2009年2月6日

任意標数におけるVeronese varietyの higher secant varietyの定義方程式について

基幹理工学研究科 数学応用数理専攻 修士課程2年 5107A007-1

伊藤 達哉

指導教員名 楫 元

1 Definition

k : **基礎体**, $k = \overline{k}$, $\mathrm{ch}(k) \geq 0$.

Veronese variety.

$V_2^m := \nu_2(\mathbb{P}^m)$: **Veronese variety**,

$\nu_2 : \mathbb{P}^m \longrightarrow \mathbb{P}^{N := \binom{m+2}{2} - 1}$; $(x_0 : x_1 : \cdots : x_m) \longmapsto (x_0^2 : x_0 x_1 : x_0 x_2 : \cdots : x_m^2)$.

1 Definition

k : **基礎体**, $k = \overline{k}$, $\mathrm{ch}(k) \geq 0$.

Veronese variety.

$V_2^m := \nu_2(\mathbb{P}^m)$: **Veronese variety**,

$\nu_2 : \mathbb{P}^m \longrightarrow \mathbb{P}^{N := \binom{m+2}{2} - 1}$; $(x_0 : x_1 : \cdots : x_m) \longmapsto (x_0^2 : x_0 x_1 : x_0 x_2 : \cdots : x_m^2)$.

Secant variety.

$X \subseteq \mathbb{P}^N$: sm. proj. var.

$$\mathrm{Sec}(X)^\circ := \bigcup_{x,y \in X,\ x \neq y} \overline{xy}, \quad \mathrm{Sec}(X) := \overline{\mathrm{Sec}(X)^\circ} \quad \subseteq \mathbb{P}^N.$$

1 Definition

k : **基礎体**, $k=\overline{k}$, $\mathrm{ch}(k)\geq 0$.

Veronese variety.

$V_2^m := \nu_2(\mathbb{P}^m)$: **Veronese variety**,
$\nu_2 : \mathbb{P}^m \longrightarrow \mathbb{P}^{N:=\binom{m+2}{2}-1}$; $(x_0 : x_1 : \cdots : x_m) \longmapsto (x_0^2 : x_0x_1 : x_0x_2 : \cdots : x_m^2)$.

Secant variety.

$X \subseteq \mathbb{P}^N$: sm. proj. var.

