2015年度　修士論文

SHGH conjecture
and
the irrationality of Seshadri constants

早稲田大学大学院基幹理工学研究科
数学応用数理専攻
学籍番号　5114A048-1

広川　未流

指導教員名　楫　元

SHGH CONJECTURE AND THE IRRATIONALITY OF SESHADRI CONSTANTS

MIRU HIROKAWA

Abstract. In this paper, we study the relation between SHGH conjecture and the irrationality of Seshadri constants on blow-ups of $\mathbb{P}^2$, and generalize the result of [5]. Moreover, by investigating SHGH conjecture on $\mathbb{P}^1 \times \mathbb{P}^1$ which was formulated in [6], we give a new form of an ample divisor on $\mathbb{P}^2$ which has an irrational Seshadri constant.

1. Introduction

We investigate the relation between Segre-Harbourne-Gimigliano-Hirschowitz conjecture and the irrationality of Seshadri constants of ample divisors on blow-ups of $\mathbb{P}^2$.

SHGH conjecture was first formulated by A. Hirschowitz in [8]:

Conjecture 1.1 (SHGH conjecture). *Let X_r be the blow-up of $\mathbb{P}^2$ at r general points with exceptional divisors $e_1, \cdots, e_r$. We denote by l the pullback to X_r of $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$. Let $d, m_1, \cdots, m_r$ be integers with $m_1 \geq \cdots \geq m_r \geq -1$ and $d \geq m_1 + m_2 + m_3$. Then the divisor*

$$D = dl - \sum_{i=1}^{r} m_i e_i$$

is nonspecial.

It is known that SHGH conjecture implies Nagata conjecture [2], and is settled for $r \leq 9$ in [3]. We reffer to [3] for more information about SHGH conjecture.

Seshadri constants were introduced originally by J.P. Demailly in [4] to study the local positivity of ample line bundles. This is a real number which is determined by each ample divisor on smooth projective varieties (see Definition 2.3). However, an ample divisor with irrational Seshadri constant has never been found in the literature.

The following is our main result of this paper.

Theorem 1.2. *Let $r \geq 9$ be an integer such that SHGH conjcture for $r+1$ on $\mathbb{P}^2$ holds. Then there exists an ample divisor A on the blow-up of $\mathbb{P}^2$ at general r points such that the Seshadri constant $\varepsilon_{\text{gen}}(A)$ is irrational.*

In [5], M. Dumnicki, A. Kronya, C. Maclean and T. Szemberg investigated the relation between SHGH conjecture and the irrationality of Seshadri constants. Our result gives a generalization of their results. Moreover, our result has a corollary which insures the existence of ample divisors with irrational Seshadri constant on "MANY" surfaces in the assumption of SHGH conjecture.

Corollary 1.3. *Let $r \geq 9$ be an integer such that SHGH conjecture for $r+1$ on $\mathbb{P}^2$ holds. Then there exists an ample divisor A on the blow-up of $\mathbb{P}^2$ at general a points such that A has a homogeneous form, $A = dl - m\sum_{i=1}^{a} e_i$ $(d, m \in \mathbb{Z})$, and $\varepsilon_{\text{gen}}(A)$ is irrational for all $a \in \{sn^2 \mid s, n \in \mathbb{N},\ 9 \leq s \leq r\}$.*

Moreover, we have a result on $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 1.4. *Let $r \geq 8$ be an integer such that SHGH conjcture for $r+1$ on $\mathbb{P}^1 \times \mathbb{P}^1$ holds. Then there exists an ample divisor A on the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at general r points such that $\varepsilon_{\text{gen}}(A)$ is irrational.*

The statement of Theorem 1.4 is similar to the one of Theorem 1.2. But Theorem 1.4 implies the existence of a new form of ample divisors which do not appear in the conclusion of Theorem 1.2 (see Remark on p.8):

Corollary 1.5. *Let $r \geq 8$ be an integer such that SHGH conjcture for $r+1$ on $\mathbb{P}^1 \times \mathbb{P}^1$ holds. Then there exists an ample divisor A on the blow-up of $\mathbb{P}^2$ at general r points such that $D = al - \sum_{i=1}^{r-2} be_i - ce_{r-1} - ce_r$, where a, b, c are nonnegative integers with $a = b + 2c$ and $\sqrt{\frac{2}{r+1}} < \frac{b}{a-b} < \sqrt{\frac{2}{r}}$.*

Acknowledgements. The author would like to thank his supervisor Professor Hajime Kaji for many discussions and helpful comments. He would also like to thank Professor Yasunari Nagai for useful suggestion. He also thanks Professor Kazunori Yasutake who gave me advice about blow-up of $\mathbb{P}^2$ at general $r \leq 8$ points, that is, del Pezzo surfaces. He also thanks Professor Taku Suzuki and Professor Daizo Ishikawa for their encouragements and all the other members of Professor Kaji and Professor Nagai's laboratories.

