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A duality property for Witt sheaves

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A DUALITY PROPERTY FOR INVERTIBLE WITT SHEAVES

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ABSTRACT. We adapt ideas from Ekedahl [Eke84] to prove a Serre-type duality property for locally free sheaves of $W\mathscr{O}_X$–modules, and attempt to illuminate its relation to Tanaka's vanishing theorem [Tan18].

CONTENTS

INTRODUCTION

In [Eke84], Ekedahl introduces a certain duality functor D, and eventually constructs an isomorphism ([Eke84, Theorem III: 2.9])

$$D(R\Gamma(W\Omega_X^\bullet))(-N)[-N] \cong R\Gamma(W\Omega_X^\bullet),$$

where $(-N)$ and $[-N]$ denote shifts in module and complex degree, respectively. He then shows that

$$D(R\Gamma(W\Omega_X^\bullet)) \cong R\operatorname{Hom}_R(R\Gamma(W\Omega_X^\bullet), \check{R}),$$

with the isomorphisms lying in $D(R)$, where R is the Raynaud Ring (which is a non-commutative $W\mathscr{O}_X$-algebra), and $\check{R}$ is some R–bi–module. The biggest problem with Ekedahl's general result is how difficult it is to use, i.e. how to actually compute $D(W\Omega_X^\bullet)$ in cases of interest. Our result is achieved by introducing a non-commutative ring ω based on similar ideas.

Theorem 0.1 (Cf. Theorem 3.6)**.** *Let k be a perfect field of characteristic $p > 0$, X^N a smooth projective variety over k, and $\mathscr{F}$ an invertible sheaf of $\mathscr{O}_X$–modules. Then*

$$R\Gamma(W\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}^\vee) \cong R\,\mathcal{H}om_\omega(R\Gamma(\underline{\mathscr{F}}), \check{\omega}[-N]), \tag{0.1}$$

where $\check{\omega}$ is a certain ω–algebra.

1. Notation

We will be using the following notations and conventions:

- A variety over k is a separated integral scheme of finite type over k.
- Throughout this paper we define $X \xrightarrow{\phi} S = \operatorname{Spec} k$, where k is a perfect field of characteristic $p > 0$.
- $W\mathscr{O}_x$ (resp. $W_n\mathscr{O}_X$) denotes the sheaf of (truncated) Witt-vectors (cf. 2.1), and WX (resp. W_nX) denotes the scheme $(X, W\mathscr{O}_X)$ (resp. $(X, W_n\mathscr{O}_X)$).
- F_X denotes the absolute Frobenius morphism on X, induced by the Frobenius automorphism on W. Usually the scheme X is understood from context, in which case we may just write F.
- If C is a complex, $C[i]$ denotes C shifted by i in complex degree.
- If M_n is an inverse system, then $\lim_n M_n$ denotes the inverse limit.

2. Preliminaries

For the reader's convenience we will recall some definitions and results. References are, for example, Local Fields by Jean-Pierre Serre [Ser95], Luc Illusie's exposition on the de Rham-Witt complex [Ill79], and Hiromu Tanaka's 2017 paper proving a vanishing theorem of Witt sheaves [Tan18].

2.1. **Witt vectors.**

2.1.1. *Motivation from p-adic integers.* The Witt vectors can be naturally motivated using the example of the p-adic integers $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is the inverse limit of $(\mathbb{Z}/p^n\mathbb{Z})^{\mathbb{N}}$ along the quotient maps (taking modulo p^n), any element of $\mathbb{Z}_p$ can be identified with a sequence $(a_i)_i, a_i \in \mathbb{Z}/p^i\mathbb{Z}$ such that for $i \leq j, a_j \equiv a_i \mod p^i$. Addition and multiplication in this ring are the normal operations performed elementwise.

An element $a \in \mathbb{Z}_p$ can then also be uniquely written as a series

$$a = \sum_{0 \leq i} \alpha_i p^i,$$

where $\alpha_i \in \mathbb{Z}/p\mathbb{Z}$, and the elements of the sequence $(a_i)_i$ are the partial sums of this series. We can therefore write $a \in \mathbb{Z}_p$ as

$$a = (\alpha_0, \alpha_1, \cdots) = (\alpha_i)_i,$$

remembering that the ring structure is given by multiplication and addition of the partial sums $a_n = \sum_{0 \leq i \leq n} \alpha_i p^i$. This way, $\mathbb{Z}_p = (\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}$ as sets.