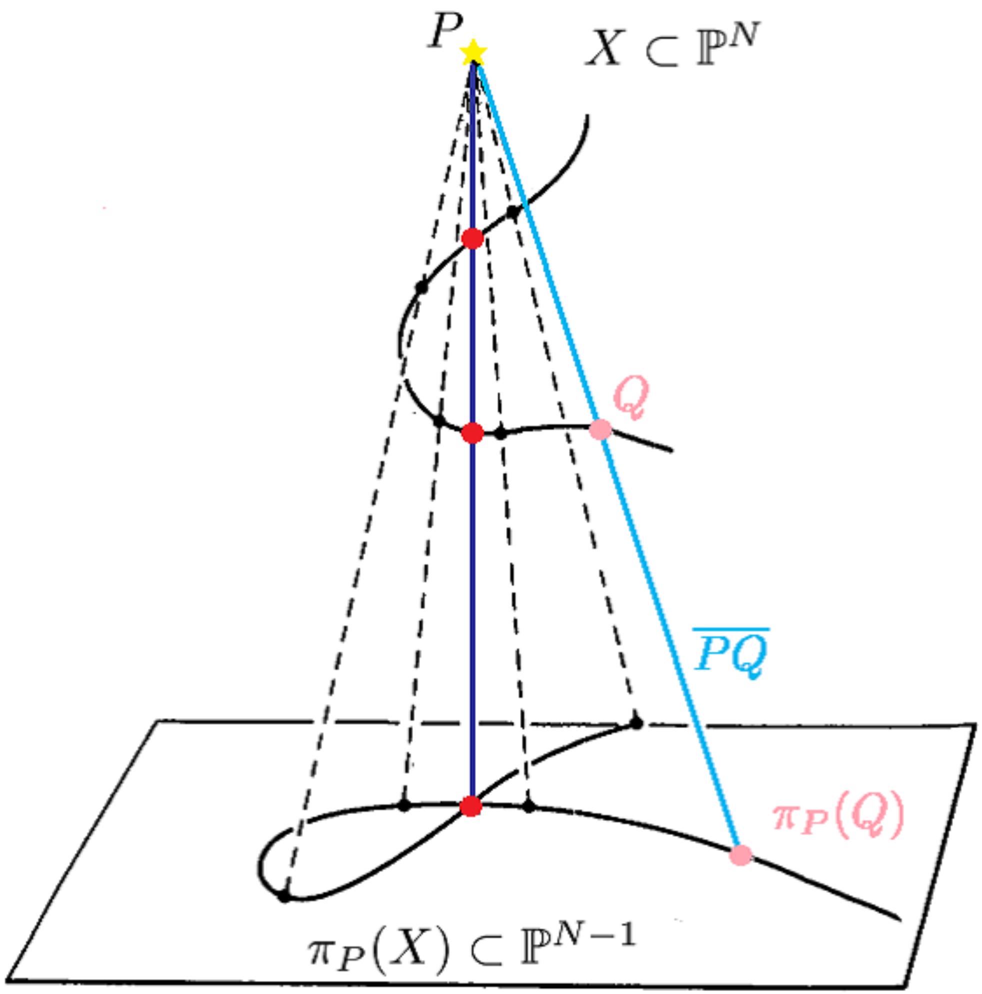
$$\mathrm{Sec}(X)^\circ := \bigcup_{x,y\in X,\ x\neq y} \overline{xy},\quad \mathrm{Sec}(X) := \overline{\mathrm{Sec}(X)^\circ} \quad \subseteq \mathbb{P}^N.$$

n-secant variety.

$$\mathrm{Sec}^n(X) := \overline{\bigcup_{x_0,\ \ldots\ ,x_n\in X} \langle x_0,\ \ldots\ ,x_n\rangle} \subseteq \mathbb{P}^N.$$

$\langle x_0,\ \ldots\ ,x_n\rangle$: $x_0,\ldots,x_n$で張られる線形部分空間.