2. Definitions and Basic properties

Notation. Let $\pi : X_r \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at r general points $p_1, \cdots, p_r$ with exceptional divisors $e_1, \cdots, e_r$ and $l := \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$.

Under this notation, we can express the Picard group of X_r as follows:

$$\operatorname{Pic} X_r = \mathbb{Z} \cdot l \oplus \mathbb{Z} \cdot e_1 \oplus \cdots \oplus \mathbb{Z} \cdot e_r$$

and the intersection theory on X_r is determined by the following rules:

$$l^2 = 1,\ l.e_i = 0,\ e_i.e_j = -\delta_{ij}\ (i, j = 1, \cdots, r).$$

Let D be a divisor on X_r. If $D.l \geq -2$ holds, then $h^2(X_r, \mathcal{O}_{X_r}(D)) = 0$ by Serre duality.

Definition 2.1. *Let $D = dl - \sum_{i=1}^{r} m_i e_i$ $(d, m_1, \cdots, m_r \in \mathbb{Z})$ be a divisor on X_r. We define the virtual dimension and the expected dimension of D as follows.*

$$v\text{-}\dim(D) := \frac{(d+1)(d+2)}{2} - \sum_{i=1}^{r} \frac{m_i(m_i+1)}{2},$$
$$e\text{-}\dim(D) := \max\{v\text{-}\dim(D),\ 0\}.$$

In general, $h^0(X_r, \mathcal{O}_{X_r}(D)) \geq e\text{-}\dim(D)$ holds. When $h^0(X_r, \mathcal{O}_{X_r}(D)) > e\text{-}\dim(D)$, D is called special. Otherwise, D is called nonspecial.

Remark. Riemann Roch theorem implies that what D is special is equaivalent to

$$h^0(X_r, \mathcal{O}_{X_r}(D)) \cdot h^1(X_r, \mathcal{O}_{X_r}(D)) \neq 0.$$

It is well known that SHGH conjecture implies Nagata conjecture. The latter concerns plane curves and is related to counterexamples of the Hilbert's 14-th problem.

Conjecture 2.2 (Nagata conjecture). *Let $p_1, \cdots, p_r$ be r general points on $\mathbb{P}^2$. Let C be an integral plane curve of degree d. If $\operatorname{mult}_{p_1} C = m_1, \cdots, \operatorname{mult}_{p_r} C = m_r$ $(m_1, \cdots, m_r \in \mathbb{Z})$, then*

$$d \geq \frac{1}{\sqrt{r}} \sum_{i=1}^{r} m_i.$$

Finally, we recall the definition of Seshadri constants.

Definition 2.3. *Given a smooth projective variety X and a nef divisor L on X, the Seshadri constant of L at a point $p \in X$ is the real number defined by*

$$\varepsilon(L;p) := \sup\{t \in \mathbb{R} \mid \mu^*L - tE\colon \text{ nef on } \mathrm{Bl}_p(X)\},$$

where $\mu : \mathrm{Bl}_p(X) \to X$ is blow-up at p with the exceptional divisor E.

It follows immediately from the difinition that $\varepsilon(L;p) \le \sqrt{L^n}$. In fact, if X is a surface and $\varepsilon(L;p)$ is a irrational number, then it must be equal to $\sqrt{L^2}$. However, any example of a divisor on a variety whose Seshadri constant at a point is irrational has not been known. It is also known that $\varepsilon(L;p)$ has a constant value at very general points p [11]. We denote this constant by $\varepsilon_{\mathrm{gen}}(L)$.

3. SHGH conjecture for $\mathbb{P}^2$ and the irrationality of Seshadri constants

First of all, we recall the property of the blow-up and the global section of line bundles.

