$\mathbb{P}^2$ の double cover における分類 と Chern Numbers

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1　概要

- $\mathbb{P}^2$ **の** double cover **である** smooth projective surface **の分類**
- $\mathbb{P}^2$ **の** double cover **である** minimal surface of general type **の** Chern Numbers **についての考察**
- $\mathbb{P}^1 \times \mathbb{P}^1$ **の** double cover **である** minimal surface of general type **の** Chern Numbers **についての考察**

基礎体は複素数体とする.

2　定義

Definition 1　X を sm. proj. surf.

有理写像　$\Phi_m : X \to \mathbb{P}^N$, $N = h^0(mK_X) - 1$,

このとき, X **の小平次元を次で**def**する**:

$$\kappa(X) := \max_{m\in\mathbb{N}} \dim \Phi_m(X) (\leq \dim(X) = 2)$$

特に,

$$X: \ \textbf{general type} \overset{\text{def}}{\Longleftrightarrow} \kappa(X) = 2.$$

Definition 2

X : surf. of general type **とする**. **このとき**,

$$X: \textbf{ minimal} \overset{\text{def}}{\iff} K_X: \text{ nef.}$$

Theorem 1 (Noether's formula)

X **を** sm. proj. surf.

$\Longrightarrow$

$$1 - q(X) + p_g(X) = \frac{1}{12}({c_1}^2(X) + c_2(X))$$

ただし, $q(X) := h^1(\mathcal{O}_X), p_g(X) := h^2(\mathcal{O}_X)$.

3 Fact

本節では, X : minimal **と仮定する**.

Theorem 2 X : general type

$\Longrightarrow$

$$c_1{}^2(X) = K_X{}^2 > 0$$

$$c_2(X) > 0$$

Theorem 3 (Bogomolov-Miyaoka-Yau **の定理**)

