射影線織曲面のイデアル自由分解について

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Introduction

$k = \mathbb{C}$ とし, 曲線は非特異かつ射影的とする.

Definition (Ruled surface)

X : C 上の ruled surface
$\Longleftrightarrow$ $\pi : X \longrightarrow C$: 全射, C : 曲線
$\forall p \in C,\ X_p := \pi^{-1}(p) \cong \mathbb{P}^1$
$\Longleftrightarrow$ $\exists \mathscr{E}$: C 上の rank 2 locally free sheaf (vector bundle)
s.t. $X \cong \mathbb{P}(\mathscr{E})$

Theorem (Alzati, Tonoli)

$X = \mathbb{P}(\mathscr{E})$: C 上の ruled surface
A : X 上の very ample divisor
$X \hookrightarrow \mathbb{P}^N$ by $|A|$, $N = h^0(X, A) - 1$ とする.
このとき, $\mathscr{I}_X$ の free resolution を計算する Macaulay 2 のアルゴリズムが存在する.

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Example (AT)

C : 楕円曲線 $(g(C)=1)$, $\deg \mathscr{E}=0$ i.e.

1. $\mathscr{E}=\mathscr{O}_C \oplus \mathscr{O}_C$ i.e. $\mathbb{P}(\mathscr{E})=C\times\mathbb{P}^1$
2. $\mathscr{E}=\mathscr{O}_C \oplus \mathscr{L}$, where $\mathscr{L}\neq\mathscr{O}_C$ and $\deg\mathscr{L}=0$
3. non-trivial extention $0 \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_C \longrightarrow 0$

$X_i=\mathbb{P}(\mathscr{E}) \hookrightarrow \mathbb{P}^5$ by $|A|=|C_0+3f|$
$(i=1,2,3)$ $(C_0$: tautological section, f : fibre $)$

このときアルゴリズムによって, ($\mathscr{O}_{\mathbb{P}^5}$ を $\mathscr{O}$ と省略)

1. $0\to\mathscr{O}(-6)\to 6\mathscr{O}(-5)\to 9\mathscr{O}(-4)\oplus 2\mathscr{O}(-3)$
   $\to 4\mathscr{O}(-3)\oplus 3\mathscr{O}(-2)\to \mathscr{I}_{X_1}\to 0$
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Motivation

(AT) では与えられた条件を満たす, ある ruled surface について Macaulay 2 のアルゴリズムで free resolution を計算.
$\rightsquigarrow$ 与えられた条件を満たす, すべての ruled surface について free resolution を計算したわけではない.

Goal

Theorem と同じ仮定 ($X = \mathbb{P}(\mathscr{E})$: C 上の ruled surface, A : X 上の very ample divisor, $X \hookrightarrow \mathbb{P}^N$ by $|A|$)
から理論的に $\mathscr{I}_X$ の free resolution を特定したい.

Main result

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$X = \mathbb{P}(\mathscr{E}) \hookrightarrow \mathbb{P}^5$ by $|A| = |C_0 + 3f|$ とすると

$$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \oplus 6\mathscr{O}(-4) \to 15\mathscr{O}(-4) \oplus 18\mathscr{O}(-3) \to 20\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_X \to 0$$

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Theorem (Eisenbud, Floystad, Schreyer)

$\mathscr{F}$: $\mathbb{P}^n$ 上の coherent sheaf

$$\exists \mathscr{B} = \cdots \longrightarrow B^{-1} \longrightarrow B^{0} \longrightarrow B^{1} \longrightarrow \cdots : \text{ complex}$$

$$\text{with} \quad B^e = \bigoplus_j H^j(\mathbb{P}^n, \mathscr{F}(e-j)) \otimes \Omega_{\mathbb{P}^n}^{j-e}(j-e) = \bigoplus_j E_1^{e-j,j}$$

$$(E_1^{pq} := H^q(\mathbb{P}^n, \mathscr{F}(p)) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p))$$

s.t. $\mathscr{B}$ は, B^0 を除いて exact で, B^0 での homology は $\mathscr{F}$ である.

