A CERTAIN TYPE OF AFFINE SURFACES WITH ISOMORPHIC CYLINDERS

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Abstract. In this paper, we study the structure of isomorphisms of S-bundles over a cetain type of prevarieties, where S is a variety with vanishing logarithmic genera (for example, $\mathbb{A}^n$ and $\mathbb{P}^n$). As in the same way, for a certain type of affine surfaces W, we determine all varieties V such that $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$.

Introduction

Let V and W be varieties over an algebraically closed field k with characteristic 0. The Zariski cancellation problem asks when the existence of an isomorphism $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ implies that $V \simeq W$. Following [6], we call a variety V a *Zariski* 1*-factor* if $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ implies $V \simeq W$ for any variety W. There are many examples of Zariski 1-factors, but in 1989, W. Danielewski found non-Zariski 1-factors by using the following property of principal $\mathbb{G}_a$-bundles.

Fact 0.1 ([2]). *Let X be a k-scheme, and let V and W be affine k-schemes which are principal $\mathbb{G}_a$-bundles over X. Then $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$.*

By using the same arguments, many examples of non-Zariski 1-factors are constructed. From these examples, we can consider the following problem.

Problem 0.2. Let Y be a prevariety, and let W be an affine variety which is a principal $\mathbb{G}_a$-bundle over Y. Then for any variety V, does $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ imply that V is affine and a principal $\mathbb{G}_a$-bundle over Y?

In this paper, we will show the following main theorem (section 3).

Theorem 0.3. *Let Y be a 1-dimensional nonsingular prevariety (a prevariety means an integral scheme of finite type over k), let Y' be a nonsingular curve with nonnegative logarithmic kodaira dimension (that is, Y' is neither $\mathbb{A}^1$ nor $\mathbb{P}^1$), let $l\colon Y \to Y'$ be a dominant morphism, and let W be an affine variety which is a principal $\mathbb{G}_a$-bundle over Y. Then for any variety V, $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$ if and only if V is affine and a principal $\mathbb{G}_a$-bundle over Y.*

From this theorem and counter examples for the cancellation problem by Dryło ([3]), we find many non-Zariski 1-factors of the above type.

To show Theorem 0.3, we use the similar arguments as Fujita-Iitaka's cancellation theorem ([8]) and Nishimura's generalized version of it ([9]). In addition to Theorem 0.3, we will slightly generalize these theorems in section 2 as follows.

Theorem 0.4. *Let X and Y be prevarieties, let Y' be a variety with $\overline{\kappa}(Y') \geq 0$ and $\dim Y' = \dim Y$, let $l\colon Y \to Y'$ be a dominant morphism, and let S_1 and S_2 be varieties with vanishing logarithmic genera and $\dim S_1 = \dim S_2$. Let $p\colon V \to X$ be a S_1-bundle, $q\colon W \to Y$ a S_2-bundle. If $\Phi\colon V \to W$ is an isomorphism, then there exists a unique isomorphism $\phi\colon X \to Y$ such that the following* *diagram is commutative.*

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "q"] \\ X \arrow[r, "\exists\phi"'] & Y \end{tikzcd}$$

Using this theorem, we show 2 corollaries (Corollary 2.5, 2.6) useful for the cancellation problem.

Before the proof of Theorem 0.3 and 0.4, we will give a criterion for principal $\mathbb{G}_a$-bundles over some good schemes to be affine (Theorem 1.6). The proof of Theorem 1.6 is almost the same as the affine criterion for principal $\mathbb{G}_a$-bundles by A. Dubouloz ([4]).

From these theorems and Dryło's examples ([3]), we can find all varieties which are non-Zariski 1-factors of the type in Theorem 0.3.

1. Affine criterion for principal $\mathbb{G}_a$-bundles over a certain type of nonseparated schemes

The purpose of this section is to prove Theorem 1.6, which gives a computable way to determine principal $\mathbb{G}_a$-bundles to be affine or not.

Definition 1.1 (principal $\mathbb{G}_a$-bundle)**.** Let X be a k-scheme, let V be a k-scheme with $\mathbb{G}_a$-action, and let $p\colon V \to X$ be a morphism of k-schemes. Then (V, p) is called a *principal $\mathbb{G}_a$-bundle over X* if the following two conditions are satisfied:

1. (1) p is $\mathbb{G}_a$-equivariant ($\mathbb{G}_a$ acts trivially on X);
2. (2) there exists an (Zariski) open covering $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ of X and a $\mathbb{G}_a$-equivariant isomorphism $g_\lambda : p^{-1}U_\lambda \to U_\lambda \times_k \mathbb{G}_a$ for each $\lambda \in \Lambda$ such that the following diagram is commutative.

$$\begin{tikzcd} p^{-1}U_\lambda \arrow[r, "g_\lambda"] \arrow[d, "p"] & U_\lambda \times \mathbb{G}_a \arrow[dl, "\mathrm{pr}"] \\ U_\lambda & \end{tikzcd}$$

Remark 1.2. Our definition of principal G-bundles for a group variety G is slightly different from the ordinaly one. But for an affine group variety, those definitions coincide.

Remark 1.3. There exists a one-to-one correspondence between isomorphic classes of principal $\mathbb{G}_a$-bundles over X and $\mathrm{H}^1(X, \mathcal{O}_X)$.

