Characterizations of projective spaces and hyperquadrics for varieties with Picard number one

楫研究室
鈴木 拓

2012 年 2 月 7 日

Question

Variety が projective space $\mathbb{P}^n$ や hyperquadric Q_n （単純な varieties）になるための条件は？

Introduction

Question

Variety が projective space $\mathbb{P}^n$ や hyperquadric Q_n （単純な varieties）になるための条件は？

- 問題の発端: Hartshorne 予想の解決

Theorem 1 [Mori,1979]

X: n-dim smooth projective variety over an algebraic closed field, tangen bundle T_X is ample $\Longrightarrow X \cong \mathbb{P}^n$.

Introduction

Question

Variety が projective space $\mathbb{P}^n$ や hyperquadric Q_n （単純な varieties）になるための条件は？

- 問題の発端: Hartshorne 予想の解決

Theorem 1 [Mori,1979]

X: n-dim smooth projective variety over an algebraic closed field, tangen bundle T_X is ample $\Longrightarrow X \cong \mathbb{P}^n$.

- 一般化された結果:

Theorem 2 [Andreatta-Wiśniewski,2001]

X: n-dim smooth complex projective variety,
$\exists \mathscr{E} \subseteq T_X$: ample vector bundle $\Longrightarrow X \cong \mathbb{P}^n$.

Introduction

- 更なる一般化の予想:

Conjecture 3 [Kovács]

X: n-dim smooth complex projective variety,
$\exists \mathscr{E}$: ample vector bundle of r-rank, $0 < \exists p \leq r$, $\exists \wedge^p \mathscr{E} \hookrightarrow \wedge^p T_X$: inclusion
$\Longrightarrow X \cong \mathbb{P}^n$ or $(p = r = n$ and$) X \cong Q_n$.

- 更なる一般化の予想:

Conjecture 3 [Kovács]

X: n-dim smooth complex projective variety,
$\exists \mathscr{E}$: ample vector bundle of r-rank, $0 < \exists p \leq r$, $\exists \wedge^p \mathscr{E} \hookrightarrow \wedge^p T_X$: inclusion
$\Longrightarrow X \cong \mathbb{P}^n$ or ($p = r = n$ and)$X \cong Q_n$.

- Kovács 予想に関する結果:

Theorem 4 [Araujo-Druel-Kovács,2008]

$\mathscr{E} = \mathscr{L}^{\oplus r}$ ($\mathscr{L}$: line bundle) の場合, Kovács 予想は肯定的である.

Main Theorem

Main Theorem

X の Picard number が 1 の場合, Kovács 予想は肯定的である.
すなわち,
X: n-dim smooth complex projective variety with Picard number 1,
$\exists\mathscr{E}$: ample vector bundle of r-rank, $0 < \exists p \leq r$, $\exists \wedge^p \mathscr{E} \hookrightarrow \wedge^p T_X$: inclusion
$\Longrightarrow X \cong \mathbb{P}^n$ or $(p = r = n$ and$) X \cong Q_n$.

Remark

- 2010 年に Ross が同じ主張の論文を出している．

Remark

- 2010 年に Ross が同じ主張の論文を出している.
- しかし Ross の証明には欠陥が有る？
  (射 $\mathscr{E} \to T_X$ の存在を仮定している？)

Conjecture 3 [Kovács]

X: n-dim smooth complex projective variety,
$\exists\mathscr{E}$: ample vector bundle of r-rank, $0 < \exists p \leq r$, $\exists \wedge^p \mathscr{E} \hookrightarrow \wedge^p T_X$: inclusion
$\Longrightarrow X \cong \mathbb{P}^n$ or $(p = r = n$ and$)X \cong Q_n$.

Remark

- 2010 年に Ross が同じ主張の論文を出している.
- しかし Ross の証明には欠陥が有る？
  (射 $\mathscr{E} \to T_X$ の存在を仮定している？)

Conjecture 3 [Kovács]

X: n-dim smooth complex projective variety,
$\exists\mathscr{E}$: ample vector bundle of r-rank, $0 < \exists p \leq r$, $\exists \wedge^p \mathscr{E} \hookrightarrow \wedge^p T_X$: inclusion
$\Longrightarrow X \cong \mathbb{P}^n$ or $(p = r = n$ and$) X \cong Q_n$.

