A certain type of affine surfaces with isomorphic cylinders

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Introduction

Problem 0.1 (Zariski's Cancellation Problem)**.** Let V and W be varieties, and let $\mathbb{A}^1$ be an affine line. Then does $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ imply that $V \simeq W$?

Fact 0.2 ([2])**.** *Let X be a k-scheme, and let V and W be affine k-schemes which are principal $\mathbb{G}_a$-bundles over X. Then $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$.*

Open Problem 0.3. *Let Y be a prevariety, and let W be an affine variety which is a principal $\mathbb{G}_a$-bundle over Y. Then for any variety V, does $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ imply that V is affine and a principal $\mathbb{G}_a$-bundle over Y?*

Definition 0.4 (principal $\mathbb{G}_a$-bundle)**.** Let X be a k-scheme, let V be a k-scheme with a $\mathbb{G}_a$-action, and let $p\colon V \longrightarrow X$ be a morphism of k-schemes. Then (V, p) is called a **principal $\mathbb{G}_a$-bundle over** X if the following two conditions are satisfied :

(1) $p : \mathbb{G}_a$-equivariant ($\mathbb{G}_a$ acts trivially on X);
(2) there exists an (zariski) open covering $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of X such that the following diagram is commutative.

$$\begin{tikzcd} p^{-1}U_\lambda \arrow[r, "\mathbb{G}_a\text{-equiv-iso}"] \arrow[d, "p"] & U_\lambda \times \mathbb{G}_a \arrow[dl, "\mathrm{pr}"] \\ U_\lambda & \end{tikzcd}$$

Remark 0.5. Our definition of principal G-bundles for a group variety G is slightly different from the ordinaly one. But in the case that G is a nonsingular affine group variety, for a nonsingular affine group variety, those definitions coincides.

Remark 0.6. (Isomorphic classes of principal $\mathbb{G}_a$-bundles over X) $\longleftrightarrow \mathrm{H}^1(X, \mathcal{O}_X)$.

Definition 0.7.

- A variety V is called a **Zariski 1-factor** if $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ implies $V \simeq W$ for any variety W.
- We say that varieties V and W **have isomorphic cylinders** if $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$.
- $\mathbb{G}_a = (\mathbb{A}^1, +)$.
- A k-scheme X is called a **prevariety** if X is an integral scheme of finite type over k.
- $\overline{\kappa}(V) := \max\{\dim \rho_{|m(K_{\overline{V}}+\partial V)|}(\overline{V}) | m \in \mathbb{N}\}$ for a nonsingular variety V, where $(\overline{V}, \partial V)$ is a smooth completion of V with boundary ∂V, and $\rho_{|m(K_{\overline{V}}+\partial V)|}$ is a rational map defined by complete linear system $|m(K_{\overline{V}} + \partial V)|$.
- For a singular variety V, $\overline{\kappa}(V) := \overline{\kappa}(V^*)$, where V^* is a nonsingular model of V
- A variety S is called a **VLG variety** if S is a nonsingular variety with $\mathrm{P}_M(S) = 0$ for all $M \in \mathbb{Z}_{\geq 0}^{\oplus\infty}$, where $\mathrm{P}_M(S) = \dim_k \mathrm{H}^0(\overline{S}, \Omega_{\overline{S}}^M(\log \partial S))$ is the logarithmic M-genus of S.

Example 0.8. $\mathbb{A}^n$and $\mathbb{P}^n$ are VLG varieties.

Main Theorems

Theorem 1 . *Let Y be a 1-dimensional nonsingular prevariety, let Y' be a nonsingular curve with $\overline{\kappa}(Y') \geq 0$ ($\Leftrightarrow Y' \neq \mathbb{A}^1$, $\mathbb{P}^1$), let $l\colon Y \to Y'$ be a dominant morphism, and let W be an affine variety which is a principal $\mathbb{G}_a$-bundle over Y. Then for any variety V, $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ if and only if V is affine and a principal $\mathbb{G}_a$-bundle over Y.*

Outline of proof.