Definition 1.4. Let X be a variety, let Z be a closed subvariety of X, let r be a natural number, and let $X_0, \ldots, X_r$ be copies of X. Then

$$X_+rZ := X \sqcup_{X\setminus Z} \underbrace{X \sqcup_{X\setminus Z} \cdots \sqcup_{X\setminus Z} X}_{r} = X_0 \sqcup_{X\setminus Z} X_1 \sqcup_{X\setminus Z} \cdots \sqcup_{X\setminus Z} X_r.$$

Namely, X_+rZ is a nonseparated k-scheme which looks like X with r copies of Z. We fix an open covering $\mathcal{X}$ of X_+rZ to be $\mathcal{X} = \{X_0, \ldots, X_r\}$.

Lemma 1.5 ([7])**.** *Let X be a scheme, let Y be an affine scheme, and let $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be an open affine covering of X. Then for any morphism $f\colon X \to Y$, f is separated if and only if*

1. (1) *$U_\mu \cap U_\lambda$ is affine for any $\mu, \lambda \in \Lambda$;*
2. (2) *$\Gamma(U_\mu \cap U_\lambda, \mathcal{O}_X)$ is generated by $\Gamma(U_\mu, \mathcal{O}_X)$ and $\Gamma(U_\lambda, \mathcal{O}_X)$.*

The following theorem is a generalization of the affine criterion for principal $\mathbb{G}_a$-bundles by A. Dubouloz ([4]), but the proof is almost the same.

Theorem 1.6. *Let $X = \mathbf{Spec} A$ be an affine variety, let $Z_1, ..., Z_m$ be hypersurfaces of X defined by prime elements $f_1, ..., f_m \in A$, and let $Z := \bigcup Z_j$. Let V be a principal $\mathbb{G}_a$-bundle over $X_+ rZ$ defined by a Čech cocycle $[\{g_{ij}\}] \in \mathrm{H}^1(\mathcal{X}, \mathcal{O}_{X_+rZ}) \simeq \mathrm{H}^1(X_+rZ, \mathcal{O}_{X_+rZ})$, where $g_{ij} \in \Gamma(X_{ij}, \mathcal{O}_{X_+rZ}) = A_{f_1\cdots f_m}$ and as an element of $A_{f_1\cdots f_m}$, g_{ij} can be written as follows; $g_{ij} = f_1^{-k_{ij,1}} \cdots f_m^{-k_{ij,m}} h_{ij}$, where $k_{ij,l} \in \mathbb{Z}_{\geq 0}$ and $h_{ij} \in A$ such that h_{ij} can not be divided by f_l if $k_{ij,l} > 0$.*

If (a) $r = 1$ or (b) $r \geq 2$ and $\emptyset \neq Z_{l_1} \cap Z_{l_2} \not\subset \bigcup_{l \neq l_1, l_2} Z_l$ for any $l_1, l_2 = 1, \ldots, m$, then the following conditions are equivalent.

1. *(1) $k_{ij,l} \geq 1$ and $(h_{ij}, f_1 \cdots f_m) = A$ for any i, $j = 0, \ldots, r$ and $l = 1, \ldots, m$.*
2. *(2) V is separated.*
3. *(3) V is affine.*

Proof. (3) $\Rightarrow$ (2) is obvious. We show that (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3).

Let us denote $(k_{ij,1}, \ldots, k_{ij,m}) \in \mathbb{Z}^m$ by $[k_{ij}]$, $(1, \ldots, 1) \in \mathbb{Z}^m$ by $\mathbf{1}$, and $f_1^{k_{ij,1}} \cdots f_m^{k_{ij,m}}$ by $\underline{\mathrm{f}}^{[k_{ij}]}$.

First of all, we give a necessaly and sufficient condition for V to be separated by using Lemma 1.5. By the definition of the open covering $\mathcal{X}$ of X_+rZ, $X_i \cap X_j$ is affine for any $i, j = 0, \ldots r$. Thus V is separated if and only if $\Gamma(X_i \cap X_j, \mathcal{O}_{X_+rZ})$ is generated by $\Gamma(X_i, \mathcal{O}_{X_+rZ})$ and $\Gamma(X_j, \mathcal{O}_{X_+rZ})$ for any $i, j = 0, \ldots r$. This condition equals to $A_{f_1\cdots f_m}[t] = A_{g_{ij}}[t]$ for any $i, j = 0, \ldots, r$ with $i \neq j$, where t is indeterminate. this implies that V is separated if and only if $A_{\underline{\mathrm{f}}} = A_{g_{ij}}$ for any $i, j = 0, \ldots, r$ with $i \neq j$.

(1) $\Rightarrow$ (2) Suppose the condition (1). Then there exists $a, b \in A$ such that $1 = ah_{ij} + b\underline{\mathrm{f}}$, and this implies that $\underline{\mathrm{f}}^{-\mathbf{1}} = a\underline{\mathrm{f}}^{[k_{ij}]-\mathbf{1}} g_{ij} + b$ and $k_{ij,l} - 1 \geq 0$. Then it follows that $A_{\underline{\mathrm{f}}} = A_{g_{ij}}$.