X : general type

$\Longrightarrow$

$$c_1^2(X) \leq 3c_2(X)$$

Theorem 4 (Noether's Inequality) X : general type

$\Longrightarrow$

$p_g(X) \leq \frac{1}{2}{c_1}^2(X) + 2$

$p_g(X) = \frac{1}{2}{c_1}^2(X) + 2$: **Noether's line**

Corollary 1 X : general type

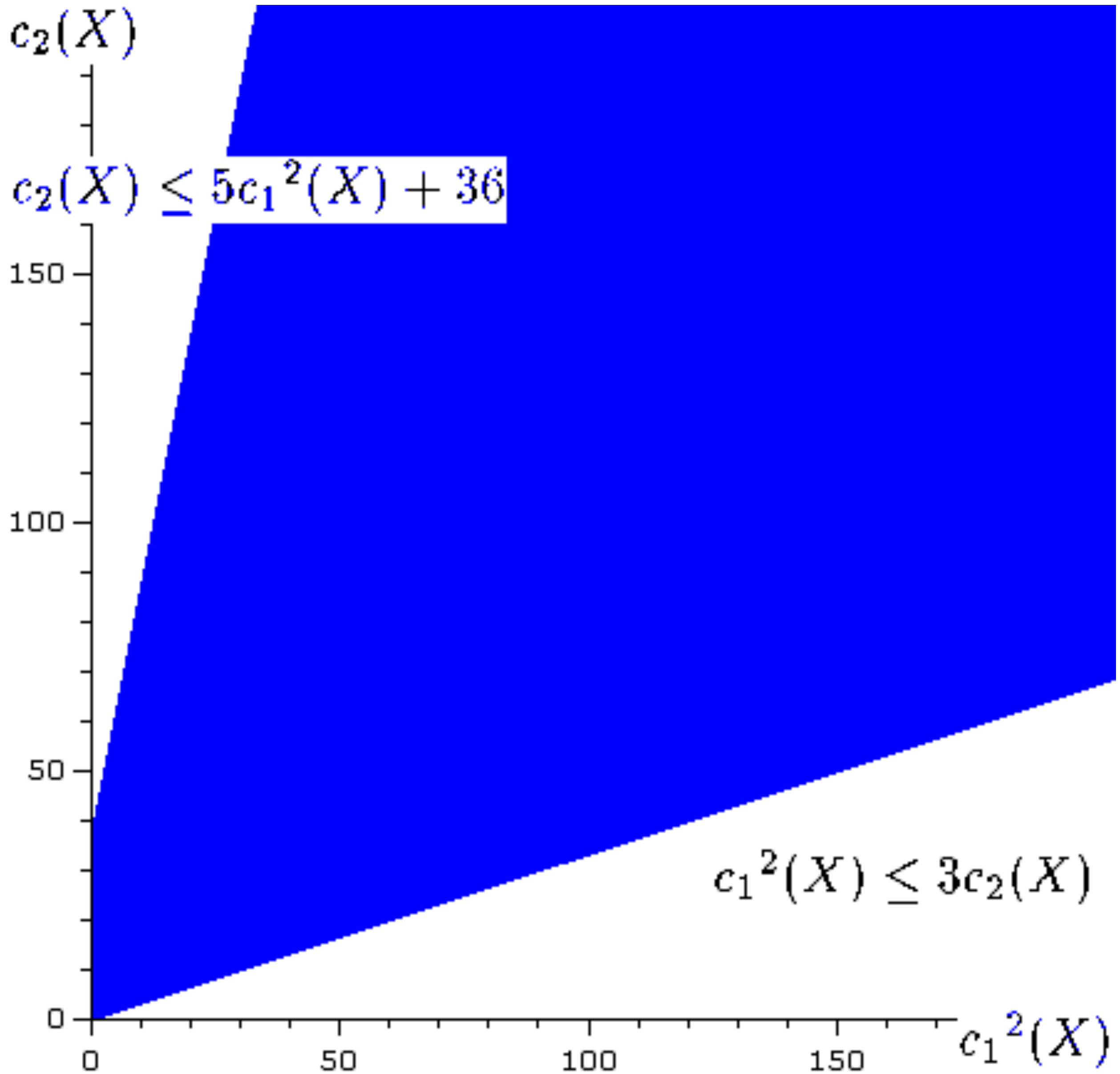
$\Longrightarrow$

$$5{c_1}^2(X) - c_2(X) + 36 \geq 0 \quad ({c_1}^2(X) \text{ : even})$$
$$5{c_1}^2(X) - c_2(X) + 30 \geq 0 \quad ({c_1}^2(X) \text{ : odd})$$

X : general type

$\Longrightarrow$

$$
\begin{aligned}
&{c_1}^2(X) = {K_X}^2 > 0 \\
&c_2(X) = 12(1 - q(X) + p_g(X)) - {K_X}^2 > 0 \\
&{c_1}^2(X) \leq 3c_2(X) \\
&5{c_1}^2(X) - c_2(X) + 36 \geq 0 \quad ({c_1}^2(X) : \text{ even })
\end{aligned}
$$


$c_2(X)$
$c_2(X) \leq 5{c_1}^2(X) + 36$
150
100
50
0
${c_1}^2(X) \leq 3c_2(X)$
0
50
100
150
${c_1}^2(X)$

4 $\mathbb{P}^2$ の double cover

Main Theorem

X は sm. proj. surf. , $\rho : X \longrightarrow \mathbb{P}^2$: double covering .

B : branch divisor , $B \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$.

(1) $b = 1$ のとき, X : $\mathbb{P}^1 \times \mathbb{P}^1$

(2) $b = 2$ のとき, X : ${K_X}^2 = 2$, Del Pezzo surf.

(3) $b = 3$ のとき, X : $K3$ surf.