Lemma 3.1. *Let X be a surface, and P a point of X. Let $\pi : \tilde{X} \to X$ be the blow-up with center P. Let $\mathcal{L}$ be a line bundle on X. Then $h^0(\tilde{X}, \pi^*\mathcal{L}) = h^0(X, \mathcal{L})$.*

proof. By the projection formula, we have $\pi_*(\mathcal{O}_{\tilde{X}} \otimes_{\tilde{X}} \pi^*\mathcal{L}) = \pi_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}$. Since π is a blow-up at one point, we have $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ [7, V Proposition 3.5]. Substituting this equation for the projection formula, we get $\pi_*\pi^*\mathcal{L} = \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L} = \mathcal{L}$. Therefore, $h^0(\tilde{X}, \pi^*\mathcal{L}) = h^0(X, \pi_*\pi^*\mathcal{L}) = h^0(X, \mathcal{L})$. □

In [5], they found the relation between SHGH conjecture and the irrationality of Seshadri constants:

Theorem 3.2 ([5])**.** *Let $r \ge 9$ be an integer such that the SHGH conjecture holds true for $r+1$ points. Then*

a) *there exists an ample line bundle on X_r whose Seshadri constant at a very general point is irrational, or*
b) *the SHGH conjecture fails for r points.*

This is the first result which refers to the relation between two problems. Moreover, they got an interesting corollary:

Corollary 3.3 ([5])**.** *If SHGH conjecture for 10 points holds true, there exists an ample line bundle on X_9 whose Seshadri constant at a very general point is irrational.*

This corollary can be obtained by the combination of their result [5, Theorem 1.1] and the fact that the SHGH conjecture for 9 points holds.

It is enough to prove the next key proposition in order to complete the proof of our main theorem by combinations of Theorem 3.2.

Proposition 3.4. *Let r be a positive integer. If SHGH conjecture for $r+1$ on $\mathbb{P}^2$ is true, then it is true also for r.*

proof. Let D be a divisor $dl - \sum_{i=1}^{r} m_i e_i$ on X_r with $m_1 \ge m_2 \ge \cdots \ge m_r \ge -1$ and $d \ge m_1 + m_2 + m_3$. We show that D is nonspecial on X_r.

When $d < 0$, D is not effective. This implies that D is nonspecial. So, we assume $d \ge 0$. We define $M(D)$ as following.

$$M(D) = \max\{m_i + m_j + m_k \mid i, j, k \text{ are distinct}\}$$

Let $\pi : X_{r+1} \to X_r$ be the blow-up of X_r at a general point.

Case 0: $m_1 \ge m_2 \ge m_3 \ge 0$.

Since $\pi^*D = dl - \sum_{i=1}^{r} m_i e_i - 0e_{r+1}$, $M(D) = m_1 + m_2 + m_3$ and SHGH conjecture for $r+1$ implies that π^*D is nonspecial on X_{r+1}. Therefore,

$$\begin{aligned} h^0(X_r, \mathcal{O}_{X_r}(D)) &= h^0(X_{r+1}, \mathcal{O}_{X_{r+1}}(\pi^*D)) \\ &= e\text{-dim}(\pi^*D) \\ &= \max\{0, \frac{(d+1)(d+2)}{2} - \sum_{i=1}^{r} \frac{m_i(m_i+1)}{2} - \frac{0 \times 1}{2}\} \\ &= \max\{0, \frac{(d+1)(d+2)}{2} - \sum_{i=1}^{r} \frac{m_i(m_i+1)}{2}\} = e\text{-dim}(D). \end{aligned}$$

We conclude that D is nonspecial.

Case 1: $0 > m_1 \geq m_2 \geq m_3$.

In this case, $m_1 = m_2 = m_3 = -1$ and $M(\pi^*D) = 0 + m_1 + m_2$. Since $d \geq 0$, we have $d \geq 0 + m_1 + m_2 = M(\pi^*D)$. SHGH conjecture for $r+1$ implies that π^*D is nonspecial. Therefore,

$$\begin{aligned} h^0(X_r, \mathcal{O}_{X_r}(D)) &= h^0(X_{r+1}, \mathcal{O}_{X_{r+1}}(\pi^*D)) \\ &= e\text{-dim}(\pi^*D) = e\text{-dim}(D). \end{aligned}$$

Case 2: $m_1 \geq 0 > m_2 \geq m_3$.

In this case, $m_2 = m_3 = -1$. If $d = 0$ and $m_1 = 0$, then $d = 0 \geq 0 + 0 + (-1) = m_1 + 0 + m_2$. SHGH conjecture for $r+1$ implies that π^*D is nonspecial. Similarly, we can conclude that D is nonspecial.

If $d > 0$ and D is not effective, then D is nonspecial. We assume that D is effective. We put $L = l - e_1$. Since L is nef, $D.L = d - m_1 \geq 0$. Therefore, $d \geq m_1 \geq m_1 - 1 = m_1 + 0 + m_2 = M(\pi^*D)$ and SHGH conjecture for $r+1$ implies that π^*D is nonspecial. Similarly, D is nonspecial.

Case 3: $m_1 \geq m_2 \geq 0 > m_3$.