(2) $\Rightarrow$ (1) Suppose the condition (2). Then $\underline{\mathrm{f}}^{-\mathbf{1}} \in A_{\underline{\mathrm{f}}} = A_{g_{ij}}$. Therefore $\underline{\mathrm{f}}^{-\mathbf{1}}$ can be written as follows;

$$\underline{\mathrm{f}}^{-\mathbf{1}} = a_0 + a_1 g_{ij} + a_2 g_{ij}^2 + \cdots + a_n g_{ij}^n,$$

where $a_0, \ldots a_s \in A$ and n is an integer. (If $n = 0$, then $f_1, \ldots, f_m$ should be units in A.) By multiplying both sides of the equation by $\underline{\mathrm{f}}^{n[k_{ij}]}$, we obtain the following equation,

$$\underline{\mathrm{f}}^{n[k_{ij}]-\mathbf{1}} = a_0 \underline{\mathrm{f}}^{n[k_{ij}]} + h_{ij} s,$$

where $s = a_1 \underline{\mathrm{f}}^{(n-1)[k_{ij}]} + \cdots + a_{n-1} h_{ij}^{n-2} \underline{\mathrm{f}}^{[k_{ij}]} + a_n h_{ij}^{n-1}$. From this equation, we can deduce that $k_{ij,l} \geq 1$ for any $l = 1, \ldots, m$ because $f_1, \ldots, f_m$ are prime elements and distinct up to units. Moreover, s can be divided by $\underline{\mathrm{f}}^{n[k_{ij}]-\mathbf{1}}$ because we take h_{ij} which is not in (f_l) for all $l = 1, \ldots, m$. Then it follows that there exists $s' \in A$ such that $1 = a_0 \underline{\mathrm{f}} + h_{ij} s$. that is, $A = (\underline{\mathrm{f}}, h_{ij})$.

(1) $\Rightarrow$ (3) Suppose the condition (1). We first observe that there exists an index $j' \in \{1, \ldots, m\}$ such that $k_{1j',l} = \max_j \{k_{1j,l}\}$ for all $l = 1, \ldots, m$. Assume that there exists indices $j_1, j_2 = 0, \ldots r$ and $l_1, l_2 = 1, \ldots, m$ such that $j_1 \neq j_2$, $l_1 \neq l_2$, $k_{1j_1,l_1} > k_{1j_2,l_1}$, and $k_{1j_1,l_2} < k_{1j_2,l_2}$ for contradiction. Put $\mu_l := \max\{k_{1j_1,l}, k_{1j_2,l}\}$ and $[\mu] := (\mu_1, \ldots \mu_m) \in \mathbb{Z}^m_{\geq 0}$. It follows from the cocycle condition $g_{j_1 j_2} = g_{1j_2} - g_{1j_1}$ that

$$\underline{\mathrm{f}}^{[\mu]-[k_{j_1 j_2}]} h_{j_1 j_2} = \underline{\mathrm{f}}^{[\mu]-[k_{1j_2}]} h_{1j_2} - \underline{\mathrm{f}}^{[\mu]-[k_{1j_1}]} h_{1j_1}.$$

By the definition of μ and h_{ij}, the right hand side of this equation is in (f_{l_1}, f_{l_2}) but not in (f_{l_1}) and (f_{l_2}). From this, it follows that $\mu_{l_1} = k_{j_1 j_2, l_1}$ and $\mu_{l_2} = k_{j_1 j_2, l_2}$. On the other hand, $\underline{\mathrm{f}}^{[\mu]-[k_{j_1 j_2}]} h_{j_1 j_2} \in (f_{l_1}, f_{l_2})$ implies $\underline{\mathrm{f}}^{[\mu]-[k_{j_1 j_2}]} \in (f_{l_1}, f_{l_2})$ because $h_{j_1 j_2}$ is a nonzero function on Z and $Z_{l_1} \cap Z_{l_2} \neq \emptyset$. But this contradicts to the assumtion $Z_{l_1} \cap Z_{l_2} \not\subset \bigcup_{l \neq l_1, l_2} Z_l$, and thus we can choose an index $j' \in \{1, \ldots, m\}$ such that $k_{1j',l} = \max_j \{k_{1j,l}\}$ for all $l = 1, \ldots, m$.

