A certain type of affine surfaces with isomorphic cylinders

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Introduction ザリスキの消去問題とは?

k を標数 0 の代数閉体とする.

Problem (ザリスキの消去問題)

V, W : *variety over* k
$\mathbb{A}^1$: *affine line*
このとき
$V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ ならば $V \simeq W$ か？

多くの場合肯定的であるが, 反例が存在する

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$\longrightarrow$

Problem

W : *variety* に対して
$V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ となる V を同型類によって分類せよ

Introduction 言葉の定義

V, W: variety

Definition

- V と W がシリンダー同型 $\Longleftrightarrow V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$
- W: ザリスキ 1 因子 $\Longleftrightarrow$ 任意の V に対して $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ ならば $V \simeq W$
- $\mathbb{G}_a = (\mathbb{A}^1, +_k)$

シリンダー同型な variety の構成方法

Fact (W. Danielewski, 1989)

X: k-scheme
V, W: affine k-scheme with principal $\mathbb{G}_a$-bundle structure over X
then
$V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$

principal $\mathbb{G}_a$-bundle = ファイバーに沿った $\mathbb{G}_a$ 作用を持つ $\mathbb{A}^1$-bundle

Introduction 非ザリスキ 1 因子 (消去問題の反例) の構成

Theorem (W. Danielewski 1989, K. H. Fieseler 1994)

$W_m := \mathbb{V}(x^m z - y^2 + 1) \subset \mathbb{A}^3 = \mathbf{Spec}\, k[x, y, z]$
$\widetilde{\mathbb{A}}^1 := \mathbb{A}^1 \sqcup_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$: *affine line with double origin*
then

1. $i \neq j \Longrightarrow W_i \not\cong W_j$
2. W_m: *principal* $\mathbb{G}_a$*-bundle over* $\widetilde{\mathbb{A}}^1$
(*thus* $W_i \times \mathbb{A}^1 \simeq W_j \times \mathbb{A}^1$)

Theorem (R. Dryło 2007)

$X = \mathbf{Spec}\, A$: *nonsingular affine variety with* $\overline{\kappa}(X) \geq 0$
$H = \bigcup_{j=1}^{m} \mathbb{V}(f_j)$: *irr decomp,* $f_1, \ldots, f_m \in A$: *prime element*
$\widetilde{X} := X \sqcup_{X \setminus H} X$
then

$$\sharp \left(\frac{\text{affine varieties with principal } \mathbb{G}_a\text{-bundle structure over } \widetilde{X}}{\text{isomorphisms of varieties}} \right) = \infty$$

Introduction open problem

これらの例から次の問題が考えられる.

Open Problem

Y : *prevariety*
W : *affine variety with principal* $\mathbb{G}_a$*-bundle structure over* Y
このとき, V : *variety* に対して
$V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1 \Longleftrightarrow V$: *affine with principal* $\mathbb{G}_a$*-bundle structure over* Y か?

prevariety = integral scheme of finite type over k, not necessarily separated
example:
$\widetilde{X} = X \sqcup_{X \setminus H} X$
$\widetilde{\mathbb{A}}^1 = \mathbb{A}^1 \sqcup_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$

Main Theorem 1

Main Theorem 1

Y : *1-dim nonsingular prevariety*

Y' : *nonsingular curve with* $\overline{\kappa}(Y') \geq 0$ $(\Longleftrightarrow Y' \neq \mathbb{A}^1, \mathbb{P}^1)$

$l : Y \to Y'$: *dominant morphism*

W : *affine variety with principal* $\mathbb{G}_a$*-bundle structure over* Y

V : *variety*

then

$V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1 \Longrightarrow V$: *affine with principal* $\mathbb{G}_a$*-bundle structure over* Y

Main Theorem 1 定理の仮定について

Main Theorem 1 の仮定

Y : 1-dim nonsingular prevariety

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$l\colon Y \to Y'$: dominant morphism

例えば,(R. Dryło (2007)) で $\dim X = 1$ の時 $\widetilde{X}$ が仮定を満たす

Theorem (R. Dryło 2007)

