] \hookrightarrow \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ is an open immersion. In addition, from 2.5, $\dim_{p'}(\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v))$: independent of the choice of $p' \in (\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v))$. Therefore,

$$\begin{aligned}\dim_{p'}[R^{N,m}_{(v_1,v_2)}(v)/\operatorname{GL}(N)] &= \dim_{p'}(\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v))\\ &= \dim(\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v))\end{aligned}$$

and,

$$\begin{aligned}\dim_{p'}[R^{N,m}_{(v_1,v_2)}(v)/\operatorname{GL}(N)] &= \dim_{p'}(R^{N,m}_{(v_1,v_2)}(v)) - N^2\\ &= \dim(V) - N^2\end{aligned}$$

t Therefore we get the claim. □

We return to the proof of the lemma. Here, because $p' \rightsquigarrow p$[14], $\exists q' \in R^{N,m}_{(v'_1,v'_2)}(v)$ s.t. $R^{N,m}_{(v'_1,v'_2)}(v) \ni q' \mapsto p' \in \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$, $q' \rightsquigarrow q$ in $R^{N,m}(v)^{\mathrm{tf}}$ where, we can think of q' as the generic point of $R^{N,m}_{(v'_1,v'_2)}(v)$, let $\overline{\{q'\}} =: V' \subseteq R^{N,m}_{(v'_1,v'_2)}(v)$, we have $\overline{V} \subseteq \overline{V'} \subseteq R^{N,m}(v)^{\mathrm{tf}}$, so $\dim(\overline{V}) = \dim(V)$, $\dim(\overline{V'}) = \dim(V')$. Because $\overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|} \subseteq \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$, $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \neq \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$, we have $\dim(V') > \dim(V)$. (if $\dim(V') = \dim(V)$, $\overline{V'} = \overline{V}$. This is a contaradiction because this two closed sets have different generic points.) Let $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$, and let the corresponding extension of E be the following.

$$0 \longrightarrow I_{Z_1}(m) \longrightarrow E \longrightarrow I_{Z_2}(n-m) \longrightarrow 0$$

where, $v(I_{Z_1}(m)) = v_1$, $v(I_{Z_2}(n-m)) = v_2$. Then,

$$\begin{aligned}\dim(\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)) &= \langle v_1, v_1\rangle + \langle v_2, v_2\rangle + \langle v_1, v_2\rangle + 2\\ &= \langle (1, mH, \frac{m^2H^2}{2} - l(Z_1) + 1)^2\rangle + \langle (1, (n-m)H, \frac{(n-m)^2H^2}{2} - l(Z_2) + 1)^2\rangle\\ &+ \langle (1, mH, \frac{m^2H^2}{2} - l(Z_1) + 1), (1, (n-m)H, \frac{(n-m)^2H^2}{2} - l(Z_2) + 1)\rangle + 2\\ &= 3(l(Z_1) + l(Z_2)) - 4 + \{m(n-m) - \frac{m^2H^2}{2} - \frac{(n-m)^2H^2}{2}\}\\ &= (3c_2 - 3m(n-m)H^2) - 4 - \frac{n^2H^2}{2} + 2mnH^2 - 2m^2H^2\\ &= (3c_2 - 4 - \frac{n^2H^2}{2}) - m(n-m)H^2\\ &= H^2(m - \frac{n}{2})^2 + (3c_2 - 4 - \frac{3n^2H^2}{4})\end{aligned}$$

However, from [20], $p \mapsto$ HNP(p) is upper semicontinuous. Because, from $p' \rightsquigarrow p$, HNP(p) $\geq$ HNP(p'), we must have $\dim(V) \geq \dim(V')$. This is a contradiction. Therefore, we get the claim. □

[1]From this, we have $R^{N,m}_{(v_1,v_2)}(v)$ is equidimensional.