2.1.2. *Generalizing to Witt vectors.* This motivates the definition of Witt vectors $W(A)$ over a ring A: let A be a commutative ring of characteristic $p > 0$. A Witt vector $a \in W(A)$ over A is an infinite sequence $(\alpha_i)_i$, where $\alpha_i \in A$. The set of Witt vectors is endowed with ring operations via addition and multiplication of the sums

$$a_n = \sum_{0 \leq i \leq n} \alpha_i^{p^{n-i}} p^i.$$

A common notation for the above as a polynomial in the elements of a is $a^{(n)}$. Since they provide the ring structure, these polynomials are called the *ghost components* of a. That is, if $a, b \in W(A)$, then $a + b = c$ and $ab = d$ are such that

$$c^{(n)} = a^{(n)} + b^{(n)},$$
$$d^{(n)} = a^{(n)} b^{(n)}.$$

It becomes apparent that in $W(A)$, $1 = (1, 0, 0, \cdots)$ and $p = (0, 1, 0, 0, \cdots)$.

2.1.3. *The maps F, V, and R.* The Frobenius endomorphism σ of A naturally induces what we call the Frobenius endomorphism F of $W(A)$ by element-wise application:

$$W(A) \xrightarrow{F} W(A)$$
$$(a_i)_i \mapsto (\sigma(a_i))_i.$$

F is an isomorphism if and only if A is perfect. If A is imperfect, F is only injective.

The so-called *Verschiebungs-map* V is the (injective) shift of the terms of a Witt vector $a \in W(A)$:

$$W(A) \xrightarrow{V} W(A)$$
$$(a_0, a_1, \cdots) \mapsto (0, a_0, a_1, \cdots a).$$

Computation shows that $FV = VF = p$.

F and V on $W(A)$ naturally induce corresponding maps on $W_n(A)$. We therefore have exact sequences for any $m \geq n$.

$$0 \to W(A) \xrightarrow{V^n} W(A) \xrightarrow{R} W_n(A) \to 0,$$
$$0 \to W_m(A) \xrightarrow{V^n} W_{m+n}(A) \xrightarrow{R} W_n(A) \to 0,$$

with the quotient map R being the natural restriction. R makes $(W_n(A))_n$ into an inverse system, the limit of which is again $W(A)$.

The *truncated* Witt vectors $W_n(A)$ are the Witt vectors truncated after the nth element: $W_n(A) := W(A)/V^n(W(A))$. In particular $W_1(A) = A$. In other words, if σ is the identity on A, they are the Witt vectors modulo p^n, and if A is perfect, they are isomorphic to the Witt vectors modulo p^n via σ. For imperfect A, however, $W(A)/p^n W(A)$ is (in some sense) much larger than $W_n(a)$.

If k is our groundfield, we shall denote $W(k)$ and $W_n(k)$ by W and W_n, respectively. Since multiplication and addition of Witt vectors are simply those of the associated ghost components which are polynomials, if A is a k–algebra, $W(A)$ is a W–algebra in the natural way.

It is worth noting that while A and $W_n(A)$ are of positive characteristic, $W(A)$ is of characteristic zero (or more precisely of mixed characteristic, since multiplication by V is injective on $W(A)$, but $W(A)/V(W(A)) = A$). It is thus not entirely surprising that, after taking the limit, properties may differ from the truncated case.

2.1.4. *Teichmüller character.* If $a_0 \in A$ is an element in our base ring, we can identify it with an element in $W(A)$ naturally in the Witt vector notation by

$$A \xrightarrow{\omega} W(A)$$
$$a_0 \mapsto \underline{a}_0 := a_0 + 0p^1 + 0p^2 + \cdots = (a_0, 0, 0, \cdots).$$

This map is multiplicative, but *not* additive. ω is called the *Teichmüller character*, $\underline{a}_0$ the Teichmüller representative of a_0.

2.2. **The de Rham-Witt complex.** We may sheafify the notion of Witt vectors. In particular, for a k-scheme $(X, \mathscr{O}_X)$ define

$$W\mathscr{O}_X(U) := W(\mathscr{O}_X(U))$$

for any open subset $U \subset X$.