2 Secant varietyの応用例



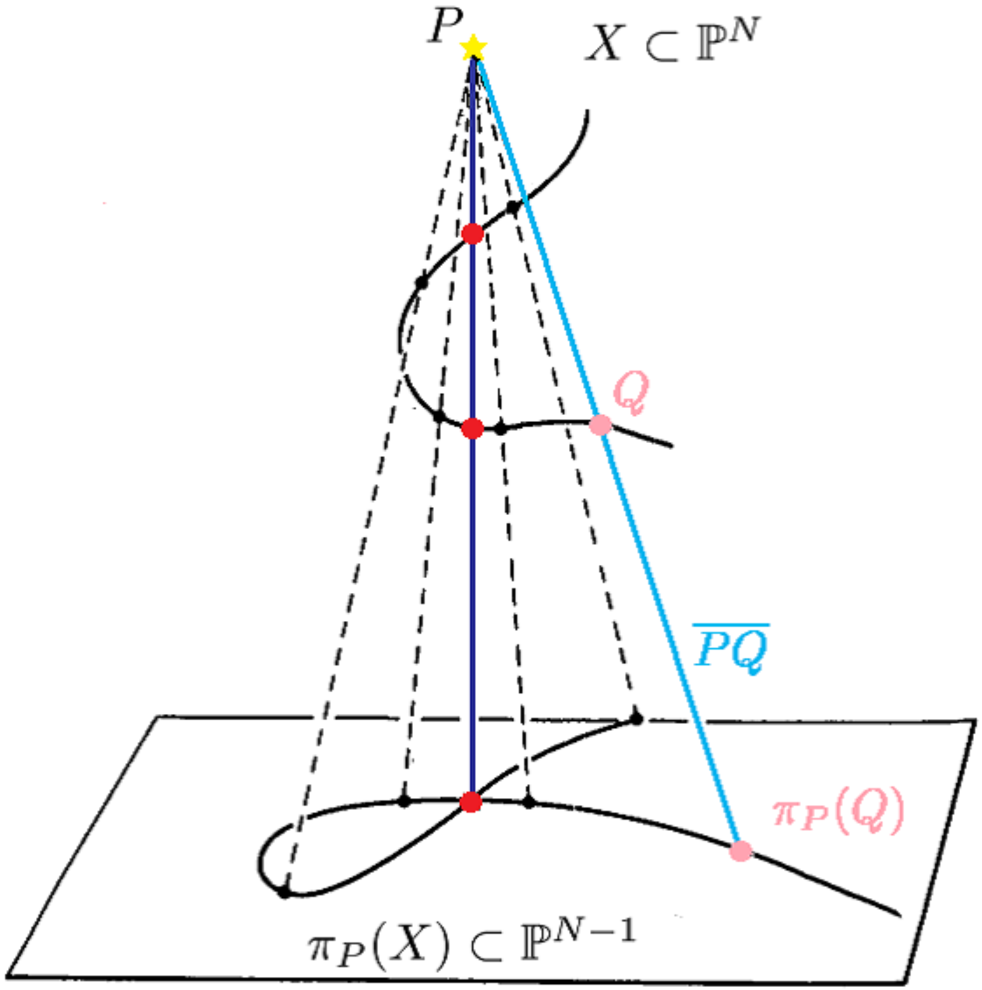
$$\begin{cases} X \subset \mathbb{P}^N : \text{sm. proj. var. }, \\ P \in \mathbb{P}^N \setminus X, \\ \pi_P : X \to \mathbb{P}^{N-1} \ (P\text{からの射影}). \end{cases}$$

Question.

$\pi_P : X \to \pi_P(X)$ **は同型になるか？**

P. Griffiths; J. Harris : *Principles of algebraic geometry.*

2 Secant varietyの応用例



$$\begin{cases} X \subset \mathbb{P}^N : \text{sm. proj. var. ,} \\ P \in \mathbb{P}^N \setminus X, \\ \pi_P : X \to \mathbb{P}^{N-1} \ (P\text{からの射影}). \end{cases}$$

Question.

$\pi_P : X \to \pi_P(X)$ **は同型になるか？**

Proposition.

$\pi_P : X \to \pi_P(X)$: 同型.

$$\Leftrightarrow \quad P \in \mathbb{P}^N \setminus \mathrm{Sec}(X).$$

P. Griffiths; J. Harris : *Principles of algebraic geometry.*

3 Fact

Proposition 1.

$$\mathrm{ch}(k) \geq 0, \qquad V_2^m = V(I_2(\Omega)).$$

$$Z = (Z_{00} : Z_{01} : Z_{02} : \cdots : Z_{m-1m} : Z_{mm}) \in \mathbb{P}^{N:=\binom{m+2}{2}-1},$$

$$\Omega(Z) := \begin{pmatrix} Z_{00} & Z_{01} & Z_{02} & \cdots & Z_{0m} \\ Z_{10} & Z_{11} & Z_{12} & \cdots & Z_{1m} \\ Z_{20} & Z_{21} & Z_{22} & \cdots & Z_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{m0} & Z_{m1} & Z_{m2} & \cdots & Z_{mm} \end{pmatrix} \quad (\text{但し}, Z_{ji} = Z_{ij}).$$

$$I_t(\Omega) := \langle\ \Omega\text{の}\, t \times t\, \text{小行列式} \rangle.$$

3 Fact

Proposition 1.