2.3 Proof of Theorem 0.1 and Corollary 0.3

Lemma 6 ([5])**.** Let $\mathscr{X}$ be a locally noetherian algebraic stack and $x \in |\mathscr{X}|$. Then, $\dim_x(\mathscr{X}) = sup_{\mathscr{T}}(\dim_x(\mathscr{T}))$, where $\mathscr{T}$ is an irreducible component of $|\mathscr{X}|$ passing through x, and we have areduced induced closed substack structure as a stack

Remark 2.12. In particular , in the above condition, if $\mathscr{X}$ is irreducible, about $\forall x \in |\mathscr{X}|$, we have $\dim_x(\mathscr{X}_{\mathrm{red}}) = \dim_x(\mathscr{X})$

Lemma 7 ([12])**.** The dimension of any irreducible component of $\dim(\mathscr{M}(v))$ is the above of $\langle v, v\rangle + 1$.

Lemma 8 ([5],[21])**.** $\mathscr{X}$ is a pseudo-catenary, jacobson, locally noetherian algebraic stack. We assume $|\mathscr{X}|$ is irreducible. Then, $\dim_x(\mathscr{X})$ is constant for $x \in |\mathscr{X}|$.

Remark 2.13. (i) any algebraic stack locally of finite type over a field holds the assumption of 8.

(ii) from 7, 8, $\dim_x(\mathscr{M}^{\mathrm{tf}}(v)) \geq \langle v, v\rangle + 1$ $(\forall x \in |\mathscr{M}^{\mathrm{tf}}(v)|)$

(iii) we can also prove (ii) of 2.9 by using 8 and (i) of 2.9.

Proof of Theorem 0.1 and Corollary 0.3 . At first, note that $|\mathscr{M}^{\mathrm{tf}}(v)|$ are a disjoint union of $|\mathscr{M}^{\mathrm{ss}}(v)|$ and $|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|$. In the case (a) and (c), under the notation of Theorem 0.1, if $\langle v_1, v_2\rangle \leq 1$,

$$\begin{aligned}\dim(\mathscr{M}^{\mathrm{ss}}(v)) &= \langle v, v\rangle + 1\\ &= \langle v_1, v_1\rangle + \langle v_2, v_2\rangle + 2\langle v_1, v_2\rangle + 1\\ &= \dim(\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)) + \langle v_1, v_2\rangle - 1\end{aligned}$$

Therefore, as with (2.11), $\overline{|\mathscr{M}^{\mathrm{ss}}(v)|} \not\supseteq \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$ and $\overline{|\mathscr{M}^{\mathrm{ss}}(v)|} \not\subseteq \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$. And, under the notation of Theorem 0.1, $\langle v_1, v_2\rangle - 1 = c_2 - \frac{n^2H^2}{2} + 3$. So, note that the map that $E \in \mathscr{M}^{\mathrm{tf}}(v)$ corresponds to its HN-polygon for Gieseker stability is also upper semicontinuous,([18]) in the same way, we get $\overline{|\mathscr{M}^{\mathrm{ss}}(v)|} \not\supseteq \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$ and $\overline{|\mathscr{M}^{\mathrm{ss}}(v)|} \not\subseteq \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$. Conversely, when $\langle v_1, v_2\rangle > 1$, we suppose $\overline{|\mathscr{M}^{\mathrm{ss}}(v)|} \not\supseteq \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$, $\forall x \in \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$, we have $\dim_x(\mathscr{M}^{\mathrm{tf}}(v)) < \langle v, v\rangle + 1$, this contradicts to (2.13), so $\overline{|\mathscr{M}^{\mathrm{ss}}(v)|} \supseteq \overline{|\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)|}$. Therefore, from (2.3), (2.5), (2.6), (2.11), the irreducible components and their general points of $\mathscr{M}^{\mathrm{tf}}(v)$ can be written as the statement of Theorem 0.1. For their dimensions(i.e. for Corollary 0.3), from (5) and (2.9), they are also written as the statement. In the case of (b), from [26, Thm0.1], we have $|\mathscr{M}^{\mathrm{ss}}(v)| = \emptyset$. Therefore we can prove the claim in the same way. $\square$

References

[1] V Balaji, Arijit Dey, and R Parthasarathi. Parabolic bundles on algebraic surfaces i—the donaldson-uhlenbeck compactification. *Proceedings Mathematical Sciences*, 118(1):43–79, 2008.