Next, we show that there exists an affine morphism $\psi \colon V \to \mathbb{A}^1$ by induction on r. If $r = 0$, then V should be isomorphic to $X \times_k \mathbb{A}^1$. Then the first (or second) projection of $X \times_k \mathbb{A}^1$ is an affine morphism. Suppose the statement holds for a natural number $r - 1$. By the assumption, there exists $s_{ij} \in A$ such that $\overline{h_{ij}s_{ij}} = 1$ in $A/(f_1 \cdots f_m)$. Define morphisms $\phi_j \colon X_j \to \mathbb{A}^1$ to be $\phi_j(x,t) = s_{1j}(\underline{\mathrm{f}}^{[k_{1j'}]}t + \underline{\mathrm{f}}^{[k_{1j'}]-[k_{1j}]}h_{1j})$ and define morphisms $\psi_j := \phi \circ g_j^{-1} \colon V_j \simeq X \times_k \mathbb{A}^1 \to \mathbb{A}^1$. By the cocycle condition, $\{\psi_j\}_{j=0,\ldots,r}$ glue to a morphism $\psi \colon V \to \mathbb{A}^1$. Define $H_j := g_j(Z \times_k \mathbb{A}^1) \subset V_j$. Then $\psi(H_1) = \phi_1 g_1^{-1} g_1(Z \times_k \mathbb{A}^1) = \{0\}$ and $\psi(H_{j'}) = \phi_{j'} g_{j'}^{-1} g_{j'}(Z \times_k \mathbb{A}^1) = \{1\}$. Thus $\psi^{-1}(\mathbb{A}^1 \setminus \{0\}) \subseteq V \setminus H_1$ and $\psi^{-1}(\mathbb{A}^1 \setminus \{1\}) \subseteq V \setminus H_{j'}$. Moreover, $V \setminus H_1$ is a principal $\mathbb{G}_a$-bundle defined by the cocycle $\{g_{ij}\}_{i,j\neq 1}$ and $V \setminus H_{j'}$ is a principal $\mathbb{G}_a$-bundle defined by the cocycle $\{g_{ij}\}_{i,j\neq j'}$ therefore it follows from the induction hypothesis that $V \setminus H_1$ and $V \setminus H_{j'}$ are affine. Therefore the restriction maps $\psi|_{V\setminus H_1} : V \setminus H_1 \to \mathbb{A}^1$ and $\psi|_{V\setminus H_{j'}} : V \setminus H_{j'} \to \mathbb{A}^1$ are affine morphisms. Then it follows that $\psi^{-1}(\mathbb{A}^1 \setminus \{0\})$ and $\psi^{-1}(\mathbb{A}^1 \setminus \{1\})$ are affine. Namely, ψ is affine. □

2. Generalization of Theorems of Fujita-Iitaka and Nishimura

The main purpose of this section is to prove Theorem 0.4. As corollaries, We obtain the uniqueness of base schemes of principal $\mathbb{G}_a$-bundles in the special case (Corollary 2.5) and a criterion for principal $\mathbb{G}_a$-bundles over a certain type of schemes to be isomorphic to each other (Corollary 2.6).

Definition 2.1.

- For a k-scheme X, we denote by $X(k)$ the set of closed points (k-valued points) of X.
- A k-scheme X is called a *prevariety* if X is an integral scheme of finite type over k.
- A variety S is called a *variety with vanishing logarithmic genera* (or *VLG variety* for short) if S is a nonsingular variety with $\mathrm{P}_M(S) = 0$ for all $M \in \mathbb{Z}_{\geq 0}^{\oplus\infty}$, where $\mathrm{P}_M(S) = \dim_k \mathrm{H}^0(\overline{S}, \Omega_{\overline{S}}^{M}(\log \partial S))$ is the logarithmic M-genus of S and $(\overline{S}, \partial S)$ is a smooth completion of S with boundary ∂S.

We will slightly generalize Fujita-Iitaka's cancellation theorem [8] and the following Nishimura's theorem.

Theorem 2.2 (Nishimura [9])**.** *Let X and Y be varieties with $\overline{\kappa}(Y) \geq 0$, and let S_1 and S_2 be VLG varieties with $\dim S_1 = \dim S_2$. Let $p \colon V \to X$ be a S_1-bundle, $q \colon W \to Y$ a S_2-bundle. If $\Phi \colon V \to W$ is an isomorphism, then there exists a unique isomorphism $\phi \colon X \to Y$ such that the following diagram is commutative.*

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "q"] \\ X \arrow[r, "\exists\phi"'] & Y \end{tikzcd}$$

Lemma 2.3. *Let X and Y be prevarieties, and let f, $g \colon X \to Y$ be morphisms of prevarieties. If f and g coincide on closed points of X, then $f = g$ as a morphism of prevarieties.*

The proof of Lemma 2.3 is the same as the one for varieties.

The next lemma is a part of the proof of Fujita-Iitaka's Cancellation Theorem ([8]).

Lemma 2.4 ([8])**.** *Let X be a variety of dimension n, let Y be a variety of dimension $n+1$ with $\overline{\kappa}(X) \geq 0$, and let S be a VLG variety. If $f \colon X \times_k S \to Y$ is a morphism, then f is not dominant.*

Proof of Theorem 0.4. We prove Theorem 0.4 with 4 steps as follows:

1. For any prime divisor C on X, $E = q\Phi p^{-1}C$ is a prime divisor on Y.
2. We can show the same statement as (1) locally (this process is necessary to deal with closed points of the prevaries X and Y as an intersection of prime divisors on some open subset).
3. We can construct a bijective map of sets $\phi' \colon X(k) \to Y(k)$ such that $\phi \circ p = q \circ \Phi$.
4. We can construct a morphism of prevarieties $\phi \colon X \to Y$ such that $\phi|_{X(k)} = \phi'$ and $\phi \circ p = q \circ \Phi$.

(1) and (4) are almost the same as [8] and [9], but (2) and (3) contain a new part to deal with closed points of non-separated schemes.