$\mathrm{ch}(k) \geq 0, \quad V_2^m = V(I_2(\Omega)).$

Proposition 2 (R. Gattazzo (1984)) .

(i) $\mathrm{ch}(k) \geq 0, \quad \mathrm{Sec}(V_2^m) \subseteq V(I_3(\Omega)).$
(ii) $\mathrm{ch}(k) \neq 2, \quad \mathrm{Sec}(V_2^m) = V(I_3(\Omega)).$

$$Z = (Z_{00} : Z_{01} : Z_{02} : \cdots : Z_{m-1m} : Z_{mm}) \in \mathbb{P}^{N := \binom{m+2}{2}-1},$$

$$\Omega(Z) := \begin{pmatrix} Z_{00} & Z_{01} & Z_{02} & \cdots & Z_{0m} \\ Z_{10} & Z_{11} & Z_{12} & \cdots & Z_{1m} \\ Z_{20} & Z_{21} & Z_{22} & \cdots & Z_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{m0} & Z_{m1} & Z_{m2} & \cdots & Z_{mm} \end{pmatrix} \quad (\text{但し}, Z_{ji} = Z_{ij}).$$

$I_t(\Omega) := \langle\ \Omega\text{の}\, t \times t\,\text{小行列式} \rangle.$

4　$\mathrm{ch}(k) = 2$の現象の考察

- $\mathrm{ch}(k) = 2,\ V(I_3(\Omega)) = \mathrm{Sec}(V_2^m)$?

(R. Gattazzo) NOTE : $\mathrm{ch}(k) = 2,\ m = 2,\ \exists R_0 \in V(I_3(\Omega)) \backslash \mathrm{Sec}(V_2^2)$.

4　$\mathrm{ch}(k) = 2$ の現象の考察

- $\mathrm{ch}(k) = 2,\ V(I_3(\Omega)) = \mathrm{Sec}(V_2^m)$?

$$\boxed{\text{(R. Gattazzo) NOTE : } \mathrm{ch}(k) = 2,\ m = 2,\ \exists R_0 \in V(I_3(\Omega)) \backslash \mathrm{Sec}(V_2^2).}$$

$$\Downarrow$$

$$\boxed{\textbf{実際は},\ \mathrm{ch}(k) = 2,\ R_0 \in \mathrm{Sec}(V_2^2) \setminus \mathrm{Sec}(V_2^2)^\circ.}$$ (注1)

(注1) $\mathrm{Sec}(X)^\circ := \bigcup_{x,y \in X,\ x \neq y} \overline{xy}.$

4　$\mathrm{ch}(k) = 2$ の現象の考察

- $\mathrm{ch}(k) = 2,\ V(I_3(\Omega)) = \mathrm{Sec}(V_2^m)$?

$$\boxed{\text{(R. Gattazzo) NOTE} : \mathrm{ch}(k) = 2,\ m = 2,\ \exists R_0 \in V(I_3(\Omega)) \setminus \mathrm{Sec}(V_2^2).}$$

$$\Downarrow$$

$$\boxed{\text{実際は},\ \mathrm{ch}(k) = 2,\ R_0 \in \mathrm{Sec}(V_2^2) \setminus \mathrm{Sec}(V_2^2)^\circ.}$$ (注 1)

$$\boxed{R_0 \in \mathrm{Tan}(V_2^2) \subseteq \mathrm{Sec}(V_2^2).}$$ (注 2)

(注 1)　$\mathrm{Sec}(X)^\circ := \bigcup_{x,y\in X,\ x\neq y} \overline{xy}$.

(注 2)　$\mathrm{Tan}(X) := \bigcup_{P\in X} \mathbb{T}_P X,\ \mathrm{Sec}(X) = \mathrm{Sec}(X)^\circ \cup \mathrm{Tan}(X)$.

