曲線のJacobi多様体における群演算の幾何的様相

〜 曲線で Let's 足し算 〜

（しっかり言う）

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例.

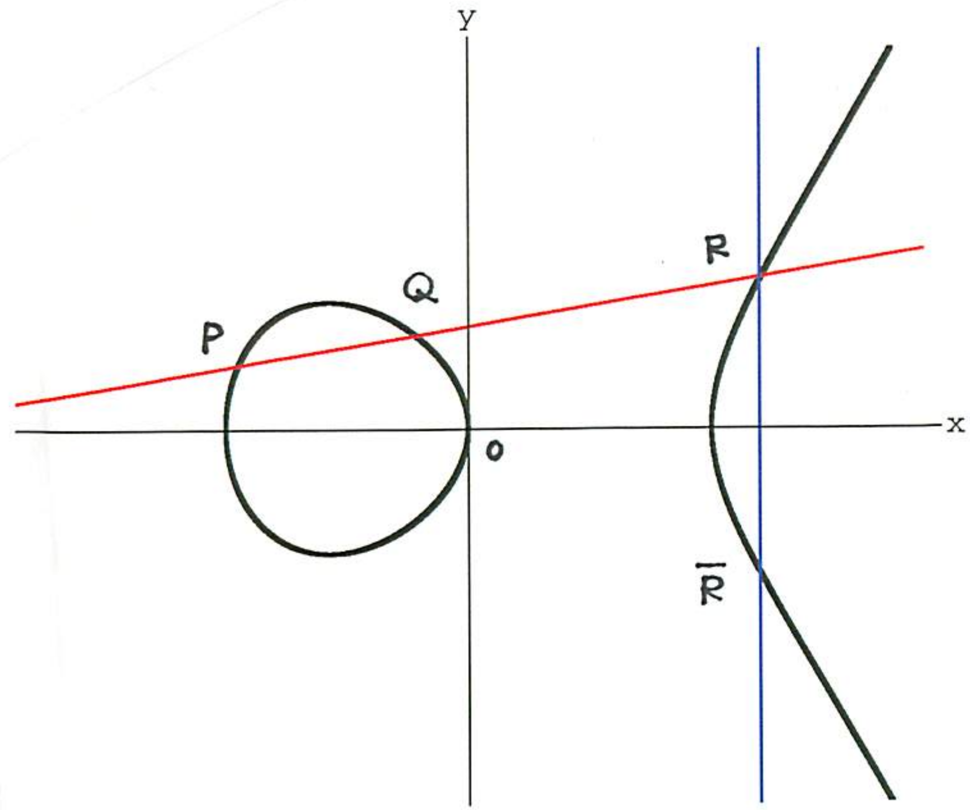
$C : (y^2 = x^3 - x) \cup \{\infty\}$: deg = 3 $g(C) = 1$ curve.

$P_0 = \infty \in C$: fix

$(P - P_0) + (Q - P_0) = \overline{R} - P_0 + (\varphi)$

$\varphi = L_1 / L_2$

L_2 は involution を与えるだけ.



$(P - \infty) + (Q - \infty) \sim (\overline{R} - \infty)$

Def

C : curve

$$\mathrm{Jac}(C) := \mathrm{Cl}^0(C) = \ker\left\{\begin{array}{rcl} \deg : \mathrm{Cl}(C) & \longrightarrow & \mathbb{Z} \\ \rotatebox{90}{$\in$} & & \\ \sum n_i P_i & \longmapsto & \sum n_i \end{array}\right\}$$

$$\mathrm{Cl}(C) = \left\{ \sum^{\text{finite}} n_i P_i \;\middle|\; n_i \in \mathbb{Z},\ P_i \in C \right\} \Big/ \sim$$

$$D_1 \sim D_2 \iff D_1 - D_2 = (\varphi) \quad \text{for } \exists \varphi \in K(C)$$

$P_0 \in C$: fix

$\mathrm{Jac}(C) \ni {}^\forall D \qquad {}^\exists n_i \in \mathbb{Z}_{>0} \qquad {}^\exists P_i \in C$

s.t. $D \sim \sum n_i P_i - \left(\sum n_i\right) \cdot P_0 \;;\; \sum n_i \leq g(C)$