- Ross の証明とは異なる証明.
  Sheaf stability の理論.
  [Araujo-Druel-Kovács,2008] の手法の応用.

Sheaf stability

Definition 5

X: n-dim projective variety, $\mathscr{H}$: fixed ample line bundle,
$\mathscr{F}$: torsion-free sheaf,

- $\mathscr{F}$ の slope を次で定義:

$$\mu(\mathscr{F}) = \frac{c_1(\mathscr{F}) \cdot c_1(\mathscr{H})^{n-1}}{\mathrm{rk}(\mathscr{F})}.$$

- $\mathscr{F}$: semistable $\iff \mu(\mathscr{E}) \leq \mu(\mathscr{F})$ for $\forall \mathscr{E} \subseteq \mathscr{F}$.

Sheaf stability

Fact 6 [Harder-Narasimhan,1975]

$\mathscr{F}$: torsion-free sheaf に対して, ∃ filtration

$$\mathscr{F} = \mathscr{F}_0 \supsetneq \mathscr{F}_1 \supsetneq \cdots \supsetneq \mathscr{F}_{k+1} = 0,$$

s.t. $\mathscr{Q}_i = \mathscr{F}_i / \mathscr{F}_{i+1}$: semistable, $\mu(\mathscr{Q}_0) < \cdots < \mu(\mathscr{Q}_k)$.
これを $\mathscr{F}$ の Harder-Narasimhan filtration という.

Outline of Proof

$\wedge^p \mathscr{E} \hookrightarrow \wedge^p T_X$ より X 上の rational curves の minimal dominating family H が存在する ([Miyaoka,1987]).

Definition 7

Irreducible component $H \subset \mathrm{RatCurves}^n(X)$ が次を満たすとき, minimal dominating family であるという:

- H-curves は X を支配し,
- General point $x \in X$ を通る H-curves の成す subvariety は proper.

Outline of Proof

$\wedge^p \mathscr{E} \hookrightarrow \wedge^p T_X$ より X 上の rational curves の minimal dominating family H が存在する ([Miyaoka,1987]).

Definition 7

Irreducible component $H \subset \mathrm{RatCurves}^n(X)$ が次を満たすとき, minimal dominating family であるという:

- H-curves は X を支配し,
- General point $x \in X$ を通る H-curves の成す subvariety は proper.

(Case1) $\deg f^*\mathscr{E} = r$ ($[f] \in H$) の場合

- [Ross,2010] で証明されている.

(Case2) $\deg f^*\mathscr{E} = r + 1$ ($[f] \in H$) の場合

- [Ross,2010] の証明に欠陥.
- Projective space になることを示す.

Outline of Proof

Key Lemma ([Araujo,2006], [Kollár,1996] 及び slope の計算)

X の Picard number が 1,
H: minimal dominating family in (Case 2),
$\exists \mathscr{D} \subseteq T_X$ s.t. $\mu(\mathscr{E}) \leq \mu(\mathscr{D}) \Longrightarrow X \cong \mathbb{P}^n$.

Outline of Proof

Key Lemma ([Araujo,2006], [Kollár,1996] 及び slope の計算)

X の Picard number が 1,
H: minimal dominating family in (Case 2),
$\exists \mathscr{D} \subseteq T_X$ s.t. $\mu(\mathscr{E}) \leq \mu(\mathscr{D}) \Longrightarrow X \cong \mathbb{P}^n$.

- T_X の Harder-Narasimhan filtration

$$T_X = \mathscr{F}_0 \supsetneq \mathscr{F}_1 \supsetneq \cdots \supsetneq \mathscr{F}_k \supsetneq \mathscr{F}_{k+1} = 0$$

をとり, $\mathscr{D} = \mathscr{F}_k = \mathscr{Q}_k$ について $\mu(\mathscr{E}) \leq \mu(\mathscr{D})$ を示す.