In this case, $m_3 = \cdots = m_r = -1$ and $D = dl - m_1e_1 - m_2e_2 + e_3 + \cdots + e_r$. Consider the ideal exact sequence:

$$0 \to \mathcal{O}_{X_r}(D - e_r) \to \mathcal{O}_{X_r}(D) \to \mathcal{O}_{e_r}(D|_{e_r}) \to 0$$

Taking the long exact sequence, we get the following.

$$0 \to H^0(X_r, \mathcal{O}_{X_r}(D - e_r)) \to H^0(X_r, \mathcal{O}_{X_r}(D)) \to H^0(e_r, \mathcal{O}_{e_r}(D|_{e_r}))$$

Since e_r is (-1)-curve, $H^0(e_r, \mathcal{O}_{e_r}(D|_{e_r})) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D.e_r)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. Therefore, $H^0(X_r, \mathcal{O}_{X_r}(D)) \simeq H^0(X_r, \mathcal{O}_{X_r}(D - e_r))$. By the induction on r, we have

$$\begin{aligned} H^0(X_r, \mathcal{O}_{X_r}(D)) &\simeq H^0(X_r, \mathcal{O}_{X_r}(dl - m_1e_1 - m_2e_2 - 0e_3 - \cdots - 0e_r)) \\ &= H^0(X_r, \mu^*(\mathcal{O}_{X_2}(dl - m_1e_1 - m_2e_2))) \\ &= H^0(X_2, \mathcal{O}_{X_2}(dl - m_1e_1 - m_2e_2)), \end{aligned}$$

where $\mu : X_r \to X_2$ is the blow-up at $p_3, \cdots, p_r$ which lie on $X_r \setminus (e_1 \cup e_2)$. Since SHGH conjecture for $r = 2$ holds, D is nonspecial. □

Remark. From the proof of [5, Theorem 1.1], we see that the divisor with irrational Seshadri constant is of the form:

$$D = dl - m\sum_{i=1}^{r} e_i,$$

where $\frac{1}{\sqrt{r+1}} \leq \frac{m}{d} \leq \frac{1}{\sqrt{s}}$.

Next, we prove that SHGH conjecture implies the existence of ample divisors with irrational Seshadri constant on "MANY" surfaces. We recall an interesting proposition proved by a method of degenerations.

Lemma 3.5 ([1])**.** *Consider the plane $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$ over the field of complex number $\mathbb{C}$. Let $D = dl - \sum_{i=1}^{r} m_i e_i - mE$ be an ample (resp. nef) divisor on X_{r+1} and $F = ml - \sum_{i=1}^{s} \alpha_i E_i$ a nef divisor on X_s. Then, the divisor $dl - \sum_{i=1}^{r} m_i e_i - \sum_{i=1}^{s} \alpha_i E_i$ is ample (resp. nef) on X_{r+s}.*

Corollary 3.6. *Let $r \geq 9$ be an integer such that SHGH conjecture for $r+1$ on $\mathbb{P}^2$ holds. Then there exists an ample divisor $A \in \mathrm{Pic}(X_a)$ which has a homogeneous form and $\varepsilon_{\mathrm{gen}}(A)$ is irrational for all $a \in \{sn^2 \mid s, n \in \mathbb{N},\ 9 \leq s \leq r\}$.*

proof . Let s be an integer with $9 \leq s \leq r$. By our main theorem, there exists an ample divisor $A = dl - \sum_{i=1}^{s} m e_i$ on X_s with irrational Seshadri constant, that is, $\varepsilon_{\mathrm{gen}}(A) = \sqrt{A^2}$. We simply denote $\varepsilon_{\mathrm{gen}}(A)$ by ε. The definition of Seshadri constant implies that $\mu^* A - \varepsilon E$ is a nef divisor on $\tilde{X}_s$, where $\mu : \tilde{X}_s \to X_s$ is blow-up of X_s at a very general point with the exceptional divisor E.

Since Nagata conjecture holds for perfect square, $N_{n^2} = l - \frac{1}{n}\sum_{j=1}^{n^2} E_j$ is nef on X_{n^2}. By applying the Lemma 3.5 to A and N_{n^2}, we obtain $\overline{A} = dl - \frac{m}{n} \sum_{1 \leq i \leq s, 1 \leq j \leq n^2} e_{ij}$ is ample on X_{sn^2}, where e_{ij} are exceptional divisors on X_{sn^2}.