(1) By composing a section of $\mathrm{pr}_V\colon V \times_k \mathbb{A}^1 \to V$ and $V \times_k \mathbb{A}^1 \simeq W \times \mathbb{A}^1 \to W \to Y$, we obtain a morphism $p\colon V \to Y$. We want to show that (V, p) is a principal $\mathbb{G}_a$-bunlde.

(2) It is easy to see that V is nonsingular, affine, and has a nontrivial $\mathbb{G}_a$-action μ. By [5, Lemma 1.1], the GIT quotient $V//\mu$ is a nonsingular affine curve. Let $p\colon V \to V//\mu$ be the quotient morphism.

(3) By the computation of logarithmic M-genus ([8]) and the cancellation criterion for $\mathbb{A}^1$-fibered surfaces over a nonsingular affine curve ([6]), we can show that p is "similar to a principal $\mathbb{G}_a$-bundle", p' is $\mathbb{A}^1$-bundle, and p and p' are "locally same". Then it follows that (V, p) is a principal $\mathbb{G}_a$-bundle.

Theorem 2 (Generarization of Theorems of Fujita-Iitaka [8] and Nishimura [9]).
Let X and Y be prevarieties, let Y' be a variety with $\overline{\kappa}(Y') \geq 0$ and $\dim Y' = \dim Y$, let $l\colon Y \to Y'$ be a dominant morphism, and let S_1, S_2 be VLG varieties with $\dim S_1 = \dim S_2$. Let $p\colon V \to X$ be a S_1-bundle, $q\colon W \to Y$ a S_2-bundle. If $\Phi\colon V \to W$ is an isomorphism, then there exists a unique isomorphism $\phi\colon X \to Y$ such that the following diagram is commutative.

$$\begin{tikzcd} V \arrow[r, "\Phi:\mathrm{iso}"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "q"] \\ X \arrow[r, "\exists\phi:\mathrm{iso}"'] & Y \end{tikzcd}$$

The proof of Main Theorem 2 is almost the same as [8] and [9], but one should note that we do not assume the separatedness for X and Y.

Corollary to Main Theorem 2 . *Let X and Y be prevarieties, let Y' be a variety with $\overline{\kappa}(Y') \geq 0$ and $\dim Y' = \dim Y$, let $l\colon Y \to Y'$ be a dominant morphism, and let V, W be principal $\mathbb{G}_a$-bundles over X, Y, respectively. Then*

(1) $V \simeq W \Rightarrow X \simeq Y$.

(2) $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1 \Rightarrow X \simeq Y$.

参考文献

[1] T. Bandman and L. Makar-Limanov, *Nonstability of the AK invariant*, Michigan Math. J. **53** (2005), no. 2, 263–281.

[2] W. Danielewski, *On a cancellation problem and automorphism groups of affine algebraic varieties*, preprint, Warsaw (1989).

[3] R. Dryło, *A note on noncancellable varieties*, Comm. Algebra **37** (2009), no. 9, 3337–3341.

[4] A Dubouloz, *Additive group actions on Danielewski varieties and the cancellation problem*, Math. Z. **255** (2007), no. 1, 77–93.

[5] K. H. Fieseler, *On complex affine surfaces with $\mathbf{C}^+$-action*, Comment. Math. Helv. **69** (1994), no. 1, 5–27.

[6] H. Flenner, S. Kaliman, and M. Zaidenberg, *Cancellation for surfaces revisited. I*, 2016. arXiv:math/1610.01805.

[7] A. Grothendieck, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. **4** (1960).

[8] S. Iitaka and T. Fujita, *Cancellation theorem for algebraic varieties*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), no. 1, 123–127.

[9] T. Nishimura, *On the existence of isomorphisms of schemes having isomorphic jet schemes*. Master's thesis, Tokyo Inst. Tech. (2017).

[10] M. Rosenlicht, *On quotient varieties and the affine embedding of certain homogeneous spaces*, Trans. Amer. Math. Soc. **101** (1961), 211–223.