The terms $W_n\Omega^\bullet_X$ of the de Rham-Witt complex are defined iteratively in n as a quotient of $\Omega^\bullet_{W_n\mathscr{O}_X}$. We therefore begin by setting $W_n\Omega^\bullet_X = 0$ for any $n \le 0$ and $W_1\Omega^\bullet_X = \Omega^\bullet_X$.

Suppose then we have $(W_i\Omega^\bullet_X, R)_{i\le n}$ with additive shift maps $W_i\Omega^\bullet_X \xrightarrow{V} W_{i+1}\Omega^\bullet_X$ for $i \le n$ satisfying the following conditions:

- (i_n) $RV(x) = VR(x)$ for any $x \in W_i\Omega^\bullet_X, i \le n-1$
- (ii_n) $W_i\Omega^0_X = W_i\mathscr{O}_X$, on which R and V are the usual restriction and shift maps
- (iii_n) $V(xdy) = V(x)dV(y)$ for any $x \in W_i\Omega^\bullet_X, i \le n-1$
- (iv_n) $V(y)d\underline{x} = V(x^{p-1}y)dV\underline{x}$ where $x \in O_X, y \in W_i\mathscr{O}_X$ for any $i \le n-1$
- (v_n) The quotient maps $\Omega^\bullet_{W_i\mathscr{O}_X} \xrightarrow{\pi_i} W_i\Omega^\bullet_X$ are surjective, $\pi^0_i = \mathrm{id}$ for any $i \le n$, and π_1 is an isomorphism.

We then set

$$W_{n+1}\Omega^\bullet_X := \Omega^\bullet_{W_{n+1}\mathscr{O}_X}/N,$$

where N is the graded differential ideal such that (i_{n+1}) through (v_{n+1}) hold and $\pi_n R(N) = 0$. Due to these definitions, two maps $W_{n+1}\Omega^\bullet_X \xrightarrow{R} W_n\Omega^\bullet_X$ and $W_n\Omega^\bullet_X \xrightarrow{V} W_{n+1}\Omega^\bullet_X$ are induced. The complex $W_\bullet\Omega^\bullet_X$ then satisfies a universal property. (For further details please consult Illusie [Ill79].)

The de Rham-Witt complex is then defined as

$$W\Omega^\bullet_X := \lim_n W_n\Omega^\bullet_X,$$

where the limit is taken along the inverse limit system given by R.

2.3. **Tanaka's vanishing.** The original Kodaira Vanishing is closely related to Hodge decomposition. Hodge decomposition in turn resembles the slope decomposition of crystalling cohomology in terms of the de Rham-Witt complex. This motivates the attempt at finding a useful vanishing theorem in the context of de Rham-Witt.

Definition 2.1 (Teichmüller lifts of line bundles, cf. [Tan18])**.** An invertible sheaf $\mathscr{F}$ on X is defined by local transition functions (f_{ji}). Tanaka defined the *Teichmüller lift* $\underline{\mathscr{F}}$ of an invertible $\mathscr{O}_X$–module to be the invertible $W\mathscr{O}_X$–module defined by the Teichmüller representatives of the transition functions $(\underline{f_{ji}})$.

We define the truncated Teichmüller lift

$$\underline{\mathscr{F}}_{\le n} := W_n\mathscr{O}_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}.$$

Theorem 2.2 (Tanaka, cf. [Tan18, Theorem 1.1])**.** *Let k be a perfect field of characteristic $p > 0$, and X be an N–dimensional smooth projective variety over k. If $\mathscr{A}$ is an ample line bundle on X, then*

(i)
- $H^j(X, \underline{\mathscr{A}}^{-s}) = 0$ *for any* $s \gg 0, j < N$,
- $H^j(X, \underline{\mathscr{A}}^{-1})_{\mathbb{Q}} = 0$ *for any* $j < N$,

(ii)
- $H^i(X, W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{A}}) = H^i(X, W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{A}}^s)$ *for any* $s > 0$,

- $H^i(X, W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{A}}) = 0$ *for any* $i > 0$.

Remark 2.3. This result appears to suggest some duality property in the context of Witt sheaves. Note that the proof of (i) is relatively simple, unlike that of (ii). So we would want a nice duality property to recover (ii) from (i).

3. Duality Theorem

In what follows, let X^N be a smooth projective variety over a perfect field k of positive characteristic p.