Similarly, $\mu'^* \overline{A} - \varepsilon E'$ is nef on $\widetilde{X_{sn^2}}$ which is the blow-up of X_{sn^2} at a point with the natural projection μ' and the exceptional divisor E'. This implies $\varepsilon \leq \varepsilon_{\mathrm{gen}}(\overline{A})$. On the other hand, since we can calculate $\overline{A}^2 = d^2 - \frac{m^2}{n^2} sn^2 = A^2$, we have $\varepsilon_{\mathrm{gen}}(\overline{A}) \leq \sqrt{\overline{A}^2} = \sqrt{A^2}$. Consequently, we get $\varepsilon_{\mathrm{gen}}(\overline{A}) = \varepsilon$. Hence the ample divisor $n\overline{A}$ has an irrational Seshadri constant and lies in $\mathrm{Pic}\, X_{sn^2}$. $\square$

4. SHGH conjecure for $\mathbb{P}^1 \times \mathbb{P}^1$ and a remark on the irrationationality of Seshadri constants on $\mathbb{P}^2$

SHGH conjecture on the Hirzebruch surfaces was formulated in [6]. We focus on $\mathbb{P}^1 \times \mathbb{P}^1$ in particular.

Let $Y_r = (\mathbb{P}^1 \times \mathbb{P}^1)_r$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points with exceptional divisors $e_1, \cdots, e_r$. We denote by l_1, l_2 the pullback to Y_r of the $\mathbb{P}^1 \times \{pt\}$, $\{pt\} \times \mathbb{P}^1$ on $\mathbb{P}^1 \times \mathbb{P}^1$. For the definitions of the special, (-1)-special divisor, we reffer to [6]. SHGH conjecture for $\mathbb{P}^1 \times \mathbb{P}^1$ is following.

Conjecture 4.1 ([6])**.** *Special divisors on Y_r are (-1)-special.*

Brian-Nagata conjecture is known as a generalized Nagata conjecture [9]. We state Brian-Nagata conjecture on $\mathbb{P}^1 \times \mathbb{P}^1$.

Conjecture 4.2 (Brian-Nagata conjecture on $\mathbb{P}^1 \times \mathbb{P}^1$)**.** *Let r be an integer with $r \geq 8$, then*

$$\varepsilon(L; r) = \sqrt{\frac{2}{r}},$$

where $\varepsilon(L; r) = \sup\{t \in \mathbb{R} \mid L - t\sum_{i=1}^{r} e_i :\ \mathrm{nef}\}$, $L = l_1 + l_2$.

We can prove that SHGH conjecture for $\mathbb{P}^1 \times \mathbb{P}^1$ implies Brian-Nagata conjecture.

Lemma 4.3. *We assume that SHGH conjecture for r on $\mathbb{P}^1 \times \mathbb{P}^1$ holds. Any integral curve C on Y_r satisfies $C^2 \geq g(C) - 1$, where $g(C)$ is the genus of C.*

proof . Since C is an integral curve, SHGH conjecture implies that C is nonspecial. Therefore $v\text{-dim}(C) = h^0(C) \geq 1$. Since $v\text{-dim}(C) = \frac{1}{2}C.(C - K_{Y_r}) + 1 = C^2 - \frac{1}{2}C.(C + K_{Y_r}) + 1$, this implies $C^2 \geq P_a(C) - 1 \geq g(C) - 1$ by the adjunction formula. □

Proposition 4.4. *SHGH conjecture for $\mathbb{P}^1 \times \mathbb{P}^1$ implies Brian-Nagata conjecture for any $r \geq 8$.*

proof. Let C be an integral curve on Y_r which belongs to the linear system $al_1 + bl_2 - \sum_{i=1}^{r} m_i e_i$.

Case1: $C^2 \geq 0$

Since $C^2 = 2ab - \sum_{i=1}^{r} m_i^2 \geq 0$, $2ab \geq \sum_{i=1}^{r} m_i^2 \geq \frac{1}{r}\left(\sum_{i=1}^{r} m_i\right)^2$. The last inequality is Cauchy-Schwarz inequality. Therefore, $\sqrt{2rab} \geq \sum_{i=1}^{r} m_i$. The arithmetic mean and the geometric mean inequality implies $\sum_{i=1}^{r} m_i \leq \sqrt{2r}\frac{a+b}{2} = \sqrt{\frac{r}{2}}(a+b)$.