Outline of Proof

- $q := \mathrm{rk} \wedge^p \mathscr{E}$ とすると, $\wedge^q \wedge^p \mathscr{E} \hookrightarrow \wedge^q \wedge^p T_X$.

Outline of Proof

- $q := \mathrm{rk} \wedge^p \mathscr{E}$ とすると, $\wedge^q \wedge^p \mathscr{E} \hookrightarrow \wedge^q \wedge^p T_X$.
- うまく curve $C \subset X$ をとって C に制限して考えると,

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}} (\wedge^{\alpha_0} \mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k} \mathscr{Q}_k)|_C$$

Outline of Proof

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}} (\wedge^{\alpha_0} \mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k} \mathscr{Q}_k)|_C$$ の証明

- $\wedge^q \wedge^p T_X$ を $\mathscr{Q}_0, \ldots, \mathscr{Q}_k$ の外積, テンソルに取り換える.

Outline of Proof

$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}}(\wedge^{\alpha_0}\mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k}\mathscr{Q}_k)|_C$ の証明

- $\wedge^q \wedge^p T_X$ を $\mathscr{Q}_0, \ldots, \mathscr{Q}_k$ の外積, テンソルに取り換える.
- まず, $0 \to \mathscr{F}_1 \to T_X \to \mathscr{Q}_0 \to 0$: exact sequence.
  $\longrightarrow \wedge^p T_X = \mathscr{G}_0 \supseteq \mathscr{G}_1 \supseteq \mathscr{G}_2 \supseteq \cdots$: filtration.

$$\wedge^q \wedge^p T_X \longrightarrow \mathscr{G}_1$$
$$\wedge^q \wedge^p T_X \longrightarrow \mathscr{F}_1, \mathscr{Q}_0$$

Outline of Proof

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}}(\wedge^{\alpha_0}\mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k}\mathscr{Q}_k)|_C$$ の証明

- $\wedge^q \wedge^p T_X$ を $\mathscr{Q}_0, \ldots, \mathscr{Q}_k$ の外積, テンソルに取り換える.
- まず, $0 \to \mathscr{F}_1 \to T_X \to \mathscr{Q}_0 \to 0$: exact sequence.
  $\longrightarrow \wedge^p T_X = \mathscr{G}_0 \supseteq \mathscr{G}_1 \supseteq \mathscr{G}_2 \supseteq \cdots$: filtration.

$$\begin{array}{ccccc} \wedge^q \wedge^p T_X & \longrightarrow & \mathscr{G}_1 & \longrightarrow & \mathscr{G}_2 \\ & \searrow & & \searrow & \\ & & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0 \end{array}$$

Outline of Proof

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}} (\wedge^{\alpha_0} \mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k} \mathscr{Q}_k)|_C$$ の証明

- $\wedge^q \wedge^p T_X$ を $\mathscr{Q}_0, \ldots, \mathscr{Q}_k$ の外積, テンソルに取り換える.
- まず, $0 \to \mathscr{F}_1 \to T_X \to \mathscr{Q}_0 \to 0$: exact sequence.
  $\longrightarrow \wedge^p T_X = \mathscr{G}_0 \supseteq \mathscr{G}_1 \supseteq \mathscr{G}_2 \supseteq \cdots$: filtration.

$$\begin{array}{ccccccc} \wedge^q \wedge^p T_X & \longrightarrow & \mathscr{G}_1 & \longrightarrow & \mathscr{G}_2 & \longrightarrow & \cdots \\ & \searrow & & \searrow & & \searrow & \\ & & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0 \end{array}$$

$$\begin{array}{ccc} \wedge^q \wedge^p T_X & \longrightarrow & \mathscr{F}_1 \\ & \searrow & \\ & & \mathscr{Q}_0 \end{array}$$

Outline of Proof

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}} (\wedge^{\alpha_0} \mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k} \mathscr{Q}_k)|_C$$ の証明

- $\wedge^q \wedge^p T_X$ を $\mathscr{Q}_0, \ldots, \mathscr{Q}_k$ の外積, テンソルに取り換える.
- まず, $0 \to \mathscr{F}_1 \to T_X \to \mathscr{Q}_0 \to 0$: exact sequence.
  $\longrightarrow \wedge^p T_X = \mathscr{G}_0 \supseteq \mathscr{G}_1 \supseteq \mathscr{G}_2 \supseteq \cdots$: filtration.