(1) Let C be a prime divisor on X. By the local triviality of p, there exists an open affine covering $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ of X such that $p^{-1}U_\lambda \simeq U_\lambda \times_k S_1$. Take $\lambda \in \Lambda$ such that $U_\lambda \cap C \neq \emptyset$ and put $C_\lambda := C \cap U_\lambda$. Then we obtain the compositon $f\colon C_\lambda \times S_1 \to Y'$ as the following diagram.

$$\begin{tikzcd}
C_\lambda \times S_1 \arrow[r, "g_\lambda:\mathrm{iso}"] \arrow[dr] \arrow[ddrrr, bend right, "f"'] & p^{-1}C_\lambda \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[r, "\Phi"] \arrow[d, "p"'] & W \arrow[d, "q"] \\
 & C_\lambda \arrow[r, hook] & X \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "l"] \\
 & & & Y'
\end{tikzcd}$$

By Lemma 2.4, f is not dominant, and thus $q \circ \Phi|_{p^{-1}C}\colon p^{-1}C \to Y$ is not dominant. Put $E_C := \overline{q\Phi p^{-1}C}^{Y}$, where the closure is taken in Y. Then E_C is irreducible closed subset of Y and $q^{-1}E_C = q^{-1}(\overline{q\Phi p^{-1}C}^{Y}) \supseteq \Phi p^{-1}C$. Then it follows that $q^{-1}E_C = \Phi p^{-1}C$ because $q^{-1}E_C$ is irreducible and $\Phi p^{-1}C$ is prime divisor of W. As a consequence, we obtain the equality $E_C = qq^{-1}E_C = q\Phi p^{-1}C$.

(2) Put $V_\lambda := p^{-1}U_\lambda$, $W_\lambda := \Phi V_\lambda$, $Y_\lambda := qW_\lambda$, and $q_\lambda := q|_{W_\lambda}$. Then Y_λ is an open subset of Y because q is locally trivial. Put $E_{C,\lambda} := \overline{q\Phi p^{-1}C_\lambda}^{Y_\lambda}$, where the closure is taken in Y_λ. Then $q^{-1}E_C \cap W_\lambda \supseteq q_\lambda^{-1}E_{C,\lambda} = q_\lambda^{-1}\overline{q\Phi p^{-1}C_\lambda}^{Y_\lambda} \supseteq \Phi p^{-1}C_\lambda$. Moreover, $q^{-1}E_C \cap W_\lambda$ and $\Phi p^{-1}C_\lambda$ are prime divisors on W_λ. Thus $q^{-1}E_{C,\lambda} = \Phi p^{-1}C_\lambda$. As a consequence, we obtain the equality $E_{C,\lambda} = qq_\lambda^{-1}E_{C,\lambda} = q\Phi p^{-1}C_\lambda$.

(3) Let $x \in X(k)$. Then there exists $\lambda \in \Lambda$ such that $x \in U_\lambda$. In U_λ, x can be expressed as an intersection of prime divisors $C_{1,\lambda}, \cdots, C_{m,\lambda}$ on U_λ since U_λ is a variety. Put $C_i := \overline{C_{i,\lambda}}^{Y}$ and $E_{C,i,\lambda} := q\Phi p^{-1}C_{i,\lambda}$. Then

$$\begin{aligned}
S_1 &\simeq p^{-1}(x) \simeq \Phi p^{-1}(x) = \Phi p^{-1}(\bigcap_{i=1}^{m} C_{j,\lambda}) = \bigcap_{i=1}^{m} \Phi p^{-1}(C_{i,\lambda}) \\
&= \bigcap_{i=1}^{m} q_\lambda^{-1}(E_{C,i,\lambda}) = q_\lambda^{-1}(\bigcap_{i=1}^{m} E_{C,i,\lambda}).
\end{aligned}$$

The fiber of q_λ is not necessarily equal to S_2, but is a nonempty open subset of S_2. Therefore $\dim \bigcap_{i=1}^{m} E_{C,i,\lambda} = \dim S_1 - \dim S_2 = 0$. Moreover, $\bigcap_{i=1}^{m} E_{C,i,\lambda} = q\Phi p^{-1}(x)$ is irreducible. Then it follows that $\bigcap_{i=1}^{m} E_{C,i,\lambda}$ is a closed point of Y, denoted by y_x. In this way, we obtain a map $\phi'\colon X(k) \to Y(k); x \mapsto y_x$ of sets. Moreover, ϕ' satisfies

$$q\Phi(v) \in q\Phi(p^{-1}p(v)) = q\Phi(\Phi^{-1}q^{-1}(y_{p(v)})) = \{y_{p(v)}\} = \{\phi'(p(v))\},$$

that is, $q \circ \Phi = \phi' \circ p$. The injectivity of ϕ' also follows because $x = p\Phi^{-1}q^{-1}q\Phi p^{-1}(x) = p\Phi^{-1}q^{-1}\phi'(x)$ for any closed point $x \in X$. Surjectivity of ϕ' is obvious.