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$\mathscr{F}$ の cohomology $\rightsquigarrow$ $\mathscr{F}$ の exact sequence

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Example

C : 楕円曲線 ($g=1$), $\mathscr{E}$: C 上の次数 0 rank 2 locally free sheaf

(手順 1) $X_i := \mathbb{P}(\mathscr{E}) \hookrightarrow \mathbb{P}^5$ $(i=1,2,3)$ by $|A| = |C_0 + 3f|$

(手順 2) $\mathscr{I}_{X_i}(2)$ に [EFS] を適用. Cohomology を計算する.

$$B^e = \bigoplus_j E_1^{e-j,j} \qquad E_1^{pq} = H^q(\mathbb{P}^5, \mathscr{I}_{X_i}(2+p)) \otimes \Omega_{\mathbb{P}^5}^{-p}(-p)$$

$\rightsquigarrow -5 \le p \le 0$, $0 \le q \le 5$ で考えれば十分.

$h^q(\mathbb{P}^5, \mathscr{I}_{X_i}(2+p))$ $(i=1,2,3)$ (縦に $0 \le q \le 5$, 横に $-5 \le p \le 0$)

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例えば

$H^3(\mathbb{P}^5, \mathscr{I}_{X_i}(2-5)) = \mathbb{C}^{18}$
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($\Omega_{\mathbb{P}^5}$ を Ω と省略, 縦に $0 \leq q \leq 5$, 横に $-5 \leq p \leq 0$)

0	0	0	0	0	0
0	0	0	0	0	0
$18\Omega^5(5)$	$6\Omega^4(4)$	0	0	0	0
0	0	0	$\Omega^2(2)$	0	0
0	0	0	0	0	0
0	0	0	0	0	$3\mathscr{O}$

(手順 3) [EFS] によって resolution を得る.

$$\rightsquigarrow \quad 0 \to \underset{\substack{= B^{-2} \\ = \oplus E_1^{-2-j,j}}}{18\Omega^5(5)} \to \underset{= B^{-1}}{6\Omega^4(4)} \to \underset{= B^0}{\Omega^2(2) \oplus 3\mathscr{O}} \to 0$$

$:\ B^0$ を除いて exact

$$\rightsquigarrow \quad 0 \to 18\Omega^5(5) \to 6\Omega^4(4) \to \Omega^2(2) \oplus 3\mathscr{O} \to \mathscr{I}_{X_i}(2) \to 0$$

: exact $(i = 1, 2, 3)$

$$E_1^{pq} = H^q(\mathbb{P}^5, \mathscr{I}_{X_i}(2+p)) \otimes \Omega^{-p}(-p) \quad (i = 1, 2, 3)$$

($\Omega_{\mathbb{P}^5}$ を Ω と省略, 縦に $0 \le q \le 5$, 横に $-5 \le p \le 0$)

0	0	0	0	0	0
0	0	0	0	0	0
$18\Omega^5(5)$	$6\Omega^4(4)$	0	0	0	0
0	0	0	$\Omega^2(2)$	0	0
0	0	0	0	0	0
0	0	0	0	0	$3\mathscr{O}$

(手順 3) [EFS] によって resolution を得る.

$$\rightsquigarrow \quad 0 \to \begin{array}{c} 18\Omega^5(5) \\ = B^{-2} \\ = \bigoplus E_1^{-2-j,j} \end{array} \to \begin{array}{c} 6\Omega^4(4) \\ = B^{-1} \end{array} \to \begin{array}{c} \Omega^2(2) \oplus 3\mathscr{O} \\ = B^0 \end{array} \to 0$$

: B^0 を除いて exact

$$\rightsquigarrow \quad 0 \to 18\Omega^5(5) \to 6\Omega^4(4) \to \Omega^2(2) \oplus 3\mathscr{O} \to \mathscr{I}_{X_i}(2) \to 0$$