Case2: $C^2 < 0$

In this case, the previous Lemma 4.3 implies $g(C) = 0$ and $C^2 = -1$. Set $N_r = \sqrt{\frac{r}{2}}(l_1 + l_2) - \sum_{i=1}^{r} e_i$. Then we have the following inequality:

$$\begin{aligned} C.N_r &= C.(N_r + K_{Y_r}) - C.K_{Y_r} \\ &= (a+b)\left(\sqrt{\frac{r}{2}} - 2\right) - (-C^2 + 2g(C) - 2) \\ &= (a+b)\left(\sqrt{\frac{r}{2}} - 2\right) + 1 \geq 1. \end{aligned}$$

The case 1 and case 2 imply that Brian-Nagata conjecture holds. □

Lemma 4.5. *If there exists a curve $C \subset Y_r$ which attains the Seshadri constant of $\mathbb{Q}$-divisor $L = l_1 + l_2 - \alpha\sum_{i=1}^{r} e_i$ ($\alpha \in \mathbb{Q}$) at p, then there exists a divisor Γ with $\operatorname{mult}_{p_1}\Gamma = \cdots = \operatorname{mult}_{p_r}\Gamma = M$ attaining the Seshadri constant of L at p, that is,*

$$\frac{L.\Gamma}{\operatorname{mult}_p \Gamma} = \frac{L.C}{\operatorname{mult}_p C} = \varepsilon(L; p).$$

proof. Apply the same discussion as [5, Lemma 2.1]. □

Lemma 4.6. *Let $r \geq 8$ be an integer. The function*

$$f(\delta) = (2\sqrt{r+1} - \sqrt{2}r)\sqrt{2 - r\delta^2} - (r\sqrt{r+1} - \sqrt{2}r)\delta + \sqrt{2}r - 2\sqrt{2}$$

takes non-negative values for any δ satisfying $\sqrt{\frac{2}{r+1}} \leq \delta \leq \sqrt{\frac{2}{r}}$.

Lemma 4.7. *Let $r \geq 8$ be an integer. The function*

$$f(r) = (r+1)^{\frac{3}{2}} - 2\sqrt{2}r + 4\sqrt{r}$$

takes positive values.

Proposition 4.8. *SHGH conjecture for $r+1$ on $\mathbb{P}^1 \times \mathbb{P}^1$ implies the conjecture for r on $\mathbb{P}^1 \times \mathbb{P}^1$.*

proof . Recall that the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point is isomorphic to the blow-up of $\mathbb{P}^2$ at two points. Via this isomorphsm, we identify Y_r with X_{r+1}. The notions defined in [6] such as "special" or "(-1)-special" for a divisor on the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ coincide with the one on the blow-up of $\mathbb{P}^2$. Therefore, Lemma 3.4 implies this Proposition 4.8. □

Now, we prove that SHGH conjecture implies the existence of ample line bundles whose Seshadri constant is irrational.

Theorem 4.9. *Let $r \geq 8$ be an integer such that SHGH conjecture for $r+1$ on $\mathbb{P}^1 \times \mathbb{P}^1$ holds. Then there exists an ample divisor $A \in \operatorname{Pic}(Y_r)$ such that $\varepsilon_{\text{gen}}(A)$ is irrational.*

proof . We first prove the following claim.
Claim: *SHGH conjecture for $r+1$ on $\mathbb{P}^1 \times \mathbb{P}^1$ implies either there exists an ample line bundle on Y_r whose Seshadri constant at a very general point is irrational, or SHGH conjecture fails for r on $\mathbb{P}^1 \times \mathbb{P}^1$.*

The combination of this and Proposition 4.8 implies our desired result.

Our proof is based on the proof of [5, Theorem 1.1]. Let δ be a rational number satisfying $\sqrt{\frac{2}{r+1}} \leq \delta \leq \sqrt{\frac{2}{r}}$. Since SHGH conjecture on $\mathbb{P}^1 \times \mathbb{P}^1$ implies Brian Nagata conjecture:

$$\varepsilon(L; r) = \sqrt{\frac{2}{r}},$$

and hence the $\mathbb{Q}$-divisor $L = l_1 + l_2 - \delta \sum_{i=1}^{r} e_i$ is ample. If $\varepsilon(L; p)$ is irrational, where p is a very general point on Y_r, then the proof is finished.