(4) For each $\lambda \in \Lambda$, we take a closed point $a_\lambda \in S_1$ and define $\phi_{\lambda,a_\lambda} := q \circ \Phi \circ g_\lambda \circ j_\lambda \circ a_\lambda : U_\lambda \to Y$, where $j_\lambda : U_\lambda \times \{a_\lambda\} \hookrightarrow U_\lambda \times S_1$ and $a_\lambda : U_\lambda \simeq U_\lambda \times \{a_\lambda\}$. Then, for any closed point $x \in U_\lambda$, $\phi_{\lambda,a_\lambda}(x) = \phi'(x)$. Therefore by the Lemma 2.3, we can glue the morphisms $\{\phi_{\lambda,a_\lambda}\}$ to a morphism $\phi : X \to Y$ such that $q \circ \Phi = \phi \circ p$. The converse morphism of ϕ can be constructed in the same way. Moreover, the uniqueness of ϕ follows from the equality $\phi(x) = \phi(pp^{-1}(x)) = q\Phi(p^{-1}(x))$ for any closed point $x \in X$ and Lemma 2.3. $\square$

Corollary 2.5. *Let X and Y be prevarieties, let Y' be a variety with $\overline{\kappa}(Y') \geq 0$ and $\dim Y' = \dim Y$, let $l\colon Y \to Y'$ be a dominant morphism, and let V, W be principal $\mathbb{G}_a$-bundles over X, Y, respectively. Then*

(1) $V \simeq W \Rightarrow X \simeq Y$.
(2) $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1 \Rightarrow X \simeq Y$.

Corollary 2.6. *Let X be an affine variety with $\overline{\kappa}(X) \geq 0$, let $Z_1, ..., Z_m$ be hypersurfaces of X defined by $f_1, ..., f_m \in A$, respectively, let $Z := \bigcup Z_j$, and for $k = 1, 2$, let V_k be a principal $\mathbb{G}_a$-bundle over X_+rZ. Then V_1 and V_2 are isomorphic if and only if they are in the same orbit of the action by $\mathrm{Aut}(X_+rZ) \times \Gamma(X, \mathcal{O}_X^\times)$.*

Proof. The computation of this proof is almost the same as in [3]. Suppose that $\Phi\colon V_1 \to V_2$ is an isomorphism. Then by Theorem 0.4, there exists a unique automorphism $\phi\colon X_+rZ \to X_+rZ$ which satisfies $\phi \circ p_1 = p_2 \circ \Phi$, where $p_k : V_k \to X_+rZ$ is the canonical projection of principal $\mathbb{G}_a$-bundles for $k = 1, 2$. Put $X'_i := \phi(X_i)$ and $\mathcal{X}' := \{X'_0, \ldots, X'_r\}$ which becomes an open covering of X_+rZ. Suppose that the principal $\mathbb{G}_a$-bundle V_1 is defined by a Čech cocycle $\{g_{ij}\} \in \mathrm{Z}^1(\mathcal{X}, \mathcal{O}_{X_+rZ})$ and V_2 is defined by a Čech cocycle $\{g'_{ij}\} \in \mathrm{Z}^1(\mathcal{X}', \mathcal{O}_{X_+rZ})$. Then the following diagram is commutative for each i, $j = 0, \ldots, r$ $(i \neq j)$;

$$\begin{tikzcd}
(X_i \cap X_j) \times_k \mathbb{A}^1 \arrow[r, "g_i"] \arrow[d, "\alpha_{ij}"] & p_1^{-1}(X_i \cap X_j) \arrow[r, "\Phi"] \arrow[d, "\mathrm{id}"] & p_2^{-1}(X'_i \cap X'_j) \arrow[r, "g_i'^{-1}"] \arrow[d, "\mathrm{id}"] & (X'_i \cap X'_j) \times_k \mathbb{A}^1 \arrow[d, "\alpha'_{ij}"] \\
(X_i \cap X_j) \times_k \mathbb{A}^1 \arrow[r, "g_j"] \arrow[dr, "\mathrm{pr}_1"'] & p_1^{-1}(X_i \cap X_j) \arrow[r, "\Phi"] \arrow[d, "p_1"] & p_2^{-1}(X'_i \cap X'_j) \arrow[r, "g_j'^{-1}"] \arrow[d, "p_2"] & (X'_i \cap X'_j) \times_k \mathbb{A}^1 \arrow[dl, "\mathrm{pr}_1"] \\
 & X_i \cap X_j \arrow[r, "\phi"] & X'_i \cap X'_j &
\end{tikzcd}$$

where $\alpha_{ij}(x, t) = (x, t + g_{ij}(x))$, $\alpha'_{ij}(x', t) = (x', t + g'_{ij}(x'))$. Moreover, by the commutativity of this diagram, there exists $a_i \in \Gamma(X_i, \mathcal{O}_{X_+rZ}^\times) = A^\times$ and $b_i \in \Gamma(X_i, \mathcal{O}_{X_+rZ})$ for each $i = 0, \ldots, r$ such that $g_i'^{-1} \circ \Phi \circ g_i(x, t) = (\phi(x), a_i t + b_i)$, $g_j'^{-1} \circ \Phi \circ g_j(x, t) = (\phi(x), a_j t + b_j)$. This is because for an isomorphism $f : A[t] \to B[t]$ of domains such that $f|_A^B : A \to B$ is an isomorphism, $f(t)$ should be equals to $at + b$ where $a \in B^\times$ and $b \in B$ by the computation of degree of t. By the commutativity of the above diagram once again, we obtain the following equation;

$$a_i(x)t + b_i(x) + g'_{ij}(\phi(x)) = a_j(x)(t + g_{ij}(x)) + b_j(x).$$

Thus, gluing $\{a_i\}$, we obtain $a \in \Gamma(X_+rZ, \mathcal{O}_{X_+rZ}^\times) \simeq \Gamma(X, \mathcal{O}_X^\times)$. Moreover, we have

$$g'_{ij}(\phi(x)) - g_{ij}(x)a(x) = b_j(x) - b_i(x),$$

that is, cocyles $\{g'_{ij}(\phi(x))\}$ and $\{g_{ij}(x)a(x)\}$ define principal $\mathbb{G}_a$-bundles isomorphic to each other. □

3. The proof of main theorem

The purpose of this section is to prove Theorem 0.3.