Proposition 3.1. *Let* $\mathscr{F}$ *be an invertible* $\mathscr{O}_X$*–module. For any* $n > 0$*,*

$$W_n\Omega^i_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n} \cong R\,\mathscr{H}om_{W_n\mathscr{O}_X}(W_n\Omega^{N-i}_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega^N_X) \tag{3.1}$$

such that, in particular,

$$H^i(W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) \cong H^{N-i}(\underline{\mathscr{F}}^\vee_{\leq n})^\vee \text{ for any } i \geq 0, n > 0. \tag{3.2}$$

Proof. Observe that, since $\underline{\mathscr{F}}_{\leq n}$ is an invertible $W_n\mathscr{O}_X$–module, $\cdot \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}$ is an exact functor. By [Eke84][Thm II: 2.2] we know that

$$W_n\Omega^i_X \cong R\,\mathscr{H}om_{W_n\mathscr{O}_X}(W_n\Omega^{N-i}_X, W_n\Omega^N_X).$$

For any invertible $\mathscr{O}_X$–module $\mathscr{F}$, we then take the derived tensor product of the above equation with $\underline{\mathscr{F}}_{\leq n}$ over $W_n\mathscr{O}_X$:

$$W_n\Omega^i_X \overset{L}{\underset{W_n\mathscr{O}_X}{\otimes}} \underline{\mathscr{F}}_{\leq n} \cong R\,\mathscr{H}om_{W_n\mathscr{O}_X}(W_n\Omega^{N-i}_X, W_n\Omega^N_X) \overset{L}{\underset{W_n\mathscr{O}_X}{\otimes}} \underline{\mathscr{F}}_{\leq n}$$

To prove Equation 3.1, consider the right hand side of the above. Choose for $W_n\Omega^{N-i}_X$ a finite projective resolution $P^\bullet$ consisting of locally free $W_n\mathscr{O}_X$–modules. This is possible because W_nX is smooth, so it has finite global dimension by Auslander-Buchsbaum-Serre. Then

$$\begin{aligned}
&R\,\mathscr{H}om_{W_n\mathscr{O}_X}(W_n\Omega^{N-i}_X, W_n\Omega^N_X) \overset{L}{\underset{W_n\mathscr{O}_X}{\otimes}} \underline{\mathscr{F}}_{\leq n} \\
\cong\, &\mathscr{H}om^\bullet_{W_n\mathscr{O}_X}(P^\bullet, W_n\Omega^N_X) \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n} \\
\cong\, &\mathscr{H}om^\bullet_{W_n\mathscr{O}_X}(P^\bullet, W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) \\
\cong\, &\mathscr{H}om^\bullet_{W_n\mathscr{O}_X}(P^\bullet \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega^N_X) \\
\cong\, &R\,\mathscr{H}om_{W_n\mathscr{O}_X}(W_n\Omega^{N-i}_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega^N_X).
\end{aligned}$$

The last isomorphism holds because $\cdot \otimes \underline{\mathscr{F}}^\vee_{\leq n}$ is exact.

To prove Equation 3.2 we take global sections of the derived push-forward.

$$\begin{aligned}
\Gamma_S(R\phi_*(W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n})) &\cong \Gamma_S(R\phi_* R\,\mathscr{H}om_{W_n\mathscr{O}_X}(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega^N_X)) \\
&\cong \Gamma_S(R\,\mathscr{H}om_{W_n\mathscr{O}_S \cong W_n}(R\phi_*(\underline{\mathscr{F}}^\vee_{\leq n}), W_n[-N])) \\
&\cong \mathrm{Hom}_{W_n\mathscr{O}_S}((R\phi_*(\underline{\mathscr{F}}^\vee_{\leq n}))[N], W_n),
\end{aligned}$$

where W_n is the constant sheaf, the second isomorphism is due to Coherent Duality and [Eke84, Theorem 4.1], and the third isomorphism is due to W_n being an injective W_n–module. In particular for all i we have isomorphisms

$$\begin{aligned} H^i(W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) &\cong \mathrm{Hom}_{W_n\mathscr{O}_S}(R^{N-i}\phi_*(\underline{\mathscr{F}}^\vee_{\leq n}), W_n) \\ &\cong \mathrm{Hom}_{W_n}(H^{N-i}(\underline{\mathscr{F}}^\vee_{\leq n}), W_n). \end{aligned}$$

□

Remark 3.2. In fact, for our current use case it would be sufficient to show that

$$R^i\,\mathscr{H}om(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega^N_X) = 0 \text{ for any } i > 0,$$