(4) $b \geq 4$ のとき, X : minimal surf. of general type

$b \geq 4$ のとき,

$2{c_1}^2(X) + 42 \leq c_2(X) \leq 5{c_1}^2(X) + 36$ を満たす

（略証）

branch divisor $B \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$

$K_X = \rho^* K_{\mathbb{P}^2} + \rho^* \mathcal{O}_{\mathbb{P}^2}(b) = \rho^* \mathcal{O}_{\mathbb{P}^2}(b-3)$

${K_X}^2 = 2(b-3)^2$

(1) $b = 1$ **のとき**, X : $\mathbb{P}^1 \times \mathbb{P}^1$

(2) $b = 2$ **のとき**, X : ${K_X}^2 = 2$, Del Pezzo surf.

(3) $b = 3$ **のとき**, $K_X = \mathcal{O}_X$,$q(X) = 0$

$\therefore X$: $K3$ surf.

(4) $b \geq 4$**のとき**, $K_X = \rho^* \mathcal{O}_{\mathbb{P}^2}(b-3)$

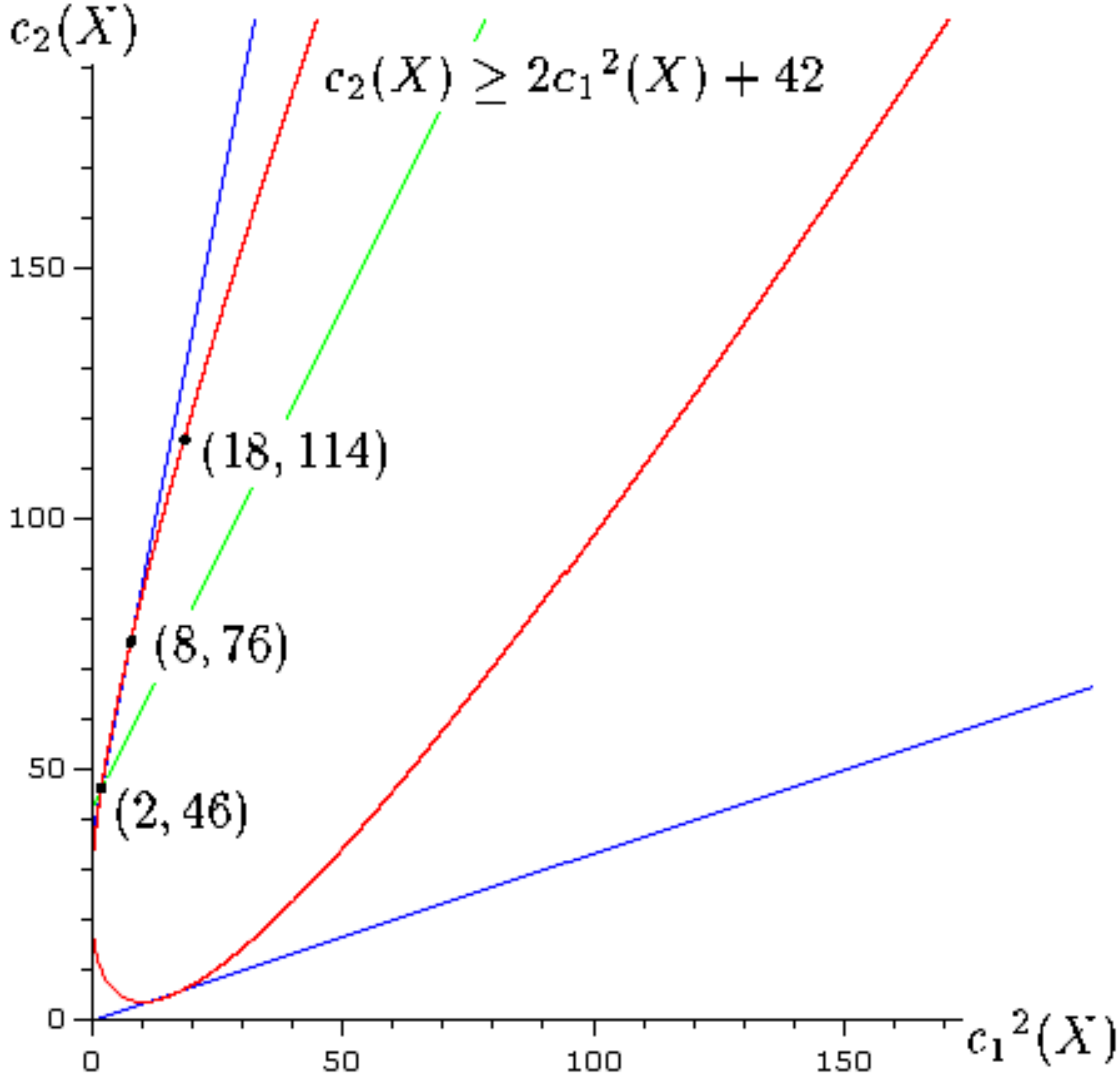
K_X : ample

$\therefore X$: general type , minimal surface.

$b \geq 4$ の ${c_1}^2(X)$, $c_2(X)$ について調べる.
${c_1}^2(X) = {K_X}^2$
$c_2(X) = 12(1 - q(X) + p_g(X)) - {K_X}^2$
