On projective varieties with group actions

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February

List of Papers

1. *Classification of polarized manifolds admitting homogeneous varieties as ample divisors*, preprint (May 2007), submitted.

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4. *A study of homogeneous varieties in the viewpoint of classification theories of polarized varieties*, preparation.

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We say that a projective variety X is *homogeneous* if there exists a group variety which acts transitively on X.

Examples of homogeneous varieties

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Dynkin diagram and a subset of its nodes $\Longrightarrow$ rational homogeneous variety.

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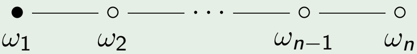
∗ $\mathbb{P}$: projective space, 

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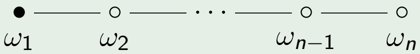
∗ $\mathbb{P}$: projective space, 

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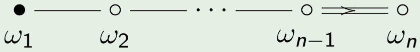

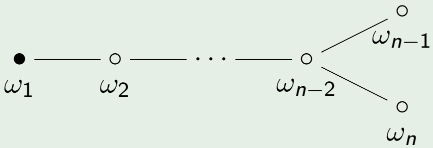

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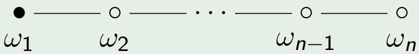
* $\mathbb{P}$: projective space, 

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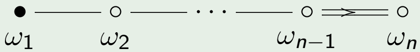

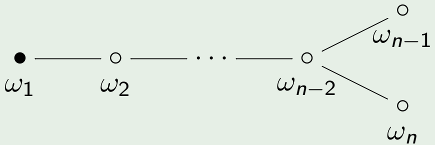

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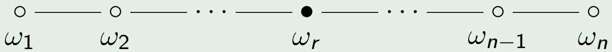

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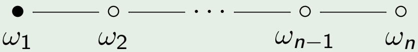

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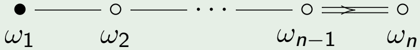

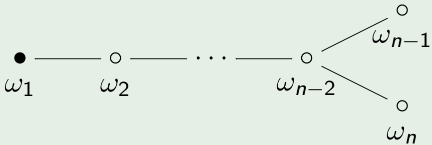

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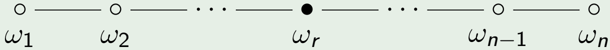

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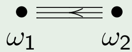
∗ 

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(6) If A and X are rational homogeneous varieties with $\rho(A) = \rho(X) = 1$, then a classification of such (X, L) is obtained by K.Konno ('88).

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(6) $(E_6(\omega_1), \mathscr{O}_{E_6(\omega_1)}(1))$.

Problem

Classify smooth projective varieties acted by some linear algebraic groups.

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Theorem (M. Andreatta, '01)

X: a smooth projective variety of dimension n
G: a simple, simply connected linear algebraic group acting regularly and non-trivially on X.
Then
(1) $n \geq r_G$,
(2) *if moreover $n = r_G$, then X is homogeneous.*

Theorem (M. Andreatta, '01)

X: a smooth projective variety of dimension n.
G: a simple, simply connected linear algebraic group of classical type acting regularly and non-trivially on X.
Assume that $n = r_G + 1$.
Then X is one of the following:
(1) $\mathbb{P}^n$,
(2) Q^n,
(3) $Y \times C$, *where* Y *is* $\mathbb{P}^{n-1}$ *or* Q^{n-1},
(4) $\mathbb{P}(\mathscr{O}_Y \oplus \mathscr{O}_Y(m))$, *where* Y *is as in* (3) *and* $m > 0$,
(5) $\mathbb{P}(T_{\mathbb{P}^2})$,
(6) $C_2(\omega_1 + \omega_2)$.

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X: a smooth projective variety of dimension n.
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Assume that $n = r_G + 1$.
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(3) $Y \times Z$, *where Y is* $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ *or* $G_2(\omega_2)$ *and Z is a smooth projective curve,*

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A linear algebraic group of Dynkin type F_4 acts on $E_6(\omega_1)$.

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Andreatta の結果と合わせると，$n = r_G + 1$ なる単純線型代数群の作用をもつ非特異射影多様体の完全な分類を得る.