(see for example [Tan18, Proposition 3.19]). Then by tensor–hom adjunction

$$W_n\Omega^N_X \otimes \underline{\mathscr{F}}_{\leq n} \cong \mathscr{H}om(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega^N_X) \cong R\,\mathscr{H}om(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega^N_X).$$

Let us now attempt to pass to the limit. Observe that for $\mathscr{F}$ an invertible sheaf of $\mathscr{O}_X$–modules,

$$\lim_n (W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) \cong R\lim_n (W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}). \tag{3.3}$$

To see this, take the exact sequence (cf. [Ill79])

$$0 \to \mathrm{gr}^n W\Omega^N_X \to W_{n+1}\Omega^N_X \to W_n\Omega^N_X \to 0,$$

where $\mathrm{gr}^n W\Omega^N_X$ is a coherent (and in fact locally free) $\mathscr{O}_X$–module. Tensoring with $\underline{\mathscr{F}}$ over $W\mathscr{O}_X$ we get an exact sequence

$$0 \to \mathrm{gr}^n W\Omega^N_X \underset{\mathscr{O}_X}{\otimes} \mathscr{F} \to W_{n+1}\Omega^N_X \underset{W_{n+1}\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n+1} \to W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n} \to 0.$$

For any $x \in X$, we can take an affine open neighborhood U_x of x. Then

$$H^1(U_x, \mathrm{gr}^n W\Omega^N_X \underset{\mathscr{O}_X}{\otimes} \mathscr{F}) = 0$$

by coherence, and therefore

(i) $H^0(U_x, W_{n+1}\Omega^N_X \underset{W_{n+1}\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n+1}) \to H^0(U_x, W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n})$ is surjective for all $n > 0$,
(ii) $H^i(U_x, W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) = 0$ for any $i > 0$.

In this fashion we can find a basis $\mathscr{U}$ for the topology of X such that the above two properties hold for all $U \in \mathscr{U}$, and so by [CR11, (1.5.1)] Equation 3.3 holds.

Now consider the limit as follows:

$$\begin{aligned} R\phi_*(W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}) &\cong R\phi_*(\lim_n (W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n})) \\ &\cong R\lim_n R\phi_*(W_n\Omega^N_X \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) && \text{(by Eq. 3.3)} \\ &\cong R\lim_n R\phi_* R\,\mathscr{H}om_{W_n\mathscr{O}_X}(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega^N_X) && \text{(by Prop. 3.1)} \\ &\cong R\lim_n R\,\mathscr{H}om_{W_n\mathscr{O}_S}(R\phi_*\underline{\mathscr{F}}^\vee_{\leq n}, W_n[-N]), \end{aligned} \tag{3.4}$$

and meditate on the last formula. A crucial ingredient to Ekedahl's result was the isomorphism in $D(W[d])$:

$$R_n \overset{L}{\underset{R}{\otimes}} R\Gamma_S(W\Omega^\bullet_X) \cong R\Gamma_S(W\Omega^\bullet_X).$$

Ideally, we would wish to employ a similar property to our case.

Illusie and Raynaud (cf. [IR83, I.1, I.3, II.1]) introduced the non-commutative Raynaud ring R which has a bi–W–module structure. To suit our needs, we shall similarly define ω to be the W-algebra generated by V, subject to the relations

$$aV = VF(a),$$
$$a \in W.$$

While as a set ω is equal to $\bigoplus_i WV^i$, it is a non-commutative ring with an evident left–W–module structure. It follows from the definition (and the fact that $k^p = k$) that every element of ω can be uniquely described by a sum

$$\sum_i a_{-i}V^i, a_i \in W.$$

Let

$$\omega_n := \omega/V^n\omega,$$

which is a (W,ω)–bimodule, since $V^n\omega$ is a sub–left–W–module of ω and a right-ω–ideal generated by V^n. We obtain two sets of right–ω–module homomorphisms: an obvious restriction map $\omega_n \xrightarrow{\pi} \omega_{n-1}$, as well as a injective map $\omega_{n-1} \xrightarrow{\varrho} \omega_n$, both induced by the respective maps R and $\varrho = \{$multiplication by p on $W_\bullet\}$. (The induced ϱ is in fact term-wise multiplication by p as well.)