テーマ：$g(C) \geq 2$ のときの φ は？

例

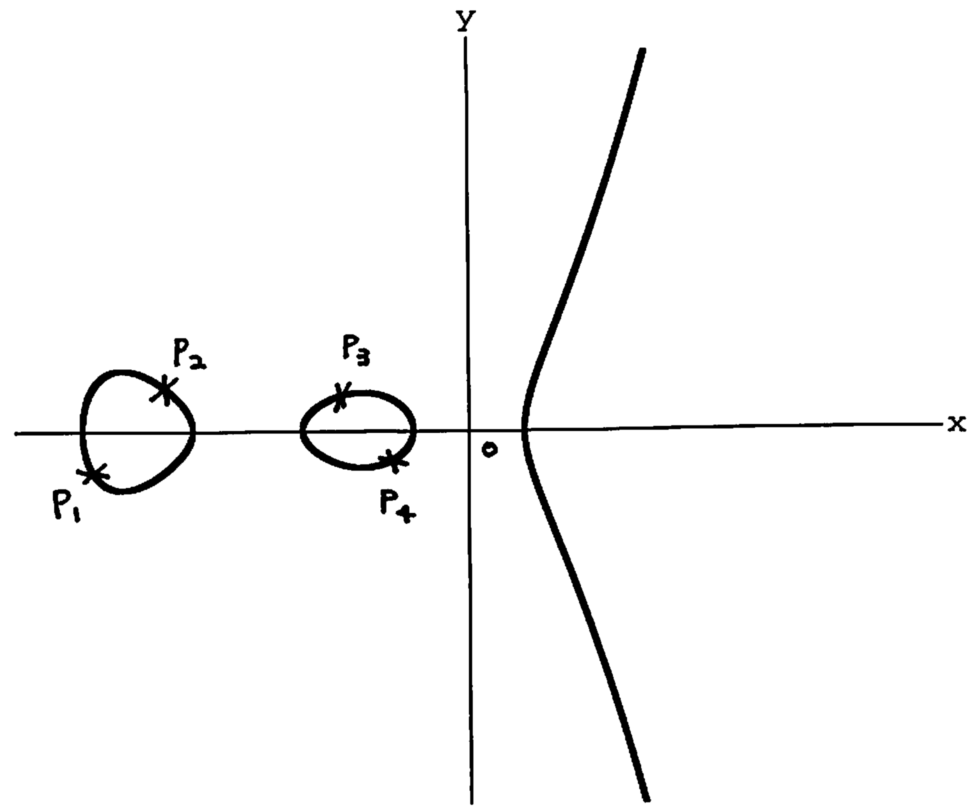
hyperelliptic curve

of $g(C) = 2$

$$\sum^{4} P_i - 4\cdot\infty$$

$$\sim \sum^{2} R_i - 2\cdot\infty$$

を与える $\varphi \in K(C)$ は？



§1. Hyperelliptic curve

$C : (y^2 = a_0 x^{2g+1} + \cdots + a_{2g+1}) \cup \{\infty\}$ hyperelliptic curve of $g(C) = g$

$P_0 = \infty$ とする.

Problem.

$D_1, D_2 \in \mathrm{Jac}(C)$

$$D_1 + D_2 = \sum^{2g} P_i - 2g \cdot \infty \sim \sum^{g} R_i - g \cdot \infty$$

このとき, linearly equivalence を与える $\varphi \in K(C)$ は?

Leitenberger による：

$\deg b(x) = [3g/2]$ ， $\deg c(x) = [g/2] - 1$

$F = b(x) - y \cdot c(x)$ s.t.