$$\begin{array}{ccccccc}
\wedge^q \wedge^p T_X & \longrightarrow & \mathscr{G}_1 & \longrightarrow & \mathscr{G}_2 & \longrightarrow & \cdots \\
 & \searrow & & \searrow & & \searrow & \\
 & & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0
\end{array}$$

- 次に, $0 \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{Q}_1 \to 0, \ldots$ について,

$$\begin{array}{ccc}
\wedge^q \wedge^p T_X & \longrightarrow & \mathscr{F}_1 \\
 & \searrow & \\
 & & \mathscr{Q}_0
\end{array}$$

Outline of Proof

$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}}(\wedge^{\alpha_0}\mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k}\mathscr{Q}_k)|_C$ の証明

- $\wedge^q \wedge^p T_X$ を $\mathscr{Q}_0, \ldots, \mathscr{Q}_k$ の外積, テンソルに取り換える.
- まず, $0 \to \mathscr{F}_1 \to T_X \to \mathscr{Q}_0 \to 0$: exact sequence.

  $\longrightarrow \wedge^p T_X = \mathscr{G}_0 \supseteq \mathscr{G}_1 \supseteq \mathscr{G}_2 \supseteq \cdots$: filtration.

$$\begin{array}{ccccccc} \wedge^q \wedge^p T_X & \longrightarrow & \mathscr{G}_1 & \longrightarrow & \mathscr{G}_2 & \longrightarrow & \cdots \\ & \searrow & & \searrow & & \searrow & \\ & & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0 \end{array}$$

- 次に, $0 \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{Q}_1 \to 0, \ldots$ について,

$$\begin{array}{ccccc} \wedge^q \wedge^p T_X & \longrightarrow & \mathscr{F}_1 & \longrightarrow & \mathscr{F}_2 \\ & \searrow & & \searrow & \\ & & \mathscr{Q}_0 & & \mathscr{Q}_1 \end{array}$$

Outline of Proof

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}} (\wedge^{\alpha_0}\mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k}\mathscr{Q}_k)|_C$$ の証明

- $\wedge^q \wedge^p T_X$ を $\mathscr{Q}_0, \ldots, \mathscr{Q}_k$ の外積, テンソルに取り換える.
- まず, $0 \to \mathscr{F}_1 \to T_X \to \mathscr{Q}_0 \to 0$: exact sequence.
  $\longrightarrow \wedge^p T_X = \mathscr{G}_0 \supseteq \mathscr{G}_1 \supseteq \mathscr{G}_2 \supseteq \cdots$: filtration.

$$\begin{array}{ccccccc}
\wedge^q \wedge^p T_X & \longrightarrow & \mathscr{G}_1 & \longrightarrow & \mathscr{G}_2 & \longrightarrow & \cdots \\
& \searrow & & \searrow & & \searrow & \\
& & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0
\end{array}$$

- 次に, $0 \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{Q}_1 \to 0, \ldots$ について,

$$\begin{array}{ccccccc}
\wedge^q \wedge^p T_X & \longrightarrow & \mathscr{F}_1 & \longrightarrow & \mathscr{F}_2 & \longrightarrow & \cdots \\
& \searrow & & \searrow & & \searrow & \\
& & \mathscr{Q}_0 & & \mathscr{Q}_1 & & \mathscr{Q}_2
\end{array}$$

Outline of Proof

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}}(\wedge^{\alpha_0}\mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k}\mathscr{Q}_k)|_C$$ の証明

- $\wedge^q \wedge^p T_X$ を $\mathscr{Q}_0, \ldots, \mathscr{Q}_k$ の外積, テンソルに取り換える.
- まず, $0 \to \mathscr{F}_1 \to T_X \to \mathscr{Q}_0 \to 0$: exact sequence.
  $\longrightarrow \wedge^p T_X = \mathscr{G}_0 \supseteq \mathscr{G}_1 \supseteq \mathscr{G}_2 \supseteq \cdots$: filtration.