$X = \mathbf{Spec} A$: *nonsingular affine variety with* $\overline{\kappa}(X) \geq 0$
$H = \bigcup_{j=1}^{m} \mathbb{V}(f_j)$: *irr decomp,* $f_1, \ldots, f_m \in A$: *prime element*
$\widetilde{X} := X \sqcup_{X \setminus H} X$
then

$$\sharp \left(\frac{\textit{affine varieties with principal } \mathbb{G}_a\textit{-bundle structure over } \tilde{X}}{\textit{isomorphisms of varieties}} \right) = \infty$$

$l\colon \tilde{X} \longrightarrow X$: dominant

Main Theorem 1 証明のアイデア

Main Theorem 1

Y : *1-dim nonsing prevariety*
Y' : *nonsing curve with* $\overline{\kappa}(Y') \geq 0$
$l\colon Y \to Y'$: *dominant morphism*
W : *affine variety with principal* $\mathbb{G}_a$*-bundle structure over* Y
V : *variety*
then
$V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1 \iff$ V : *affine with principal* $\mathbb{G}_a$*-bundle structure over* Y

(基本方針)$p\colon V \longrightarrow Y$ をつくり, principal $\mathbb{G}_a$-bundle であることを示す.

1. $p\colon V \simeq V \times_k \{a\} \hookrightarrow V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1 \to W \to Y$
2. $p'\colon V \longrightarrow V//\mathbb{G}_a = \mathrm{Spec}\mathcal{O}_V(V)^{\mathbb{G}_a}$: GIT quotient morphism, $V//\mathbb{G}_a$: nonsing affine curve
3. Y と $V//\mathbb{G}_a$, $p\colon V \longrightarrow Y$ と $p'\colon V \longrightarrow V//\mathbb{G}_a$ が "局所的には同じ" であることを示す

Main Theorem 2 Corollary

Corollary to Main Theorem 2

X, Y : prevariety
Y' : variety with $\overline{\kappa}(Y') \geq 0$ and $\dim Y' = \dim Y$
$l \colon Y \to Y'$: dominant morphism
V : principal $\mathbb{G}_a$-bundle over X
W : principal $\mathbb{G}_a$-bundle over Y
then

1. $V \simeq W \Longrightarrow X \simeq Y$
2. $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1 \Longrightarrow X \simeq Y$

すなわち,Y 上の principal $\mathbb{G}_a$-bundle とシリンダー同型な principal $\mathbb{G}_a$-bundle は Y 上のものに限る

Main Theorem 2

Main Theorem 2 (Generarization of Fujita-Iitaka, Nishimura)

X, Y : *prevariety*

Y' : *variety with* $\overline{\kappa}(Y') \geq 0$ *and* $\dim Y' = \dim Y$

$l\colon Y \to Y'$: *dominant morphism*

S_1, S_2: *VLG variety with* $\dim S_1 = \dim S_2$

$p\colon V \longrightarrow X$: S_1*-bundle*

$q\colon W \longrightarrow Y$: S_2*-bundle*

If $\Phi\colon V \to W$: *isomorphism*

then $\exists \phi\colon X \to Y$: *isomorphism s.t.*

$$\begin{array}{ccc} V & \xrightarrow{\Phi:\ iso} & W \\ {\scriptstyle p}\downarrow & \circlearrowright & \downarrow{\scriptstyle q} \\ X & \xrightarrow[\exists \phi:\ iso]{} & Y \end{array}$$

今後の課題

Main Theorem 1

Y : 1-dim nonsingular prevariety
Y' : nonsingular curve with $\overline{\kappa}(Y') \geq 0$
$l\colon Y \to Y'$: dominant morphism
W : affine variety with principal $\mathbb{G}_a$-bundle structure over Y
V : variety
then
$V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1 \Longrightarrow V$ *: affine with principal $\mathbb{G}_a$-bundle structure over Y*

1. Y 上の affine principal $\mathbb{G}_a$-bundle の分類
2. Main Theorem 1 の高次元化
3. ログ小平次元 $\overline{\kappa}(Y')$ の仮定を外す

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