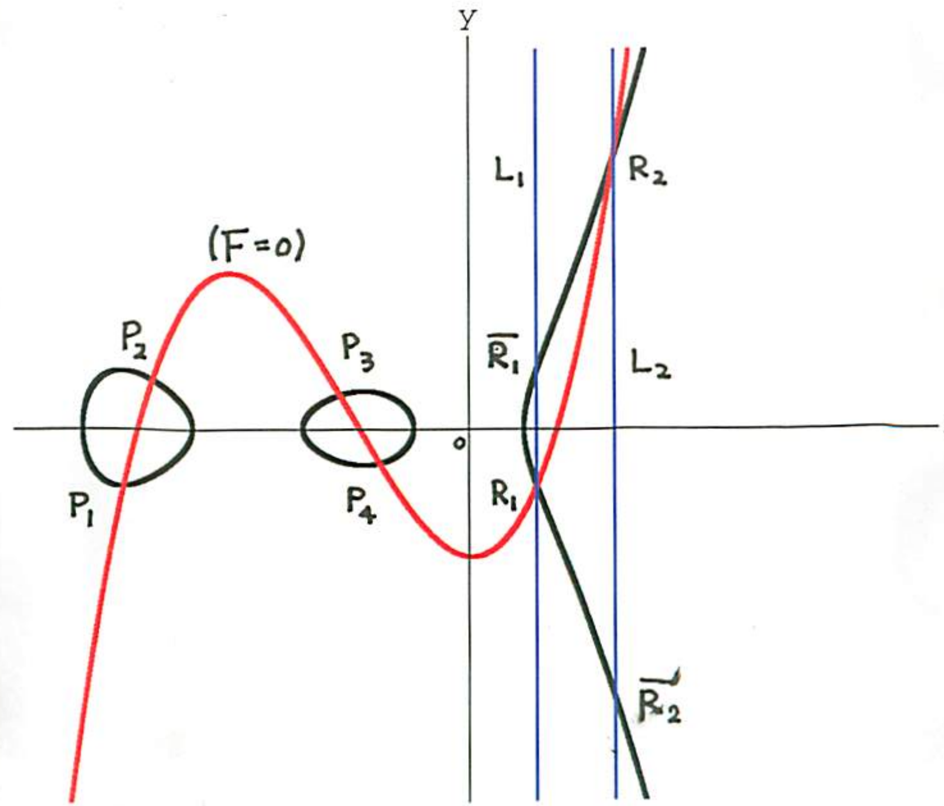
$P_1 \sim P_{2g} \in Z(F)$

<u>ex</u>. $g = 2$ のとき

$F = b(x) - y$.

$\sum^4 P_i - 4\cdot\infty \sim \sum^2 \overline{R_i} - 2\cdot\infty$

$\varphi = F / L_1 \cdot L_2$



Thm (F. Leitenberger)

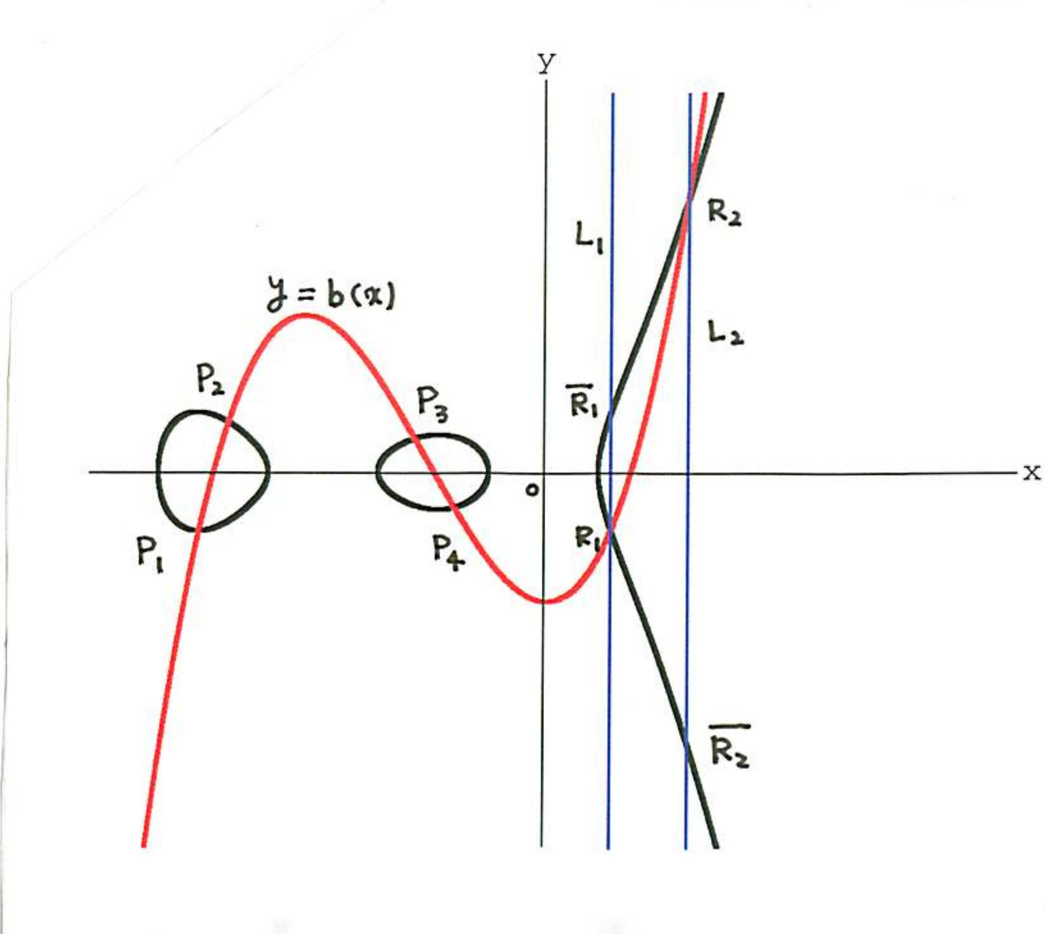
$g = 2$ のとき, $D_1 + D_2 = \sum_{1}^{4} P_i - 4\cdot\infty$

$(P_i \neq P_j, \overline{P_j} \ (\text{if } i \neq j))$

$$b(x) := \sum_{i=1}^{4} y_i \cdot \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

$F = y - b(x)$.