4 $\mathrm{ch}(k)=2$の現象の考察

- $\mathrm{ch}(k)=2,\ V(I_3(\Omega))=\mathrm{Sec}(V_2^m)$?

$$\boxed{\text{(R. Gattazzo) NOTE : } \mathrm{ch}(k)=2,\ m=2,\ \exists R_0\in V(I_3(\Omega))\setminus\mathrm{Sec}(V_2^2).}$$

$$\Downarrow$$

$$\boxed{\text{実際は},\ \mathrm{ch}(k)=2,\ R_0\in\mathrm{Sec}(V_2^2)\setminus\mathrm{Sec}(V_2^2)^\circ.}$$ (注1)

$$\boxed{R_0\in\mathrm{Tan}(V_2^2)\subseteq\mathrm{Sec}(V_2^2).}$$ (注2)

$\mathrm{ch}(k)=2$ の現象.

$$\mathrm{ch}(k)\neq 2\Rightarrow\mathrm{Sec}(V_2^m)=\mathrm{Sec}(V_2^m)^\circ.$$
$$\mathrm{ch}(k)=2\Rightarrow\mathrm{Sec}(V_2^m)\neq\mathrm{Sec}(V_2^m)^\circ.$$

(注1) $\mathrm{Sec}(X)^\circ := \bigcup_{x,y\in X,\ x\neq y}\overline{xy}.$

(注2) $\mathrm{Tan}(X) := \bigcup_{P\in X}\mathbb{T}_P X,\ \mathrm{Sec}(X)=\mathrm{Sec}(X)^\circ\cup\mathrm{Tan}(X).$

5 Main Theorem

Main Theorem.

$\mathrm{ch}(k) \geq 0,\ V_2^m \subset \mathbb{P}^{N:=\binom{m+2}{2}-1}$: Veronese variety.

$$\mathrm{Sec}^n(V_2^m) = V(I_{n+2}(\Omega)) \quad (n \geq 0).$$

$$Z = (Z_{00} : Z_{01} : Z_{02} : \cdots : Z_{m-1m} : Z_{mm}) \in \mathbb{P}^N,$$

$$\Omega(Z) := \begin{pmatrix} Z_{00} & Z_{01} & Z_{02} & \cdots & Z_{0m} \\ Z_{10} & Z_{11} & Z_{12} & \cdots & Z_{1m} \\ Z_{20} & Z_{21} & Z_{22} & \cdots & Z_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{m0} & Z_{m1} & Z_{m2} & \cdots & Z_{mm} \end{pmatrix} \quad (\text{但し}, Z_{ji} = Z_{ij}).$$

$$I_t(\Omega) := \langle\ \Omega\text{の}\, t \times t\, \text{小行列式} \rangle.$$

5 Main Theorem

Main Theorem.

$\mathrm{ch}(k) \geq 0$, $V_2^m \subset \mathbb{P}^{N := \binom{m+2}{2} - 1}$: Veronese variety.

$$\mathrm{Sec}^{n}(V_2^m) = V(I_{n+2}(\Omega)) \quad (n \geq 0).$$

$$Z = (Z_{00} : Z_{01} : Z_{02} : \cdots : Z_{m-1m} : Z_{mm}) \in \mathbb{P}^N,$$

$$\Omega(Z) := \begin{pmatrix} Z_{00} & Z_{01} & Z_{02} & \cdots & Z_{0m} \\ Z_{10} & Z_{11} & Z_{12} & \cdots & Z_{1m} \\ Z_{20} & Z_{21} & Z_{22} & \cdots & Z_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{m0} & Z_{m1} & Z_{m2} & \cdots & Z_{mm} \end{pmatrix} \quad (\text{但し}, Z_{ji} = Z_{ij}).$$