Proposition 3.3. *Let A be a k-algebra. Then $W(A)$ has a natural structure of left–ω–modules and there is an isomorphism of left–W–modules*

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} W(A) \cong W_n(A).$$

In particular, for a sheaf of left–ω–modules $\underline{\mathscr{F}}$ on X,

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma(\underline{\mathscr{F}}) \cong R\Gamma(\underline{\mathscr{F}}_{\leq n})$$

Proof. The left–ω–module structure on $W(A)$ is given by

$$\omega \times W(A) \xrightarrow{\quad\cdot\quad} W(A)$$
$$(\Sigma_i \underline{a_i} V^i, b) \longmapsto \Sigma_i \underline{a_i} V^i(b).$$

To compute the derived tensor product

$$D(\omega - \mathfrak{lmod}) \xrightarrow{\omega_n \overset{L}{\underset{\omega}{\otimes}} \cdot} D(\mathfrak{ab}),$$

take a projective resolution $P^\bullet$ of ω_n:

$$0 \to \omega \xrightarrow{V^n\cdot} \omega \to \omega_n \to 0.$$

This clearly is a homomorphism of right–ω–modules. It yields a complex $P^\bullet \otimes_\omega W(A)$:

$$0 \to W(A) \xrightarrow{V^n} W(A) \to 0.$$

To see that this represents $W_n(A)$ simply observe that the map induced by $\omega \xrightarrow{V^n\cdot} \omega$ via the tensor product is precisely the n-fold Verschiebungs-map on $W(A)$:

$$\begin{array}{ccccccc} W(A) & \xrightarrow{\sim} & \omega \underset{\omega}{\otimes} W(A) & \xrightarrow{V} & \omega \underset{\omega}{\otimes} W(A) & \xrightarrow{\sim} & W(A) \\ a & \longmapsto & 1 \otimes a & \longmapsto & V \otimes a & \longmapsto & V \cdot a = V(a). \end{array}$$

Moreover, since ω_n is a left–W–module, so is $\omega_n \otimes^L_\omega W(A)$, and the W–module homomorphisms on $W(A)$ induced by π and ϱ on ω are R and multiplication by p, respectively. Lastly, to see that the $D(\mathfrak{ab})$–isomorphism is in fact in $D(W-\mathfrak{lmod})$, simply observe that the left–W–module structures on both sides coincide via the isomorphism.

For the second statement, let $M \in (X, \omega)$, that is a sheaf of left–ω–modules on X. Let $P^\bullet$ be the projetive resolution of ω_n

$$0 \to \omega \xrightarrow{V^n\cdot} \omega \to \omega_n \to 0.$$

Then, since P^i is projective for all i,

$$\omega_n \otimes^L_\omega R\Gamma(M) \cong P^\bullet \underset{\omega}{\otimes} R\Gamma(M) \cong R\Gamma(P^\bullet \underset{\omega}{\otimes} M) \cong R\Gamma(\omega_n \otimes^L_\omega M) \cong R\Gamma(M_n).$$

Observe that just like $W(A)$, $W\mathscr{O}_X$ — and therefore $\underline{\mathscr{F}}$ — have a natural structure of (sheaves of) left–ω–modules, and so the statement follows. □

Continuing from Equation 3.4, we now have

$$\begin{aligned} R\phi_*(W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}^\vee) &\cong R\lim_n R\,\mathcal{H}om_{W_n\mathscr{O}_S}(R\phi_*\underline{\mathscr{F}}_{\leq n}, W_n[-N]) \\ &\cong R\lim_n R\operatorname{Hom}_{W_n}(\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma(\underline{\mathscr{F}}), W_n[-N]) \\ &\cong R\lim_n R\operatorname{Hom}_\omega(R\Gamma(\underline{\mathscr{F}}), R\operatorname{Hom}_{W_n}(\omega_n, W_n[-N])) \\ &\cong R\operatorname{Hom}_\omega(R\Gamma(\underline{\mathscr{F}}), R\lim_n \operatorname{Hom}_{W_n}(\omega_n, W_n[-N])). \end{aligned} \tag{3.5}$$