$$\begin{array}{ccccccc}
\wedge^q \wedge^p T_X & \longrightarrow & \mathscr{G}_1 & \longrightarrow & \mathscr{G}_2 & \longrightarrow & \cdots \\
& \searrow & & \searrow & & \searrow & \\
& & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0 & & \mathscr{F}_1, \mathscr{Q}_0
\end{array}$$

- 次に, $0 \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{Q}_1 \to 0, \ldots$ について,

$$\begin{array}{ccccccc}
\wedge^q \wedge^p T_X & \longrightarrow & \mathscr{F}_1 & \longrightarrow & \mathscr{F}_2 & \longrightarrow & \cdots \\
& \searrow & & \searrow & & \searrow & \\
& & \mathscr{Q}_0 & & \mathscr{Q}_1 & & \mathscr{Q}_2
\end{array}$$

Now we consider about sheaves on the curve C. From the previous step, we have

$$\wedge^q \wedge^p \mathscr{E} \hookrightarrow \wedge^{q-k_0}(\wedge^p \mathscr{Q}_0 \otimes \wedge^0 \mathscr{F}_1) \otimes \wedge^{k_0}\mathscr{G}_1.$$

By the property of exterior products, the short exact sequence

$$0 \to \mathscr{F}_1 \to T_X \to \mathscr{Q}_0 \to 0$$

yields a filtration

$$\wedge^p T_X = \mathscr{G}_0 \supseteq \mathscr{G}_1 \supseteq \cdots \supseteq \mathscr{G}_{p+1} = 0$$

such that $\mathscr{G}_j/\mathscr{G}_{j+1} \cong \wedge^{p-j}\mathscr{Q}_0 \otimes \wedge^j \mathscr{F}_1$. Again the short exact sequence

$$0 \to \mathscr{G}_2 \to \mathscr{G}_1 \to \wedge^{p-1}\mathscr{Q}_0 \otimes \wedge^1 \mathscr{F}_1 \to 0$$

yields a filtration

$$\wedge^{k_0}\mathscr{G}_1 = \mathscr{H}_0 \supseteq \mathscr{H}_1 \supseteq \cdots \supseteq \mathscr{H}_{k_0+1} = 0$$

such that $\mathscr{H}_k/\mathscr{H}_{k+1} \cong \wedge^{k_0-k}(\wedge^{p-1}\mathscr{Q}_0 \otimes \wedge^1 \mathscr{F}_1) \otimes \wedge^k \mathscr{G}_2$.

From the filtration

$$\begin{aligned}\wedge^{q-k_0}(\wedge^p \mathscr{Q}_0 \otimes \wedge^0 \mathscr{F}_1) \otimes \wedge^{k_0}\mathscr{G}_1 &= \wedge^{q-k_0}(\wedge^p \mathscr{Q}_0 \otimes \wedge^0 \mathscr{F}_1) \otimes \mathscr{H}_0 \\ &\supseteq \wedge^{q-k_0}(\wedge^p \mathscr{Q}_0 \otimes \wedge^0 \mathscr{F}_1) \otimes \mathscr{H}_1 \\ &\supseteq \cdots,\end{aligned}$$

we have an integer k_1 such that

$$\wedge^q \wedge^p \mathscr{E} \subseteq \wedge^{q-k_0}(\wedge^p \mathscr{Q}_0 \otimes \wedge^0 \mathscr{F}_1) \otimes \mathscr{H}_{k_1},$$
$$\wedge^q \wedge^p \mathscr{E} \not\subseteq \wedge^{q-k_0}(\wedge^p \mathscr{Q}_0 \otimes \wedge^0 \mathscr{F}_1) \otimes \mathscr{H}_{k_1+1}.$$