$q(X) = h^1(\mathcal{O}_{\mathbb{P}^2}) + h^1(\mathcal{O}_{\mathbb{P}^2}(-b)) = 0$
$p_g(X) = h^2(\mathcal{O}_{\mathbb{P}^2}) + h^2(\mathcal{O}_{\mathbb{P}^2}(-b)) = h^0(K_{\mathbb{P}^2} - \mathcal{O}_{\mathbb{P}^2}(-b))$
$= h^0(\mathcal{O}_{\mathbb{P}^2}(b-3)) = \frac{(b-1)(b-2)}{2}$
$\therefore c_2(X) = 2(2b^2 - 3b + 3)$
以上から

$$b = t \qquad ({c_1}^2(X)\ ,\ c_2(X)) = (2(t-3)^2, 2(2t^2 - 3t + 3))$$

これらの軌跡を求めると,
$({2c_1}^2(X) - c_2(X) + 24)^2 = 162{c_1}^2(X)$



$$(2{c_1}^2(X) - c_2(X) + 24)^2 = 162{c_1}^2(X)$$

以上より, $b \geq 4$ **であるならば**,
$2{c_1}^2(X) + 42 \leq c_2(X) \leq 5{c_1}^2(X) + 36$ **を満たす**.　　Q.E.D.

5　$\mathbb{P}^1 \times \mathbb{P}^1$ のdouble cover

$\rho : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$: double covering.

branch divisor $B \in |\mathcal{O}(2b, 2b)|$ とする.