Note that the injective left–W–linear maps $\omega_{n-1} \xrightarrow{\varrho} \omega_n$ form a direct system. The $\operatorname{Hom}_{W_n}(\omega_n, W_n[-N])$ then form an inverse system (cf. [Eke84, III.2.3.*]) with boundary maps π defined by the commutativity of the diagram

$$\begin{tikzcd} \mathcal{H}om_{W_n}(\omega_n, W_n[-N]) \arrow[r, "\varrho^*"] \arrow[d, "\pi"] & \mathcal{H}om_{W_n}(j_{n,*}\omega_{n-1}, W_n[-N]) \\ j_{n,*}\,\mathcal{H}om_{W_{n-1}}(\omega_{n-1}, W_{n-1}[-N]). \arrow[ur, "\varrho_*"] & \end{tikzcd} \tag{3.6}$$

Here $W_{n-1}S \xrightarrow{j_n} W_nS$ is the natural immersion. There exist unique such maps π because by coherent duality and the fact that $W_{n-1} \cong j_n^! W_n$, ϱ_* is an isomorphism. Since the ϱ are injective, the π are surjective.

Proposition 3.4. *In fact, $\omega \cong \bigoplus_i WV^i \cong \bigoplus_\infty W$ as left–W–modules (similarly, $\omega_n \cong \bigoplus_{i<n} W_{n-i}V^i \cong \bigoplus_\infty W_{n-i}$ as left–W_n–modules). Hence,*

$$\mathcal{H}om_{W_n}(\omega_n, W_n) \cong \omega_n.$$

Proof. Since $k = k^p$, elements $a \in \omega$ can be uniquely written as

$$a = \sum_i a_i V^i, a_i \in W.$$

Let $a = \sum_i a_i V^i \in \omega$. The natural identification is clearly additive and bijective:

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ \omega \arrow[rr, "\sim"] & & \bigoplus_\infty W \end{tikzcd}$$

$$\sum_i a_i V^i \longmapsto \sum_i a_i.$$

Since the left–W–module structure of ω is simply multiplication on the left, it is W–linear (on the left). We therefore have

$$\mathcal{H}om_{W_n}(\omega_n, W_n) \cong \bigoplus_{i<n} \mathcal{H}om_{W_n}(W_{n-i}V^i, W_n) \cong \bigoplus_{i<n} W_{n-i}V^i.$$

□

Remark 3.5. An argument analogue to the above shows that $\omega \cong W$ as left–ω–modules.

It then promptly follows

Theorem 3.6. *Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$. Write $\breve{\omega} := \prod_i WV^i$. Then for any invertible $\mathscr{O}_X$–module $\mathscr{F}$ on X,*

$$R\phi_*(W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}^\vee) \cong R\,\mathcal{H}om_\omega(R\phi_*\underline{\mathscr{F}}, \breve{\omega}[-N]).$$

Proof. Let $0 \neq w \in \mathrm{Hom}(W_{n-i}, W_n) \cong W_i$. Since π is induced by the commutative Diagram 3.6, $\pi(w)$ is the unique map such that the following diagram commutes:

$$\begin{tikzcd} W_{n-i} \arrow[r, "w"] & W_n \\ W_{n-i-1} \arrow[u, "\varrho"] \arrow[r, "\pi(w)"] & W_{n-1} \arrow[u, "\varrho"'] \end{tikzcd} .$$

This unique map is $R(w)$ (since for $\tau \in W_{n-i-1}, \varrho(R(w)(\tau)) = w\varrho(\tau) \in W_n$). The induced maps $W_n \xrightarrow{\pi} W_{n-1}$ are therefore precisely the term-wise restriction maps R. Hence, taking the limit we have

$$R\lim_n \mathcal{H}om_{W_n}(\omega_n, W_n) \cong R\lim_n \bigoplus_{i<n} W_{n-i} \cong \prod_\infty W.$$

The theorem then follows from Equation 3.5. □

4. Open questions

- Let $\mathscr{A}$ be an ample invertible sheaf on X. Twisting $\mathscr{A}$ via Frobenius, by Theorem 3.6

  $R\Gamma(X, W\Omega^N_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{A}}) \cong R\,\mathcal{H}om_\omega(H^N(X, \underline{\mathscr{A}}^{-s}), \breve{\omega})$ for any $s > 0$.

It is not yet clear how part (ii) and (i) follow from eachother using this duality. I.e. we would like to show that the $R^i\,\mathcal{H}om$ on the right hand side above vanish for $s, i > 0$.

- It is not yet known whether Tanaka's theorem holds for $\mathscr{A}$ nef and big instead of ample.

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