Lemma 3.1 ([5])**.** *Let V be an affine $\mathbb{G}_a$-surface. Then the GIT quotient $V//\mathbb{G}_a$ is a nonsingular affine curve and there exists an open affine subset $C \subseteq V//\mathbb{G}_a$ such that $p^{-1}C$ is $\mathbb{G}_a$-equivariantly isomorphic to $C \times_k \mathbb{G}_a$, where $p\colon V \to V//\mathbb{G}_a$ is the quotient morphism.*

Theorem 3.2 ([6])**.** *Let X be a nonsingular affine curve, let W be an $\mathbb{A}^1$-fibered affine surface over X. Then W is Zariski 1-factor if and only if W is a line bundle over X.*

Proof of Theorem 0.3. It is easy to see that V is affine and nonsingular. Take a closed point $a \in \mathbb{A}^1$ and define a morphism $p' : V \to Y$ to be the composition $V \simeq V \times_k \{a\} \hookrightarrow V \times_k \mathbb{A}^1 \to W \times_k \mathbb{A}^1 \to W \to Y$. We show that p' satisfies the condition of the structure morphism of a principal $\mathbb{G}_a$-bundle over Y.

Suppose that V has no nontrivial $\mathbb{G}_a$-action for contradiction. Then $V \simeq W$ holds by the cancellation theorem for varieties with no nontrivial $\mathbb{G}_a$-action ([1]), but this contradicts to that V has no nontrivial $\mathbb{G}_a$-action. Thus V has a nontrivial $\mathbb{G}_a$-action. We fix a nontrivial $\mathbb{G}_a$-action on V, denoted by $\mu : \mathbb{G}_a \times V \to V$.

Let $B := V//\mu$ be the GIT quotient of μ and $p : V \to B$ be its quotient morphism. By Lemma 3.1, B is a nonsingular affine curve and there exists a nonempty open subset $C \subseteq B$ such that the following diagram is commutative.

$$\begin{tikzcd}
C \times \mathbb{A}^1 \arrow[r, "\text{iso}"] \arrow[dr] & P^{-1}C \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "p"] \\
 & C \arrow[r, hook] & B
\end{tikzcd}$$

.

Using the same arguments as Theorem 0.4, we obtain an injective morphism $\phi : C \to Y$ such that the following diagram commutes.

$$\begin{tikzcd}
 & p^{-1}C \times_k \mathbb{A}^1 \arrow[r, hook] \arrow[d, "\text{pr}"] \arrow[dr, phantom, "\circlearrowleft"] & V \times_k \mathbb{A}^1 \arrow[r, "\Phi\colon \text{iso}"] \arrow[d, "\text{pr}"] & W \times_k \mathbb{A}^1 \arrow[d, "\text{pr}"] \\
C \times \mathbb{A}^1 \arrow[r, "\text{iso}"] \arrow[dr] & p^{-1}C \arrow[r, hook] \arrow[d, "p"] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "p"] \arrow[r, phantom, "\circlearrowleft"] & W \arrow[d, "q"] \\
 & C \arrow[r, hook] \arrow[rr, bend right, "\exists\phi"'] & B & Y \arrow[d, "l"] \\
 & & & Y'
\end{tikzcd}$$

Next, we construct a morphism $\psi : Y \to B$. At first, we show that there exists a map $\psi' : Y(k) \to B(k)$ of sets such that $\psi' \circ q \circ \text{pr}_W = p \circ \text{pr}_V \circ \Phi^{-1}$. For any closed point $y \in \phi(C) \subseteq Y$, there exists a unique closed point $y \in C$ such that $x = \phi(y)$. On the other hand, for any closed point $y \in Y \setminus \phi(C)$, a morphism

$$\tau_y := p \circ \text{pr}_V \circ \Phi^{-1} \circ j_y : \text{pr}^{-1}q^{-1}(y) \hookrightarrow W \times_k \mathbb{A}^1 \simeq V \times_k \mathbb{A}^1 \to V \to B$$

is not dominant. (If τ_y is dominant, then $C \cap \tau_y \text{pr}_V^{-1} q^{-1}(y) \neq \emptyset$, but this contradicts to the construction of ϕ.) Then it follows that the image of τ_y is a closed point of B. we define a morphism $\psi' \colon Y(k) \to B(k)$ of sets to be $\psi(y) = \text{Im}\tau_y$ for each $y \in Y(k)$. Then we have the equality $\psi' \circ q \circ \text{pr}_W = p \circ \text{pr}_V \circ \Phi^{-1}$. Let $\{Y_\lambda\}_{\lambda \in \Lambda}$ be an open affine covering of Y. Put $W_\lambda := q^{-1}Y_\lambda (\simeq Y_\lambda \times_k \mathbb{A}^1)$, $(V \times_k \mathbb{A}^1)_\lambda := \Phi^{-1}(W_\lambda \times_k \mathbb{A}^1)$, $V_\lambda := \text{pr}_V((V \times_k \mathbb{A}^1)_\lambda)$ and $X_\lambda := pV_\lambda$.