$$I_t(\Omega) := \langle\ \Omega \text{の} t \times t \text{小行列式} \rangle.$$

6　Application(Rank と Betti 数の関係について)

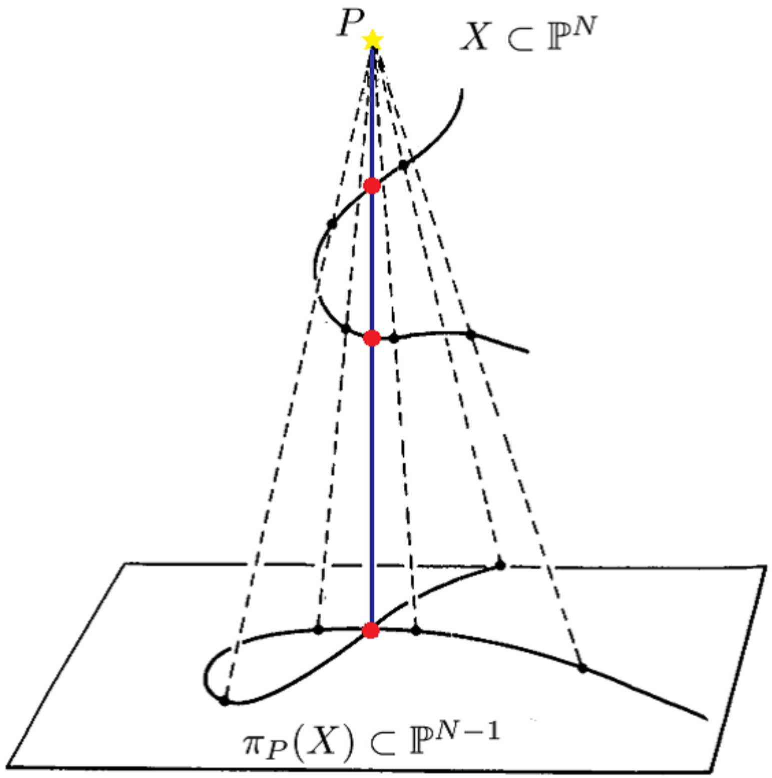
$X \subseteq \mathbb{P}^N$: sm. proj. var. , $P \in \mathbb{P}^N \setminus X$.

Rank.

$$\mathrm{rk}_X(P) := \min\{n \mid P \in \mathrm{Sec}^n(X)\}.$$

$$X \subseteq \mathrm{Sec}(X) \subseteq \mathrm{Sec}^2(X) \subseteq \cdots \subseteq \mathrm{Sec}^n(X) \subseteq \cdots \subseteq \mathbb{P}^N.$$

$\pi_P : X \to \mathbb{P}^{N-1}$.



6 Application(RankとBetti数の関係について)

Proposition (E. Park (2007)).

$\mathrm{ch}(k) \geq 0,\ P_1,\ P_2 \in \mathbb{P}^d \setminus \mathrm{Sec}(X),\ X := V_d^1,$

$$\mathrm{rk}_X(P_1) = \mathrm{rk}_X(P_2) \iff B(\pi_{P_1}(X)) = B(\pi_{P_2}(X)).$$

$$V_d^1 := \nu_d(\mathbb{P}^1),\ \nu_d : \mathbb{P}^1 \to \mathbb{P}^d; (x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : \cdots : x_1^d).$$

次数付きBetti数.

$$B(X) := (\beta_{ij}(X)).$$

$$F_i = \bigoplus\nolimits_j R(-j)^{\beta_{ij}(X)}, \qquad \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_0 \longrightarrow I_X \longrightarrow 0.$$

$$\mathrm{rk}_X(P) := \min\{n \mid P \in \mathrm{Sec}^n(X)\}.$$

6 Application(Rank とBetti数の関係について)

Proposition (E. Park (2007)).

$\mathrm{ch}(k) \geq 0$, $P_1, P_2 \in \mathbb{P}^d \setminus \mathrm{Sec}(X)$, $X := V_d^1$,

$$\mathrm{rk}_X(P_1) = \mathrm{rk}_X(P_2) \iff B(\pi_{P_1}(X)) = B(\pi_{P_2}(X)).$$

$$\rightsquigarrow$$

Question.