$K_X = \rho^* \mathcal{O}(b-2, b-2)$

${K_X}^2 = 4(b-2)^2$

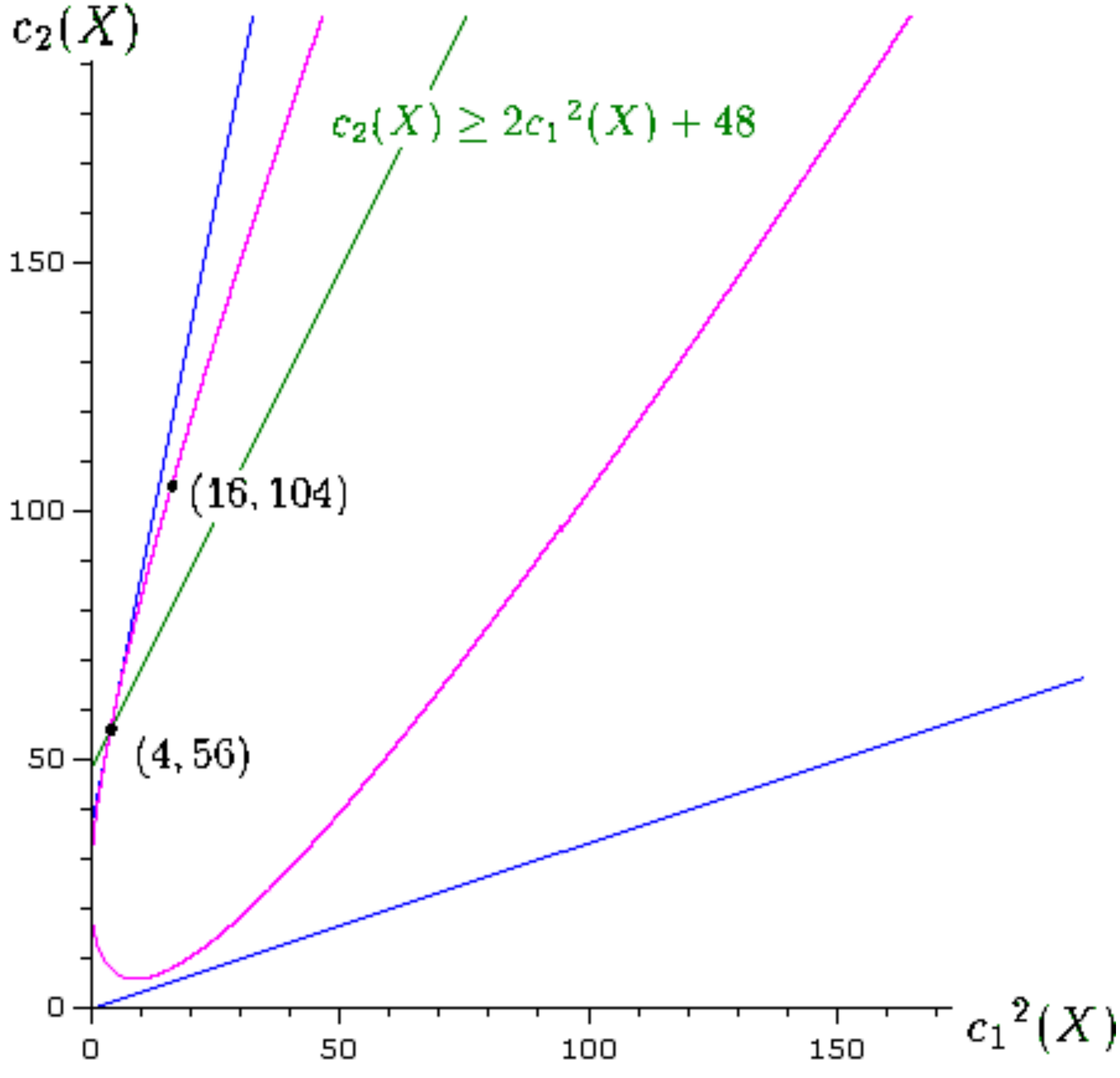
$b \geq 3$ のとき, X : minimal surf. of general type

