A DUALITY PROPERTY FOR INVERTIBLE WITT SHEAVES

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Introduction

Ordinary Kodaira Vanishing is closely related to the Hodge decomposition. In the context of Witt Sheaves there is an analogue to the Hodge decomposition, the slope decomposition of crystalline cohomology. This motivated Tanaka [Tan18] to investigate Kodaira-like vanishing properties in the same context.

Theorem 0.1 (Tanaka, cf. [Tan18, Theorem 1.1])**.** *Let k be a perfect field of characteristic $p > 0$, and X be an N–dimensional smooth projective variety over k. Let $\mathscr{A}$ be an ample invertible sheaf, $\underline{\mathscr{A}}$ its Teichmüller lift and $W\Omega_X^\bullet$ the de Rham-Witt complex. Then*

(i)
- $H^j(X, \underline{\mathscr{A}}^{-s}) = 0$ *for any* $s \gg 0, j < N$,
- $H^j(X, \underline{\mathscr{A}}^{-1}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = 0$ *for any* $j < N$,

(ii)
- $H^i(X, W\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{A}}) = H^i(X, W\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{A}}^s) = 0$ *for any* $s, i > 0$,

We shall show a Serre-type duality property in this context.

1. Preliminaries

1.1. **Notation.** We will be using the following notations and definitions:

- Throughout this paper we define $X \xrightarrow{\phi} S = \operatorname{Spec} k$, where k is a perfect field of characteristic $p > 0$.
- If A is a commutative ring, $W(A)$ denotes the ring of Witt vectors. As a set, $W(A) = A^{\mathbb{N}}$. The truncated Witt vectors are denoted $W_n(A)$.
- $W\mathscr{O}_x$ (resp. $W_n\mathscr{O}_X$) denotes the sheaf of (truncated) Witt-vectors, and WX (resp. W_nX) denotes the scheme $(X, W\mathscr{O}_X)$ (resp. $(X, W_n\mathscr{O}_X)$).
- F_X denotes the absolute Frobenius morphism on X, induced by the Frobenius automorphism on W. We may also simply write F.
- V denotes the Verschiebungs map
$$W(A) \xrightarrow{V} W(A)$$
$$V(a_0, a_1, \cdots) = (0, a_0, a_1, \cdots).$$
- The Teichmüller lift of a line bundle on X is defined by the Teichmüller characters $\underline{f_{ji}}$ of its defining transition functions f_{ji}. The Teichmüller character of an element $a \in A$ is $(a, 0, 0, \cdots) \in W(A)$. $\underline{\mathscr{F}}_{\leq n} := W_n\mathscr{O}_X \otimes_{W\mathscr{O}_X} \underline{\mathscr{F}}$.
- If C is a complex of modules, $C[i]$ will denote the shift of C by i.
- If M_n is an inverse system, then $\lim_n$ will denote the inverse limit.

2. Duality Theorems

In what follows, let X be a smooth projective k-variety, with k a perfect field of positive characteristic.

Proposition 2.1. *Let $\mathscr{F}$ be an invertible $\mathscr{O}_X$-module. For any $n > 0$,*

$$W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n} \cong R\,\mathcal{H}om_{W_n\mathscr{O}_X}(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega^N_X).$$

Define ω to be the W-algebra generated by V, subject to the relation

$$aV = VF(a), a \in W.$$

ω is a non-commutative ring, and it has an evident W-module structure. Let $\omega_n := \omega/V^n\omega$, which is a (W, ω)-bimodule, since $V^n\omega$ is a sub-left-W-module of ω and a right-ω-ideal generated by V^n. As sets (and in fact as left–W–modules), $\omega \cong \bigoplus_i WV^i$.

Proposition 2.2. *Let A be a k-algebra. Then $W(A)$ has a natural structure of left-ω-modules and there is an isomorphism of left-W-modules*

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma(\underline{\mathscr{F}}) \cong R\Gamma(\underline{\mathscr{F}}_{\leq n})$$

We find that $R\lim_n \mathcal{H}om_{W_n}(\omega_n, W_n) \cong \prod_i WV^i =: \breve{\omega}$. This and Proposition 2.2 lead to

Theorem 2.3. *Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$. Then for any invertible $\mathscr{O}_X$-module $\mathscr{F}$ on X,*

$$R\Gamma(W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}^\vee) \cong R\operatorname{Hom}_\omega(R\Gamma(\underline{\mathscr{F}}), \breve{\omega}[-N]).$$

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