$\mathrm{ch}(k) \geq 0$, $P_1, P_2 \in \mathbb{P}^d \setminus \mathrm{Sec}(X)$, $X := V_2^m$,

$$\mathrm{rk}_X(P_1) = \mathrm{rk}_X(P_2) \iff B(\pi_{P_1}(X)) = B(\pi_{P_2}(X)). \quad ?$$

$V_d^1 := \nu_d(\mathbb{P}^1)$, $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d; (x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : \cdots : x_1^d)$.
$V_2^m := \nu_2(\mathbb{P}^m)$, $\nu_2 : \mathbb{P}^m \to \mathbb{P}^N; (x_0 : \cdots : x_m) \mapsto (x_0^2 : x_0x_1 \cdots : x_m^2)$.

6 Application(RankとBetti数の関係について)

Example.

$X := V_2^4,\ \exists P_1,\ P_2 \in \mathbb{P}^{14} \setminus \mathrm{Sec}(X)$ s.t. $\mathrm{rk}_X(P_1) = \mathrm{rk}_X(P_2) = 3,$

(i) $\mathrm{ch}(k) = 2, \quad B(\pi_{P_1}(X)) \neq B(\pi_{P_2}(X)).$

(ii) $\mathrm{ch}(k) = 0, 3, \quad B(\pi_{P_1}(X)) = B(\pi_{P_2}(X)).$

Question.

$\mathrm{ch}(k) \geq 0,\ P_1,\ P_2 \in \mathbb{P}^d \setminus \mathrm{Sec}(X),\ X := V_2^m,$

$$\mathrm{rk}_X(P_1) = \mathrm{rk}_X(P_2) \iff B(\pi_{P_1}(X)) = B(\pi_{P_2}(X)). \quad \textbf{?}$$

$\mathrm{rk}_X(P) := \min\{n \mid P \in \mathrm{Sec}^n(X)\}.$

$X \subseteq \mathrm{Sec}(X) \subseteq \mathrm{Sec}^2(X) \subseteq \cdots \subseteq \mathrm{Sec}^n(X) \subseteq \cdots \subseteq \mathbb{P}^N.$

7 まとめ

まとめ.

$$\mathrm{Sec}^{n}(V_2^m) = V(I_{n+2}(\Omega)) \quad (n \geq 0).$$

- **Main Theorem.**

  未解決だった$\mathrm{ch}(k) = 2$**の部分を証明した**.

- **Application.**

  $\mathrm{ch}(k) = 2$**において**, 1**次元の場合に成り立つ**rank**と次数付き**Betti**数の関係が**, **高次元で成り立たない例を与えた**.

7 まとめ

まとめ.

$$\mathrm{Sec}^{n}(V_2^m) = V(I_{n+2}(\Omega)) \quad (n \geq 0).$$

- **Main Theorem.**

  未解決だった$\mathrm{ch}(k) = 2$**の部分を証明した**.

- **Application.**

  $\mathrm{ch}(k) = 2$**において**, 1**次元の場合に成り立つ**rank**と次数付き**Betti**数の関係が**, **高次元で成り立たない例を与えた**.

以上です. **ありがとうございました**.

References

[1] D. Eisenbud : *The geometry of syzygies*. A second course in commutative algebra and algebraic geometry. Graduate Texts in Mathematics, 229. Springer-Verlag, New York, 2005.

[2] R. Gattazzo : *In characteristic $p = 2$ the Veronese variety $V^m \subset \mathbb{P}^{m(m+3)/2}$ and each of its generic projection is set-theoretic complete intersection*, Complete intersections (Acireale, 1983), 221–228, Lecture Notes in Math., 1092, Springer, Berlin, 1984.

[3] P. Griffiths; J. Harris : *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978.