従って, $\varphi = F / L_1 \cdot L_2$



Thm 1. ($g(C) =: g$ が一般のとき)

$D_1 + D_2 = \sum (P_i - \infty)$; $P_i \neq P_j, \overline{P_j}$ (if $i \neq j$)

$n := [(g+1)/2]$

$\{P_i\}_{i=1}^{2g}$ のうち branch pt. が $2g-n$ 点以下のとき

$$b(x) := \sum_{1 \leq i_1 < \cdots < i_n \leq 2g} y_{i_1} \cdots y_{i_n} \cdot \prod_{k \neq i_1, \cdots, i_n} \frac{x - x_k}{(x_{i_1} - x_k) \cdots (x_{i_n} - x_k)}$$

$$c(x) := \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq 2g} y_{j_1} \cdots y_{j_{n-1}} \cdot \frac{\prod_{l=1}^{n-1} (x_{j_l} - x)}{\prod_{k \neq j_1, \cdots, j_{n-1}} (x_{j_1} - x_k) \cdots (x_{j_{n-1}} - x_k)}$$

$$F = b(x) - y \cdot c(x), \quad \varphi = F / \prod_{i=1}^{g} L_i$$

§2. 空間曲線

$C \subset \mathbb{P}^3$; smooth, nondegenerate curve
$\deg = d$, gen $= g$

$P_0 \in C$: fix

$$\sum_{i=1}^{g+1} P_i - (g+1) P_0 \sim \sum_{i=1}^{g} R_i - g \cdot P_0$$

を定めるような rational function φ

を与える :

Lemma

$$N_m := \binom{m+3}{3} - 1$$

$\{P_i\}_{i=1}^{N_m} \subset \mathbb{P}^3$

$\{P_i\}_{i=1}^{N_m} \subset H_m$ なる m 次 surface H_m の

定義方程式は、

$$F_m(\{P_i\}) := \sum_{i=1}^{N_m} \det(v_1 \cdots \overset{i}{\breve{v}_{N_m+1}} \cdots v_{N_m}) \cdot M_i + \det A \cdot M_{N_m+1}$$

但し、$M_1 \sim M_{N_m+1}$ を $k[x,y,z,w]$ の m 次 monomial,

$v_j = {}^t(M_j(P_1), \cdots, M_j(P_{N_m}))$, $A = (v_1 \cdots v_{N_m})$

Thm 2.

$C \subset \mathbb{P}^3$: smooth, nondegenerate, $\deg = d$, $g(C) = g$

$P_0 \in C$ を fix. $P_1 \sim P_{g+1} \in C$ に対し,

$$\sum_{i=1}^{g+1} P_i - (g+1)\cdot P_0 \sim \sum_{i=1}^{g} R_i - g\cdot P_0$$ を与える φ は

${}^\exists\{Q_i\}_{i=1}^{md-(g+1)}$, ${}^\exists\{R_i\}_{i=1}^{g} \subset C$

${}^\exists\{S_i\}_{i=1}^{Nm-md}$, ${}^\exists\{T_i\}_{i=1}^{Nm-md} \subset \mathbb{P}^3 \setminus C$

s.t.

$$\varphi = \frac{F_m(\{P_i\}, \{Q_i\}, \{S_i\})}{F_m(\{Q_i\}, \{R_i\}, \{T_i\}, P_0)}$$

proof.

- $N_m \geq md \geq g+1$ となるように $m \in \mathbb{Z}_{\geq 1}$ をとる.
- m次surface H_m について, $H_m . C = \{ md$ 点$\}$

$$H_m := Z(F_m(\{P_i\}, \{Q_i\}, \{S_i\}))$$
$$H'_m := Z(F_m(\{Q_i\}, \{R_i\}, \{T_i\}, P_0))$$ とすると,

$$H_m . C = \sum^{g+1} P_i + \sum^{md-(g+1)} Q_i$$
$$H'_m . C = \sum^{md-(g+1)} Q_i + P_0 + \sum^{g} R_i$$
$$\therefore \sum^{g+1} P_i - (g+1) P_0 \sim P_0 + \sum^{g} R_i - (g+1) \cdot P_0$$
$$= \sum^{g} R_i - g \cdot P_0$$

この linearly equivalent を与える $\varphi = \dfrac{F_m(\{P_i\}, \{Q_i\}, \{S_i\})}{F_m(\{Q_i\}, \{R_i\}, \{T_i\}, P_0)}$ ∎