: exact $(i = 1, 2, 3)$

$$E_1^{pq} = H^q(\mathbb{P}^5, \mathscr{I}_{X_i}(2+p)) \otimes \Omega^{-p}(-p) \quad (i=1,2,3)$$

($\Omega_{\mathbb{P}^5}$ を Ω と省略, 縦に $0 \le q \le 5$, 横に $-5 \le p \le 0$)

0	0	0	0	0	0
0	0	0	0	0	0
$18\Omega^5(5)$	$6\Omega^4(4)$	0	0	0	0
0	0	0	$\Omega^2(2)$	0	0
0	0	0	0	0	0
0	0	0	0	0	$3\mathscr{O}$

(手順 3) [EFS] によって resolution を得る.

$$\rightsquigarrow \quad 0 \to \begin{array}{c} 18\Omega^5(5) \\ = B^{-2} \\ = \bigoplus E_1^{-2-j,j} \end{array} \to \begin{array}{c} 6\Omega^4(4) \\ = B^{-1} \\ \end{array} \to \begin{array}{c} \Omega^2(2) \oplus 3\mathscr{O} \\ = B^0 \\ : B^0 \text{ を除いて exact} \end{array} \to 0$$

$$\rightsquigarrow 0 \to 18\Omega^5(5) \to 6\Omega^4(4) \to \Omega^2(2) \oplus 3\mathscr{O} \to \mathscr{I}_{X_i}(2) \to 0$$

: exact $(i = 1,2,3)$

$$E_1^{pq} = H^q(\mathbb{P}^5, \mathscr{I}_{X_i}(2+p)) \otimes \Omega^{-p}(-p) \quad (i = 1,2,3)$$

($\Omega_{\mathbb{P}^5}$ を Ω と省略, 縦に $0 \le q \le 5$, 横に $-5 \le p \le 0$)

0	0	0	0	0	0
0	0	0	0	0	0
$18\Omega^5(5)$	$6\Omega^4(4)$	0	0	0	0
0	0	0	$\Omega^2(2)$	0	0
0	0	0	0	0	0
0	0	0	0	0	$3\mathscr{O}$

(手順 3) [EFS] によって resolution を得る.

$$\rightsquigarrow \quad 0 \to \begin{array}{c} 18\Omega^5(5) \\ = B^{-2} \\ = \bigoplus E_1^{-2-j,j} \end{array} \to \begin{array}{c} 6\Omega^4(4) \\ = B^{-1} \end{array} \to \begin{array}{c} \Omega^2(2) \oplus 3\mathscr{O} \\ = B^0 \end{array} \to 0$$

: B^0 を除いて exact

$$\rightsquigarrow \quad 0 \to 18\Omega^5(5) \to 6\Omega^4(4) \to \Omega^2(2) \oplus 3\mathscr{O} \to \mathscr{I}_{X_i}(2) \to 0$$

: exact $(i = 1,2,3)$

$$E_1^{pq} = H^q(\mathbb{P}^5, \mathscr{I}_{X_i}(2+p)) \otimes \Omega^{-p}(-p) \quad (i=1,2,3)$$

($\Omega_{\mathbb{P}^5}$ を Ω と省略, 縦に $0 \le q \le 5$, 横に $-5 \le p \le 0$)

0	0	0	0	0	0
0	0	0	0	0	0
$18\Omega^5(5)$	$6\Omega^4(4)$	0	0	0	0
0	0	0	$\Omega^2(2)$	0	0
0	0	0	0	0	0
0	0	0	0	0	$3\mathscr{O}$

(手順 3) [EFS] によって resolution を得る.

$$\rightsquigarrow \quad 0 \to \underset{\substack{=B^{-2} \\ =\bigoplus E_1^{-2-j,j}}}{18\Omega^5(5)} \to \underset{=B^{-1}}{6\Omega^4(4)} \to \underset{=B^0}{\Omega^2(2)\oplus 3\mathscr{O}} \to 0$$

: B^0 を除いて exact

$$\rightsquigarrow \quad 0 \to 18\Omega^5(5) \to 6\Omega^4(4) \to \Omega^2(2)\oplus 3\mathscr{O} \to \mathscr{I}_{X_i}(2) \to 0$$