[4] J. Harris : *Algebraic Geometry: A First Course*, Graduate Texts in Mathematics, 133. Springer-Verlag, New York, 1992.

[5] R. Hartshorne : *Algebraic Geometry*, Graduate Texts in Mathematics, 52. Springer-Verlag, New York-Heidelberg, 1977.

[6] H. Kaji : **随伴多様体の射影幾何的魅力，数理解析研究所講究録** 1460**巻** 2005**年** 23-32.

[7] V. Kanev : Chordal varieties of Veronese varieties and catalecticant matrices. Algebraic geometry, 9. *J. Math. Sci. (New York)* 94 (1999), no. 1, 1114–1125.

[8] A. Micali, O. E. Villamayor : Sur les algèbres de Clifford, *Ann. Sci. École Norm. Sup. (4)* **1** (1968), 271-304

[9] E. Park: Projective curves of degree = codimension+2. *Math. Z.* 256 (2007), no. 3, 685–697.

[10] M. Pucci : The Veronese variety and catalecticant matrices. *J. Algebra* 202 (1998), no. 1, 72–95.

[11] F. L. Zak : *Tangents and Secants of Algebraic Varieties*, Translated from the Russian manuscript by the author. Translations of Mathematical Monographs, 127. American Mathematical Society, Providence, RI, 1993.

8 定理の証明に関して

Lemma 1.

$$\mathrm{Sec}^n(V_2^m) \subseteq V(I_{n+2}(\Omega)) \quad (n \geq 0).$$

Lemma 2.

$R = (r_{ij}) \in V(I_3(\Omega)) \subset \mathbb{P}^N$,

(i) $\exists i,\ r_{ii} \neq 0 \Rightarrow \exists P \in V_2^m, \exists Q \in V_2^m$ s.t. $R \in \langle P, Q \rangle$.

(ii) $\forall i,\ r_{ii} = 0 \Rightarrow \exists P \in \mathbb{P}^m$ s.t. $R \in \mathbb{T}_P V_2^m \subseteq \mathrm{Tan}(V_2^m) = \mathrm{Sec}(V_2^m)$.

Lemma 3.

$R = (r_{ij}) \in V(I_{n+2}(\Omega)) \subset \mathbb{P}^N,\ n \geq 2$,

(i) $\exists i,\ r_{ii} \neq 0 \Rightarrow \exists P \in V_2^m\ ,\ \exists Q \in \mathrm{Sec}^{n-1}(V_2^m)$ s.t. $R \in \langle P,\ Q \rangle$.

(ii) $\forall i,\ r_{ii} = 0 \Rightarrow \exists P \in \mathrm{Tan}(V_2^m)\ ,\ \exists Q \in \mathrm{Sec}^{n-2}(V_2^m)$ s.t. $R \in \langle P,\ Q \rangle$.

9　$d=3$についての予想

Conjecture.

$$\mathrm{Sec}^n(V_3^m) = V(I_{n+2}(\mathrm{Cat}(1,\ 3-1;\ m+1))).$$

但し, $\mathrm{Cat}(1,\ 3-1;\ m+1)$**は**Catalecticant matrices**のこと**.

$\mathrm{ch}(k)=0,\ n=1$ならこれは正しいことが知られている.

$$\mathrm{Cat}(1,\ 3-1;\ 2+1) := \begin{pmatrix} Z_{300} & Z_{210} & Z_{201} & Z_{120} & Z_{111} & Z_{102} \\ Z_{210} & Z_{120} & Z_{111} & Z_{030} & Z_{021} & Z_{012} \\ Z_{201} & Z_{111} & Z_{102} & Z_{021} & Z_{012} & Z_{003} \end{pmatrix},$$

$$Z := (Z_{300} : Z_{210} : Z_{201} : Z_{120} : Z_{111} : Z_{102} : Z_{030} : Z_{021} : Z_{012} : Z_{003}) \in \mathbb{P}^9.$$