So we can assume that $\varepsilon(L; p)$ is rational, and not equal to $\sqrt{L^2}$. In this situation, basic properties of Seshadri constants and Lemma 4.5 imply that there is a divisor $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ of type (a, b) with $M = \operatorname{mult}_{p_1} \Gamma = \cdots = \operatorname{mult}_{p_r} \Gamma$ and $m = \operatorname{mult}_p \Gamma$ whose proper transform $\tilde{\Gamma}$ on X_r attains the Seshadri constant of L at p:

$$\varepsilon(L; p) = \frac{L.\tilde{\Gamma}}{m} = \frac{a + b - r\delta M}{m} < \sqrt{2 - r\delta^2}.$$

Set $\gamma = a + b$. Then,

$$\gamma < m\sqrt{2 - r\delta^2} + r\delta M. \tag{1}$$

Now, we suppose that SHGH conjecture on $\mathbb{P}^1 \times \mathbb{P}^1$ holds for $r+1$. So, Brian-Nagata conjecture on $\mathbb{P}^1 \times \mathbb{P}^1$ also holds for $r+1$. Therefore,

$$\frac{\gamma}{rM + m} \geq \sqrt{\frac{2}{r+1}} \sum_{i=1}^{r} e_i. \tag{2}$$

This is because $(l_1 + l_2 - \sqrt{\frac{2}{r+1}}).\tilde{\Gamma} \geq 0$.

We claim that $\delta \geq 2M + m$. Suppose not:

$$\gamma < 2M + m.$$

Put

$$\alpha = \frac{2\sqrt{r+1} - \sqrt{2}r}{2 - r\delta}, \ \beta = \frac{\sqrt{2}r - \delta r\sqrt{r+1}}{2 - r\delta}$$

which are positive real numbers. The formulas (1) and (2) imply

$$\sqrt{2} < \beta + \alpha\sqrt{2 - r\delta^2}.$$

Substituting α and β for (1), (2), we obtain that:

$$\sqrt{2}(2-r\delta) > \sqrt{2}r - r\delta\sqrt{r+1} + (2\sqrt{r+1}-\sqrt{2}r)\sqrt{2-r\delta^2}.$$

This contradicts Lemma 4.6. So, we obtain that $\delta \geq 2M+m$.

By SHGH conjecture for $r+1$ on $\mathbb{P}^1 \times \mathbb{P}^1$, the effective linear system

$$al_1 + bl_2 - M\sum_{i=1}^{r} e_i - me_{r+1}$$

is nonspecial on X_{r+1}. Indeed $\gamma \geq 2M+m$ by the previous discussion and $\gamma \geq 3M$ is satisfied because of $\frac{\gamma}{rM} > \sqrt{\frac{2}{r}}$ and $r \geq 8$.

We have

$$0 \leq 2(ab+a+b) - rM(M+1) - m(m+1).$$

The formula (1) is equivalent to

$$a+b < m\sqrt{2-r\delta^2} + r\delta M,$$

and

$$2ab \leq (a+b)^2.$$

Those inequalities impily

$$0 < r(r\delta^2-1)M^2 + 2r\delta\sqrt{2-r\delta^2}mM + (1-r\delta^2)m^2 + r(2\delta-1)M + (2\sqrt{2-r\delta^2}-1)m. \tag{3}$$

Now, the quadratic terms in M and m in (3) are negative definite. Indeed, if we set

$$A = \begin{pmatrix} r(r\delta^2-1) & r\delta\sqrt{2-r\delta^2} \\ r\delta\sqrt{2-r\delta^2} & 1-r\delta^2 \end{pmatrix},$$

then,

$$\det A \leq r\left(-1 + \frac{2\sqrt{2r}(\sqrt{r}-\sqrt{2})}{(r+1)\sqrt{r+1}}\right).$$

Lemma 4.7 implies that $\det A < 0$. Moreover, we have $r(r\delta^2-1) > 0$.

Furthermore, the linear terms in (3) are also negative:

$$2\delta - 1 < 0,\ 2\sqrt{2-r\delta^2} < 0,$$

because $r \geq 8$.

This is the desired contradiction. □

Corollary 4.10. *Let $r \geq 8$ be an integer such that SHGH conjecture for $r+1$ on $\mathbb{P}^1 \times \mathbb{P}^1$ holds, then there exists an ample divisor A on the blow-up of $\mathbb{P}^2$ at general r points such that $A = al - \sum_{i=1}^{r-2} be_i - ce_{r-1} - ce_r$, where $a, b, c \in \mathbb{Z}_{\geq 0}$ and $a = b + 2c$, $\sqrt{\frac{2}{r+1}} < \frac{b}{a-b} < \sqrt{\frac{2}{r}}$.*

proof . Since the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point is isomorphic to the blow-up of $\mathbb{P}^2$ at two points, their Picard groups are isomorphic. We identify that the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at p with the exceptional divisor E_p with $\mathbb{P}^2$ at q_1, q_2 with exceptional divisors E_{q_1}, E_{q_2}, where the strict transform of the two lines l_1, l_2 through p on $\mathbb{P}^1 \times \mathbb{P}^1$ coincides with E_{q_1}, E_{q_2} and strict transform of the line passing through q_1, q_2 on $\mathbb{P}^2$ is E_p.