: exact $(i=1,2,3)$

$$E_1^{pq} = H^q(\mathbb{P}^5, \mathscr{I}_{X_i}(2+p)) \otimes \Omega^{-p}(-p) \quad (i = 1,2,3)$$

($\Omega_{\mathbb{P}^5}$ を Ω と省略, 縦に $0 \le q \le 5$, 横に $-5 \le p \le 0$)

0	0	0	0	0	0
0	0	0	0	0	0
$18\Omega^5(5)$	$6\Omega^4(4)$	0	0	0	0
0	0	0	$\Omega^2(2)$	0	0
0	0	0	0	0	0
0	0	0	0	0	$3\mathscr{O}$

(手順 3) [EFS] によって resolution を得る.

$$\rightsquigarrow \quad 0 \to \begin{array}{c} 18\Omega^5(5) \\ = B^{-2} \\ = \bigoplus E_1^{-2-j,j} \end{array} \to \begin{array}{c} 6\Omega^4(4) \\ = B^{-1} \end{array} \to \begin{array}{c} \Omega^2(2) \oplus 3\mathscr{O} \\ = B^0 \end{array} \to 0$$

: B^0 を除いて exact

$$\rightsquigarrow \quad 0 \to 18\Omega^5(5) \to 6\Omega^4(4) \to \Omega^2(2) \oplus 3\mathscr{O} \to \mathscr{I}_{X_i}(2) \to 0$$

: exact $(i = 1,2,3)$

$$E_1^{pq} = H^q(\mathbb{P}^5, \mathscr{I}_{X_i}(2+p)) \otimes \Omega^{-p}(-p) \quad (i=1,2,3)$$

($\Omega_{\mathbb{P}^5}$ を Ω と省略, 縦に $0 \le q \le 5$, 横に $-5 \le p \le 0$)

0	0	0	0	0	0
0	0	0	0	0	0
$18\Omega^5(5)$	$6\Omega^4(4)$	0	0	0	0
0	0	0	$\Omega^2(2)$	0	0
0	0	0	0	0	0
0	0	0	0	0	$3\mathscr{O}$

(手順 3) [EFS] によって resolution を得る.

$$\rightsquigarrow \quad 0 \to \begin{array}{c} 18\Omega^5(5) \\ = B^{-2} \\ = \bigoplus E_1^{-2-j,j} \end{array} \to \begin{array}{c} 6\Omega^4(4) \\ = B^{-1} \\ \end{array} \to \begin{array}{c} \Omega^2(2) \oplus 3\mathscr{O} \\ = B^0 \\ \end{array} \to 0$$

: B^0 を除いて exact

$$\rightsquigarrow \quad 0 \to 18\Omega^5(5) \to 6\Omega^4(4) \to \Omega^2(2) \oplus 3\mathscr{O} \to \mathscr{I}_{X_i}(2) \to 0$$

: exact $(i = 1, 2, 3)$

$$E_1^{pq} = H^q(\mathbb{P}^5, \mathscr{I}_{X_i}(2+p)) \otimes \Omega^{-p}(-p) \quad (i = 1, 2, 3)$$

($\Omega_{\mathbb{P}^5}$ を Ω と省略, 縦に $0 \le q \le 5$, 横に $-5 \le p \le 0$)

0	0	0	0	0	0
0	0	0	0	0	0
$18\Omega^5(5)$	$6\Omega^4(4)$	0	0	0	0
0	0	0	$\Omega^2(2)$	0	0
0	0	0	0	0	0
0	0	0	0	0	$3\mathscr{O}$

(手順 3) [EFS] によって resolution を得る.

$$\rightsquigarrow \quad 0 \to \begin{matrix} 18\Omega^5(5) \\ = B^{-2} \\ = \bigoplus E_1^{-2-j,j} \end{matrix} \to \begin{matrix} 6\Omega^4(4) \\ = B^{-1} \end{matrix} \to \begin{matrix} \Omega^2(2) \oplus 3\mathscr{O} \\ = B^0 \end{matrix} \to 0$$