Next we observe that $(V \times_k \mathbb{A}^1)_\lambda = V_\lambda \times_k \mathbb{A}^1$. Put $K_{y,\lambda} := \overline{\text{pr}_V \circ \Phi^{-1} \circ Q^{-1}(y)}^{V_\lambda}$. Then $(V \times_k \mathbb{A}^1)_\lambda = \bigcup_{y \in Y_\lambda} \Phi^{-1}Q^{-1}(y)$ and $V_\lambda = \bigcup_{y \in Y_\lambda} K_{y,\lambda}$. Moreover, we have $\text{pr}^{-1}K_{y,\lambda} \supseteq \Phi^{-1}Q^{-1}(y)$, and thus $\text{pr}^{-1}K_{y,\lambda} = \Phi^{-1}Q^{-1}(y)$. As a consequence, we obtain the equality $(V \times_k \mathbb{A}^1)_\lambda = V_\lambda \times_k \mathbb{A}^1$.

For each $\lambda \in \Lambda$, we take a closed point $(a_{1,\lambda}, a_{2,\lambda}) \in \mathbb{A}^2$ and define

$$\begin{aligned}\psi_{\lambda,a_\lambda} \colon Y_\lambda \ &\xrightarrow{a_{1,\lambda}}\ Y \times_k \{a_{1,\lambda}\} \hookrightarrow Y_\lambda \times \mathbb{A}^1 \to W_\lambda \xrightarrow{a_{2,\lambda}} W_\lambda \times_k \{a_{2,\lambda}\} \\ &\hookrightarrow\ W_\lambda \times_k \mathbb{A}^1 \to V_\lambda \times_k \mathbb{A}^1 \to V_\lambda \to B.\end{aligned}$$

Then for any closed point $y \in Y_\lambda$, $\psi_{\lambda,a_{1,\lambda}}(y) = \psi'(y)$. Therefore by Lemma 2.3, we can glue the morphisms $\{\psi_{\lambda,a_{1,\lambda}}\}$ to a morphism $\psi : Y \to B$. Moreover, ψ is a surjective birational morphism and satisfies $q \circ \mathrm{pr}_V \circ \Phi^{-1} = \psi \circ p \circ \mathrm{pr}_W$.

The variety W_λ is isomorphic to $Y_\lambda \times_k \mathbb{A}^1$ because W_λ is a principal $\mathbb{G}_a$-bundle over Y_λ and Y_λ is affine. By Theorem 3.2, we obtain an isomorphism $F_\lambda \colon V_\lambda \simeq W_\lambda$. In the same way as ψ, we obtain a surjective birational morphism $f_\lambda \colon Y_\lambda \to B_\lambda$ such that $f_\lambda \circ q = F_\lambda^{-1} \circ p$. Moreover Y_λ and B_λ are nonsingular curves. Thus f_λ (and $\psi_\lambda := \psi|_{Y_\lambda}^{B_\lambda}$) should be an isomorphism. In conclusion, we obtain the following commutative diagram.

$$\begin{tikzcd}
V_\lambda \arrow[r, "F_\lambda"] \arrow[d, "p_\lambda"] \arrow[dr, phantom, "\circlearrowleft"] & W_\lambda \arrow[r, "g_\lambda"] \arrow[d, "q"] & Y_\lambda \times_k \mathbb{A}^1 \arrow[r, "f_\lambda \times \mathrm{id}_{\mathbb{A}^1}"] \arrow[dl, "\mathrm{pr}_{Y_\lambda}"'] & B_\lambda \times_k \mathbb{A}^1 \arrow[dll, "\mathrm{pr}_{Y_\lambda} \circ (f_\lambda^{-1} \times \mathrm{id}_{\mathbb{A}^1})"] \\
B_\lambda & Y_\lambda \arrow[l, "f_\lambda"] & &
\end{tikzcd}$$

For a closed point $x \in C_\lambda = B_\lambda \cap C$, the fiber $p_\lambda^{-1}(x)$ is just the orbit of μ by the construction of p and p_λ. For a closed point $x \in B_\lambda \setminus C_\lambda$, we can not say that the fiber $p_\lambda^{-1}(x)$ is just the orbit only by the construction. But thanks to the fact that each $\mathbb{G}_a$-orbits are closed ([10]) and are isomorphic to either $\mathbb{A}^1$ or a closed point, we can deduce that $p_\lambda^{-1}(x)$ is just the μ-orbit. Therefore, the action μ on V can be restricted on V_λ, denoted by $\mu|_{V_\lambda}$, and its quotient morphism is just $p_\lambda : V_\lambda \to B_\lambda$. By the commutativity of the above diagram, such a $\mathbb{G}_a$-action should be $\mathbb{G}_a$-equivariantly trivial, that is, $p_\lambda : V_\lambda \to B_\lambda$ is a $\mathbb{G}_a$-equivariantly trivial morphism.

By the construction of p' and ψ, we have the equality $p'_\lambda = \psi_\lambda^{-1} \circ p_\lambda$. Then it follows that p' satisfies the condition of the structure morphism of a principal $\mathbb{G}_a$-bundle over Y. $\square$

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