Namely,

$$l_i - E_p = E_{q_i}\ (i = 1, 2),\ l - E_{q_1} - E_{q_2} = E_p.$$

Via this isomorphism, we have

$$al_1 + bl_2 - nE_p = (a+b-n)l - (b-n)E_{q_1} - (a-n)E_{q_2}.$$

in $\mathrm{Pic}(Y_1) = \mathrm{Pic}(X_2)$, where $a, b, n \in \mathbb{Z}$.

Especially,

$$ml_1 + ml_2 - nE_p = (2m-n)l - (m-n)E_{q_1} - (m-n)E_{q_2}.$$

This implies Corollary 4.10. $\square$

Remark. In an assumption of SHGH conjecture, the existence of a homogeneous ample divisor with irrational Seshadri constant has already been known by Theorem 1.2 and [5]. However, corollary 4.10 insures the existence of an ample divisor with an irrational Seshadri constant which is "NOT" a homogeneous form.

Finally, we provide proofs of Lemmas 4.6 and 4.7.

Proof of Lemma 4.6

Since $f(\sqrt{\frac{2}{r+1}}) = 0$, it is enough to show that $f(\delta)$ is increasing on the interval $\sqrt{\frac{2}{r+1}} \leq \delta \leq \sqrt{\frac{2}{r}}$. The derivation of $f(\delta)$ is following.

$$f'(\delta) = r\left(\sqrt{2} + \frac{8}{\sqrt{2-r\delta^2}}(\sqrt{2}r - 2\sqrt{r+1}) - \sqrt{r+1}\right).$$

Now, if $\sqrt{\frac{2}{r+1}} \leq \delta \leq \sqrt{\frac{2}{r}}$, then

$$\frac{\delta}{\sqrt{2-r\delta^2}} \geq 1.$$

Therefore,

$$\begin{aligned} f'(\delta) &\geq r(\sqrt{2}r + 2\sqrt{r+1}) - \sqrt{r+1} \\ &= r(\sqrt{2} + \sqrt{2}r - 3\sqrt{r+1}) \geq 0, \end{aligned}$$

because $r \geq 8$. This implies that $f(\delta) \geq 0$.

Proof of Lemma 4.7

The proof of this lemma is obtained by an elementary calculus.

References

[1] P. Biran, "Constructing new ample divisors out of old ones", Duke Math. J. **98** (1999), no. 1, 113–135.

[2] C. Ciliberto, B. Harbourne, R. Miranda and J. Roe, "Variations on Nagata's Conjecture", arXiv:1202.0475.

[3] C. Ciliberto, "Geometric aspects of polynomial interpolation in more variables and of Waring's problem", European Congress of Mathematics, Vol. I (Barcelona, 2000), 289–316, Progr. Math. **201**, Birkhuser, Basel, 2001.

[4] Demailly, J.-P., "Singular Hermitian metrics on positive line bundles", Complex algebraic vari-eties (Bayreuth, 1990), Lect. Notes Math. **1507**, Springer-Verlag, 1992, pp. 87–104.

[5] M. Dumnicki, A. Küronya, C. Maclean, T. Szemberg, "Seshadri constants via Okounkov functions and theSegre-Harbourne-Gimigliano-Hirschowitz Conjecture", arXiv:1304.0249.

[6] M. Dumnicki, "Special homogeneous linear systems on Hirzebruch surfaces", Geometriae Dedicata **147**(1), 283–311, 2010.

[7] R. Hartshorne,57 "Algebraic Geometry", GTM52, Springer, 1977.

[8] A. Hirschowitz, "Une conjecture pour la cohomologie des diviseurs sur les surfaces rationelles génériques, J. Reine Angew.Math. **397** (1989), 208–213.

[9] R.Lazarsfeld, "Positivity in Algebraic Geometry I", Springer-Verlag, 2004.

[10] M. Nagata, "On the fourteenth problem of Hilbert", Amer. J. Math. **81** (1959), 766–772.

[11] T. Sano, "Seshadri constants on rational surfaces with anticanonical pencils", J. Pure Appl. Algebra **218** (2014), no. 4, 602–617.

[12] T. Bauer, Th., S. Di Rocco, B. Harbourne, M. Kapustka, A. Knutsen, W. Syzdek, T. Szemberg "A primer on Seshadri constants", Interactions of Classical and Numerical Algebraic Geometry, Proceedings of a conference in honor of A. J. Sommese, held at Notre Dame, May 2224 2008. Contemporary Mathematics vol. **496**, (2009), 33–70.