: B^0 を除いて exact

$$\rightsquigarrow \quad 0 \to 18\Omega^5(5) \to 6\Omega^4(4) \to \Omega^2(2) \oplus 3\mathscr{O} \to \mathscr{I}_{X_i}(2) \to 0$$

: exact $(i = 1, 2, 3)$

(手順 4) Free resolution にする.

$$
\begin{array}{c}
0 \\
\downarrow \\
18\Omega^5(5) \\
\downarrow \alpha \\
6\Omega^4(4) \\
\downarrow \\
\text{Coker } \alpha \\
\downarrow \\
0
\end{array}
$$

$$
\begin{array}{c}
0 \\
\downarrow \\
\text{Coker } \alpha \\
\downarrow \\
\Omega^2(2) \oplus 3\mathscr{O} \\
\downarrow \\
\mathscr{I}_{X_i}(2) \\
\downarrow \\
0
\end{array}
$$

(手順 4) Free resolution にする.

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & \to & 18\mathscr{O}(-1) & \to & 18\Omega^5(5) & \to & 0 \\
 & & & & & & \downarrow \alpha & & \\
0 & \to & 6\mathscr{O}(-2) & \to & 36\mathscr{O}(-1) & \to & 6\Omega^4(4) & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & \text{Coker } \alpha & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

$$\begin{array}{c}
0 \\
\downarrow \\
\text{Coker } \alpha \\
\downarrow
\end{array}$$

$$0 \to \mathscr{O}(-4) \to 6\mathscr{O}(-3) \to 15\mathscr{O}(-2) \to 20\mathscr{O}(-1) \oplus 3\mathscr{O} \to \Omega^2(2) \oplus 3\mathscr{O} \to 0$$

$$\begin{array}{c}
\downarrow \\
\mathscr{I}_{X_i}(2) \\
\downarrow \\
0
\end{array}$$

(手順 4) Free resolution にする.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & 18\Omega^5(5) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & 6\mathscr{O}(-2) \arrow[r] & 36\mathscr{O}(-1) \arrow[r] & 6\Omega^4(4) \arrow[r] \arrow[d] & 0 \\
 & & & \text{Coker } \alpha \arrow[d] & \\
 & & & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & & & & \text{Coker } \alpha \arrow[d] & \\
0 \arrow[r] & \mathscr{O}(-4) \arrow[r] & 6\mathscr{O}(-3) \arrow[r] & 15\mathscr{O}(-2) \arrow[r] & 20\mathscr{O}(-1) \oplus 3\mathscr{O} \arrow[r] & \Omega^2(2) \oplus 3\mathscr{O} \arrow[r] \arrow[d] & 0 \\
 & & & & & \mathscr{I}_{X_i}(2) \arrow[d] & \\
 & & & & & 0 &
\end{tikzcd}
$$

(手順 4) Free resolution にする.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 18\mathscr{O}(-1) & \rightarrow & 18\Omega^5(5) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \alpha & & \\
0 & \rightarrow & 6\mathscr{O}(-2) & \rightarrow & 36\mathscr{O}(-1) & \rightarrow & 6\Omega^4(4) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 6\mathscr{O}(-2) & \rightarrow & 18\mathscr{O}(-1) & \rightarrow & \text{Coker } \alpha & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{c}
0 \\
\downarrow \\
\text{Coker } \alpha \\
\downarrow
\end{array}
$$

$$
0 \rightarrow \mathscr{O}(-4) \rightarrow 6\mathscr{O}(-3) \rightarrow 15\mathscr{O}(-2) \rightarrow 20\mathscr{O}(-1) \oplus 3\mathscr{O} \rightarrow \Omega^2(2) \oplus 3\mathscr{O} \rightarrow 0
$$

$$
\begin{array}{c}
\downarrow \\
\mathscr{I}_{X_i}(2) \\
\downarrow \\
0
\end{array}
$$