[2] José Bertin. Algebraic stacks with a view toward moduli stacks of covers. In Pierre Dèbes, Michel Emsalem, Matthieu Romagny, and A. Muhammed Uludağ, editors, *Arithmetic and Geometry Around Galois Theory*, pages 1–148, Basel, 2013. Springer Basel.

[3] Izzet Coskun and Jack Huizenga. The moduli spaces of sheaves on surfaces, pathologies and brill-noether problems. In *The Abel Symposium*, pages 75–105. Springer, 2017.

[4] Laura Costa and Rosa Maria Miró-Roig. Brill-noether theory for moduli spaces of sheaves on algebraic varieties. In *Forum Mathematicum*, volume 22, pages 411–432. Walter de Gruyter GmbH & Co. KG, 2010.

[5] Matthew Emerton and Toby Gee. Dimension theory and components of algebraic stacks. *arXiv preprint arXiv:1704.07654*, 2017.

[6] Barbara Fantechi, Lothar Gottsche, and Luc Illusie. *Fundamental algebraic geometry: Grothendieck's FGA explained*. Number 123. American Mathematical Soc., 2005.

[7] L Göttsche and D Huybrechts. Hodge numbers of moduli spaces of stable bundles on K 3 surfaces. *International Journal of Mathematics*, 7(3):359–372, 1996.

[8] Daniel Halpern-Leistner. θ-stratifications, θ-reductive stacks, and applications. *Algebraic Geometry: Salt Lake City 2015*, 97:349, 2018.

[9] Daniel Huybrechts. *Lectures on K3 Surfaces*, volume 158. Cambridge University Press, 2016.

[10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge University Press, 2010.

[11] Dominic Joyce and Yinan Song. A theory of generalized donaldson-thomas invariants. *arXiv preprint arXiv:0810.5645*, 2008.

[12] Masanori Kimura and Kōta Yoshioka. Birational maps of moduli spaces of vector bundles on $k3$ surfaces. *Tokyo J. Math.*, 34(2):473–491, 12 2011.

[13] Koichi Kurihara and Kōta Yoshioka. Holomorphic vector bundles on non-algebraic tori of dimension 2. *manuscripta mathematica*, 126(2):143–166, 2008.

[14] G Laumon. Moret-bailly, champs algébriques, ergeb. math., vol. 39, 2000.

[15] Maxim Leyenson. On the brill-noether theory for k3 surfaces. *Open Mathematics*, 10(4):1486–1540, 2012.

[16] Shigeru Mukai. On the moduli space of bundles on k3 surfaces. i. *Vector bundles on algebraic varieties (Bombay, 1984)*, 11:341–413, 1984.

[17] Shigeru Mukai. Symplectic structure of the moduli space of sheaves on an abelian or k3 surface. *Inventiones mathematicae*, 77(1):101–116, 1984.

[18] Nitin Nitsure. Schematic harder–narasimhan stratification. *International Journal of Mathematics*, 22(10):1365–1373, 2011.

[19] Christian Okonek, Michael Schneider, Heinz Spindler, and Sergeĭ Gelfand. *Vector bundles on complex projective spaces.*

[20] Stephen S Shatz. The decomposition and specialization of algebraic families of vector bundles. *Compositio Mathematica*, 35(2):163–187, 1977.

[21] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.

[22] Stein Arild Strømme. Deforming vector bundles on the projective plane. *Mathematische Annalen*, 263(3):385–397, 1983.

[23] Charles H Walter. Components of the stack of torsion-free sheaves of rank 2 on ruled surfaces. *Math. Ann*, 301:699–715, 1995.

[24] Kōta Yoshioka. Some examples of mukai's reflections on k3 surfaces. *J. Reine Angew. Math.*, 515:97–123, 1999.

[25] Kōta Yoshioka. Twisted stability and fourier–mukai transform i. *Compositio Mathematica*, 138(3):261–288, 2003.

[26] Kōta Yoshioka. Moduli spaces of vector bundles on algebraic surfaces. *Suugaku*, 56(3):225–247, 2004.