$\therefore {c_1}^2(X) = {K_X}^2$, $c_2(X) = 8(b^2 - b + 1)$

以上から

$$b = t \qquad ({c_1}^2(X)\ ,\ c_2(X)) = (4(t-2)^2, 8(t^2 - t + 1))$$

軌跡は,$(2{c_1}^2(X) - c_2(X) + 24)^2 = 144{c_1}^2(X)$

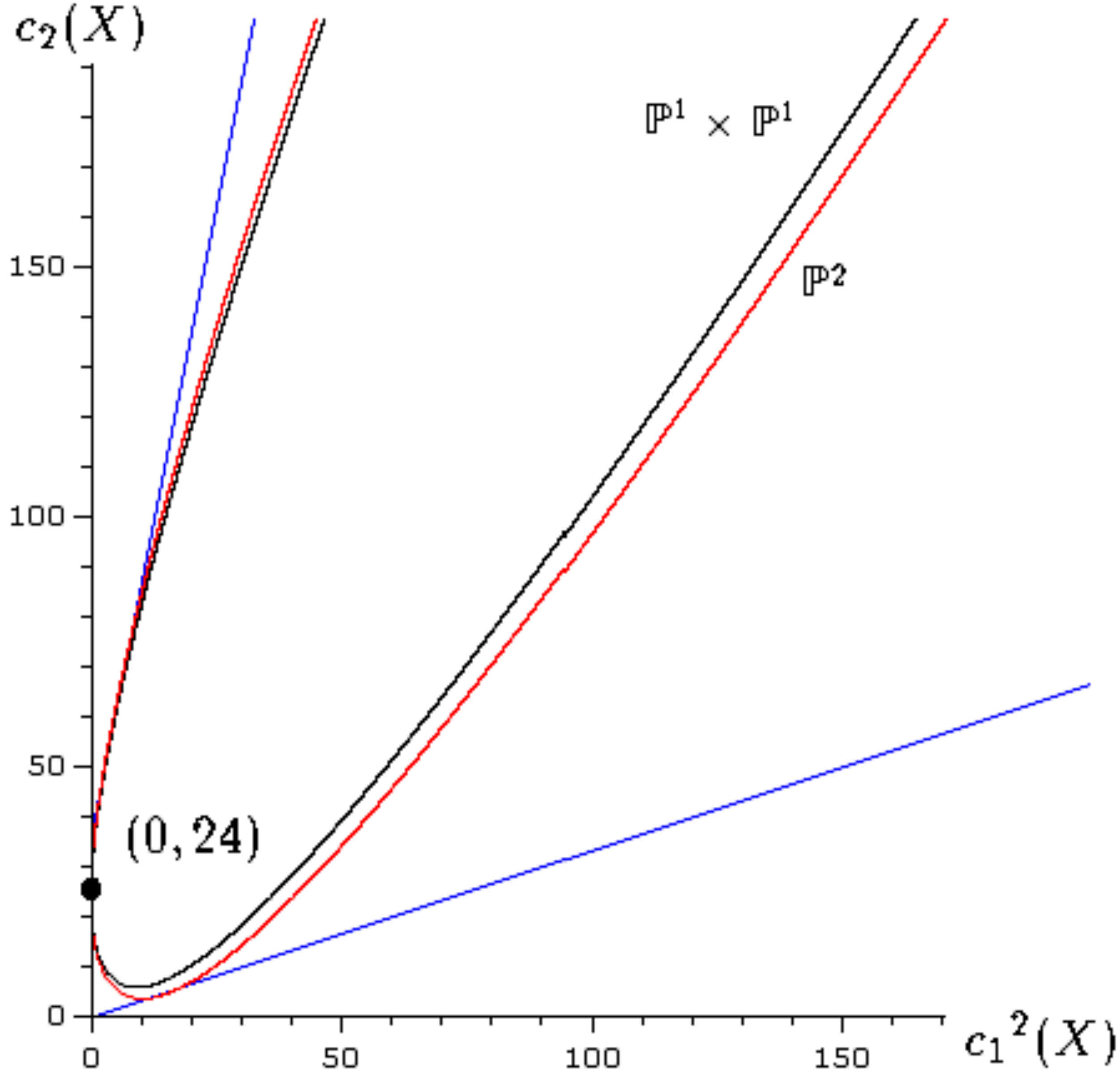


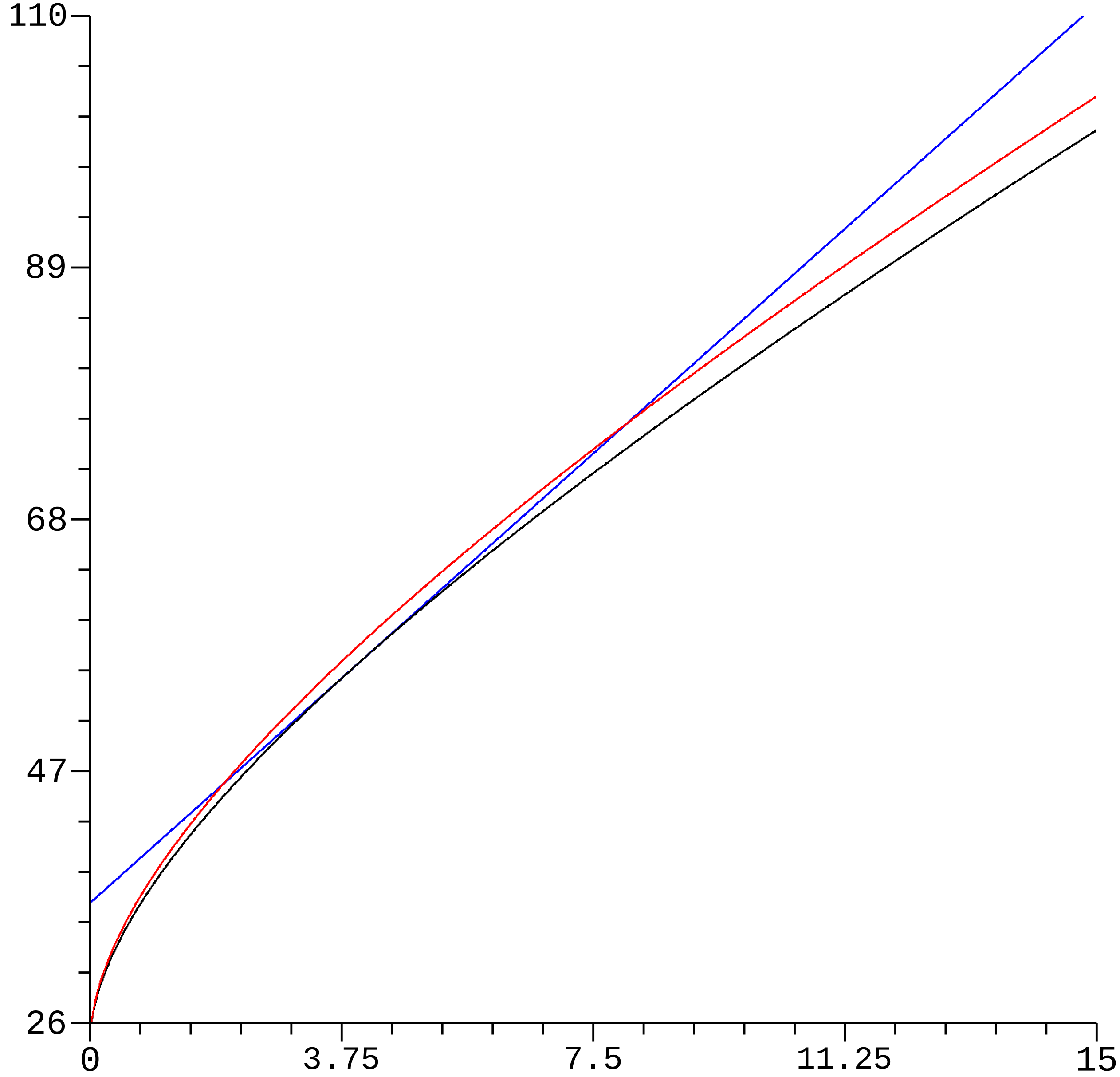
$$(2c_1{}^2(X) - c_2(X) + 24)^2 = 144c_1{}^2(X)$$

以上より,$b \geq 3$ であるならば,

$2{c_1}^2(X) + 48 \leq c_2(X) \leq 5{c_1}^2(X) + 36$ を満たしている.

6 $\mathbb{P}^2$ と $\mathbb{P}^1 \times \mathbb{P}^1$ の比較





110
89
68
47
26
0
3.75
7.5
11.25
15

7　まとめ

X : sm. proj. surf. , B : branch divisor

(1) $\rho : X \longrightarrow \mathbb{P}^2$: double covering
$B \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$ ($b \geq 4$) **としたとき**,
$2{c_1}^2(X) + 42 \leq c_2(X) \leq 5{c_1}^2(X) + 36$ **を満たす**
minimal surf. of general type

(2) $\rho : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$: double covering
$B \in |\mathcal{O}(2b, 2b)|$ ($b \geq 3$) **としたとき**,
$2{c_1}^2(X) + 48 \leq c_2(X) \leq 5{c_1}^2(X) + 36$ **を満たす**
minimal surf. of general type

参考文献

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