(手順 4) Free resolution にする.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 18\mathscr{O}(-1) & \to & 18\Omega^5(5) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \alpha & & \\
0 & \to & 6\mathscr{O}(-2) & \to & 36\mathscr{O}(-1) & \to & 6\Omega^4(4) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 6\mathscr{O}(-2) & \to & 18\mathscr{O}(-1) & \to & \text{Coker } \alpha & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{c}
 0 \\
 \downarrow \\
 {\color{red} 0 \to 6\mathscr{O}(-2) \to 18\mathscr{O}(-1) \to \text{Coker } \alpha \to 0} \\
 \downarrow \\
 0 \to \mathscr{O}(-4) \to 6\mathscr{O}(-3) \to 15\mathscr{O}(-2) \to 20\mathscr{O}(-1) \oplus 3\mathscr{O} \to \Omega^2(2) \oplus 3\mathscr{O} \to 0 \\
 \downarrow \\
 \mathscr{I}_{X_i}(2) \\
 \downarrow \\
 0
\end{array}
$$

(手順 4) Free resolution にする.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & 18\Omega^5(5) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & 6\mathscr{O}(-2) \arrow[r] \arrow[d] & 36\mathscr{O}(-1) \arrow[r] \arrow[d] & 6\Omega^4(4) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 6\mathscr{O}(-2) \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & \text{Coker } \alpha \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & 6\mathscr{O}(-2) \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & \text{Coker } \alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}(-4) \arrow[r] & 6\mathscr{O}(-3) \arrow[r] & 15\mathscr{O}(-2) \arrow[r] & 20\mathscr{O}(-1) \oplus 3\mathscr{O} \arrow[r] \arrow[dr] & \Omega^2(2) \oplus 3\mathscr{O} \arrow[r] \arrow[d] & 0 \\
 & & & & & \mathscr{I}_{X_i}(2) \arrow[d] & \\
 & & & & & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & 6\mathscr{O}(-2) \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & \text{Coker } \alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}(-4) \arrow[r] & 6\mathscr{O}(-3) \arrow[r] & 15\mathscr{O}(-2) \arrow[r] & 20\mathscr{O}(-1) \oplus 3\mathscr{O} \arrow[r] \arrow[dr] & \Omega^2(2) \oplus 3\mathscr{O} \arrow[r] \arrow[d] & 0 \\
 & & & & & \mathscr{I}_{X_i}(2) \arrow[d] & \\
 & & & & & 0 &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & 6\mathscr{O}(-2) \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & \text{Coker } \alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}(-4) \arrow[r] & 6\mathscr{O}(-3) \arrow[r] \arrow[ur, dash, dotted, "\oplus" description] & 15\mathscr{O}(-2) \arrow[r] \arrow[ur, dash, dotted, "\oplus" description] & 20\mathscr{O}(-1) \oplus 3\mathscr{O} \arrow[r] \arrow[dr] & \Omega^2(2) \oplus 3\mathscr{O} \arrow[r] \arrow[d] & 0 \\
 & & & & & \mathscr{I}_{X_i}(2) \arrow[d] & \\
 & & & & & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & & & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & 6\mathscr{O}(-2) \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & \text{Coker } \alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}(-4) \arrow[r] & 6\mathscr{O}(-3) \arrow[r] & 15\mathscr{O}(-2) \arrow[r] & 20\mathscr{O}(-1) \oplus 3\mathscr{O} \arrow[r] \arrow[dr] & \Omega^2(2) \oplus 3\mathscr{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}(-4) \arrow[r] & \begin{matrix} 6\mathscr{O}(-2) \\ \oplus \\ 6\mathscr{O}(-3) \end{matrix} \arrow[r] & \begin{matrix} 18\mathscr{O}(-1) \\ \oplus \\ 15\mathscr{O}(-2) \end{matrix} \arrow[r] & \begin{matrix} 3\mathscr{O} \\ \oplus \\ 20\mathscr{O}(-1) \end{matrix} \arrow[r] & \mathscr{I}_{X_i}(2) \arrow[r] \arrow[d] & 0 \\
 & & & & & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & 6\mathscr{O}(-2) \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & \text{Coker } \alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}(-4) \arrow[r] & 6\mathscr{O}(-3) \arrow[r] & 15\mathscr{O}(-2) \arrow[r] & 20\mathscr{O}(-1) \oplus 3\mathscr{O} \arrow[r] \arrow[dr] & \Omega^2(2) \oplus 3\mathscr{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}(-4) \arrow[r] & \begin{matrix} 6\mathscr{O}(-2) \\ \oplus \\ 6\mathscr{O}(-3) \end{matrix} \arrow[r] & \begin{matrix} 18\mathscr{O}(-1) \\ \oplus \\ 15\mathscr{O}(-2) \end{matrix} \arrow[r] & \begin{matrix} 3\mathscr{O} \\ \oplus \\ 20\mathscr{O}(-1) \end{matrix} \arrow[r] & \mathscr{I}_{X_i}(2) \arrow[r] \arrow[d] & 0 \\
 & & & & & 0 &
\end{tikzcd}
$$