Then an inclusion

$$\begin{aligned}\wedge^q \wedge^p \mathscr{E} &\hookrightarrow (\wedge^{q-k_0}(\wedge^p \mathscr{Q}_0 \otimes \wedge^0 \mathscr{F}_1) \otimes \mathscr{H}_{k_1}) \\ &\quad /(\wedge^{q-k_0}(\wedge^p \mathscr{Q}_0 \otimes \wedge^0 \mathscr{F}_1) \otimes \mathscr{H}_{k_1+1}) \\ &\cong \wedge^{q-k_0}(\wedge^p \mathscr{Q}_0 \otimes \wedge^0 \mathscr{F}_1) \otimes \wedge^{k_0-k_1}(\wedge^{p-1} \mathscr{Q}_0 \otimes \wedge^1 \mathscr{F}_1) \\ &\quad \otimes \wedge^{k_1} \mathscr{G}_2\end{aligned}$$

is induced.

Outline of Proof

- $q := \mathrm{rk} \wedge^p \mathscr{E}$ とすると, $\wedge^q \wedge^p \mathscr{E} \hookrightarrow \wedge^q \wedge^p T_X$.
- うまく curve $C \subset X$ をとって C に制限して考えると,

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}} (\wedge^{\alpha_0} \mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k} \mathscr{Q}_k)|_C$$

Outline of Proof

- $q := \mathrm{rk} \wedge^p \mathscr{E}$ とすると, $\wedge^q \wedge^p \mathscr{E} \hookrightarrow \wedge^q \wedge^p T_X$.
- うまく curve $C \subset X$ をとって C に制限して考えると,

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}} (\wedge^{\alpha_0} \mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k} \mathscr{Q}_k)|_C$$

$\mathscr{Q}_i$:semistable $\longrightarrow$ 右辺: semistable.

- slope の大小比較

Outline of Proof

- $q := \mathrm{rk} \wedge^p \mathscr{E}$ とすると, $\wedge^q \wedge^p \mathscr{E} \hookrightarrow \wedge^q \wedge^p T_X$.
- うまく curve $C \subset X$ をとって C に制限して考えると,

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}} (\wedge^{\alpha_0} \mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k} \mathscr{Q}_k)|_C$$

$\mathscr{Q}_i$:semistable $\longrightarrow$ 右辺: semistable.

- slope の大小比較

Fact 8

$\mu(\mathscr{F} \otimes \mathscr{G}) = \mu(\mathscr{F}) + \mu(\mathscr{G})$.
$\mu(\wedge^a \mathscr{F}) = a\mu(\mathscr{F})$.

Outline of Proof

- $q := \mathrm{rk} \wedge^p \mathscr{E}$ とすると, $\wedge^q \wedge^p \mathscr{E} \hookrightarrow \wedge^q \wedge^p T_X$.
- うまく curve $C \subset X$ をとって C に制限して考えると,

$$\wedge^q \wedge^p \mathscr{E}|_C \hookrightarrow \bigotimes_{\alpha_0+\cdots+\alpha_k=q} \wedge^{a_{\alpha_0\cdots\alpha_k}}(\wedge^{\alpha_0}\mathscr{Q}_0 \otimes \cdots \otimes \wedge^{\alpha_k}\mathscr{Q}_k)|_C$$

$\mathscr{Q}_i$:semistable $\longrightarrow$ 右辺: semistable.

- slope の大小比較

Fact 8

$\mu(\mathscr{F} \otimes \mathscr{G}) = \mu(\mathscr{F}) + \mu(\mathscr{G})$.
$\mu(\wedge^a \mathscr{F}) = a\mu(\mathscr{F})$.

$\mu(\mathscr{Q}_0) < \cdots < \mu(\mathscr{Q}_k) \longrightarrow \mu(\mathscr{E}) \leq \mu(\mathscr{Q}_k)$.
したがって, $X \cong \mathbb{P}^n$.

Main Theorem

Main Theorem

X の Picard number が 1 の場合, Kovács 予想は肯定的である.
すなわち,
X: n-dim smooth complex projective variety with Picard number 1,
$\exists \mathscr{E}$: ample vector bundle of r-rank, $0 < \exists p \leq r$, $\exists \wedge^p \mathscr{E} \hookrightarrow \wedge^p T_X$: inclusion
$\Longrightarrow X \cong \mathbb{P}^n$ or $(p = r = n$ and$) X \cong Q_n$.