Mapping cone によって, 任意の $i = 1, 2, 3$ で

$$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \oplus 6\mathscr{O}(-4) \to 15\mathscr{O}(-4) \oplus 18\mathscr{O}(-3) \to 20\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_i} \to 0$$

Example (AT)

1. $0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \oplus 2\mathscr{O}(-3) \to 4\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_1} \to 0$
2. $0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \to 2\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_2} \to 0$
3. $0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \to 2\mathscr{O}(-3) \to 3\mathscr{O}(-2) \to \mathscr{I}_{X_3} \to 0$

Mapping cone によって, 任意の $i = 1, 2, 3$ で

$$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \oplus 6\mathscr{O}(-4) \to 15\mathscr{O}(-4) \oplus 18\mathscr{O}(-3) \to 20\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_i} \to 0$$

Example (AT)

1. $0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \oplus 2\mathscr{O}(-3) \to 4\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_1} \to 0$
2. $0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \to 2\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_2} \to 0$
3. $0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \to 2\mathscr{O}(-3) \to 3\mathscr{O}(-2) \to \mathscr{I}_{X_3} \to 0$

Mapping cone によって, 任意の $i = 1, 2, 3$ で

$$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \oplus 6\mathscr{O}(-4) \to 15\mathscr{O}(-4) \oplus 18\mathscr{O}(-3) \to 20\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_i} \to 0$$

Example (AT)

1. $0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \oplus 2\mathscr{O}(-3) \to 4\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_1} \to 0$
2. $0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \to 2\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_2} \to 0$
3. $0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \to 2\mathscr{O}(-3) \to 3\mathscr{O}(-2) \to \mathscr{I}_{X_3} \to 0$

Mapping cone によって, 任意の $i = 1, 2, 3$ で

$$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \oplus 6\mathscr{O}(-4) \to 15\mathscr{O}(-4) \oplus 18\mathscr{O}(-3) \to 20\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_i} \to 0$$

Example (AT)

1. $$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \oplus 2\mathscr{O}(-3) \to 4\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_1} \to 0$$
2. $$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \to 2\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_2} \to 0$$
3. $$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \to 2\mathscr{O}(-3) \to 3\mathscr{O}(-2) \to \mathscr{I}_{X_3} \to 0$$

Mapping cone によって, 任意の $i = 1, 2, 3$ で

$$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \oplus 6\mathscr{O}(-4) \to 15\mathscr{O}(-4) \oplus 18\mathscr{O}(-3) \to 20\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_i} \to 0$$

Example (AT)

1. $$0 \to \mathscr{O}(-6) \to 6\mathscr{O}(-5) \to 9\mathscr{O}(-4) \oplus 2\mathscr{O}(-3) \to 4\mathscr{O}(-3) \oplus 3\mathscr{O}(-2) \to \mathscr{I}_{X_1} \to 0$$
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Problems

- なぜ $i = 1, 2, 3$ で異なる free resolution になるか?
- どうすれば無駄のない free resolution にできるか?

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$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & 0 & \to & 6\mathscr{O}(-2) & \to & 18\mathscr{O}(-1) & \to & \text{Coker } \alpha & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to \mathscr{O}(-4) & \to & 6\mathscr{O}(-3) & \to & 15\mathscr{O}(-2) & \to & 20\mathscr{O}(-1) \oplus 3\mathscr{O} & \to & \Omega^2(2) \oplus 3\mathscr{O} & \to & 0 \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & \mathscr{I}_{X_i}(2) & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & 0 & &
\end{array}
$$

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$$
\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & 6\mathscr{O}(-2) \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & \text{Coker } \alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}(-4) \arrow[r] & 6\mathscr{O}(-3) \arrow[r] & 15\mathscr{O}(-2) \arrow[r] & 20\mathscr{O}(-1) \oplus 3\mathscr{O} \arrow[r] & \Omega^2(2) \oplus 3\mathscr{O} \arrow[r] \arrow[d] & 0 \\
 & & & & & \mathscr{I}_{X_i}(2) \arrow[d] & \\
 & & & & & 0 &
\end{tikzcd}
$$

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\begin{tikzcd}[column sep=small]
 & & & & & 0 \arrow[d] & \\
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 & & & & & 0 &
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 & & & & & 0 \arrow[d] & \\
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 & & & & & \mathscr{I}_{X_i}(2) \arrow[d] & \\
 & & & & & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & 18\Omega^5(5) \arrow[r] & 6\Omega^4(4) \arrow[rr] \arrow[dr] & & 2\Omega^2(2) \oplus 3\mathscr{O} \arrow[r] & \mathscr{I}_{X_i}(2) \arrow[r] & 0 \\
 & & 0 \arrow[r] & \text{Coker } \alpha \arrow[r] \arrow[ur] & 0 & &
\end{tikzcd}
$$

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$$
\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & 6\mathscr{O}(-2) \arrow[r] \arrow[d] & 18\mathscr{O}(-1) \arrow[r] \arrow[d] & \text{Coker } \alpha \arrow[r] \arrow[d] & 0 \\
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 & & & & & \mathscr{I}_{X_i}(2) \arrow[d] & \\
 & & & & & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & 18\Omega^5(5) \arrow[r] & 6\Omega^4(4) \arrow[rr, red] \arrow[dr] & & 2\Omega^2(2) \oplus 3\mathscr{O} \arrow[r] & \mathscr{I}_{X_i}(2) \arrow[r] & 0 \\
 & & 0 \arrow[r] & \text{Coker } \alpha \arrow[r] \arrow[ur] & 0 & &
\end{tikzcd}
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$$\begin{array}{rcl} 6\Omega^4(4) & \to & \Omega^2(2) \oplus 3\mathscr{O} \\ \text{i.e. } H^3(\mathscr{I}_{X_i}(-2)) \otimes \Omega^4(4) & \to & (H^2(\mathscr{I}_{X_i}) \otimes \Omega^2(2)) \oplus (H^0(\mathscr{I}_{X_i}(2)) \otimes \mathscr{O}) \end{array}$$

は各 $\mathscr{I}_{X_i}$ $(i = 1, 2, 3)$ で異なる．この写像の具体的な対応を調べれば, Main result の free resolution を無駄のない形に出来る.

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Main result

C : 楕円曲線 $(g(C)=1)$, $\deg\mathscr{E}=0$,
$X=\mathbb{P}(\mathscr{E})\hookrightarrow\mathbb{P}^5$ by $|A|=|C_0+3f|$ とすると

$$0\to\mathscr{O}(-6)\to 6\mathscr{O}(-5)\oplus 6\mathscr{O}(-4)\to 15\mathscr{O}(-4)\oplus 18\mathscr{O}(-3)$$
$$\to 20\mathscr{O}(-3)\oplus 3\mathscr{O}(-2)\to\mathscr{I}_X\to 0$$