Duality, vanishing, and a counterexample for Witt divisorial sheaves

因子的 Witt 層の双対性、消滅と反例

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1 Introduction

It is well-known that Kodaira Vanishing and its generalizations do not hold in positive characteristic. One possible approach to overcome this is to use Witt vectors to move from positive characteristic to a characteristic zero setting. This can be attempted in at least two different ways.

Given an effective Cartier divisor D on a projective variety X, the divisorial sheaf $\mathscr{O}_X(-D)$ can be thought of in two equivalent ways:

- as the ideal sheaf $\mathscr{I}_D$ of $\mathscr{O}_X$,
- as the invertible $\mathscr{O}_X$-module locally generated by the local equations of D.

In the case of Witt vectors, however, these two characterizations are no longer equivalent.

Recent work exploring the first approach includes papers by Ren and Ruelling [RR24] as well as Baudin [Bau25]. This approach has the advantage that the Witt vector endomorphisms V and F naturally extend to the Witt sheaf. In order to accomplish this for the second approach, we will need to consider the infinite direct sum $\bigoplus_{\mathbb{Z}}^{t} W\mathscr{O}(p^t D)$ (cf. section 3.2.3).

This work discusses the second approach, based on previous work by Tanaka [Tan22]. This approach has the advantage of being more easily generalized beyond effective divisors. Tanaka proposed a Kodaira-like vanishing theorem using the second approach, which holds for ample divisors in positive characteristic.

Theorem 1.0.1 ([Tan22, Corollary 4.16, Theorem 4.17, Theorem 5.3 Step 5])**.** *Let k be a perfect field of positive characteristic p, and $X \xrightarrow{\phi}$ Spec k be an N-dimensional smooth projective variety. If A is an ample $\mathbb{Q}$-divisor with simple normal crossing support on X, then*

(i)
- $H^j(X, W\mathscr{O}_X(-A))$ *is p^t-torsion, for some t, for any $j < N$.*
- $H^j(X, W\mathscr{O}_X(-sA))$ *is torsion free for large enough s, all j.*

(ii)
- $R^i\phi_* \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(-A), W\Omega_X^N) = 0$ *for $0 < i$ if A is a $\mathbb{Z}$-divisor.*
- *the above term is at most torsion if A is a $\mathbb{Q}$-divisor.*

At first glance, this is reminiscent of the two formulations of Kodaira vanishing due to Serre duality:

$$H^i(X, \mathscr{O}(-A)) \cong H^{N-i}(X, \mathscr{O}(A) \otimes \Omega_X^N)^\vee = 0.$$

It is therefore a natural question, whether there is a Serre-like dual relationship between (i) and (ii). In fact, the proof of Theorem 1.0.1(i) follows relatively directly from Serre vanishing, while that of (ii) is rather involved. Therefore it may prove particularly helpful for future developments to show a relationship of the type $(i) \Rightarrow (ii)$. For applications in birational geometry, for example, we would ideally want it to hold for nef and big divisors. The main motivation for this work is to establish a duality property for the Witt divisorial sheaf $W\mathscr{O}_X(D)$ associated to a $\mathbb{Q}$-divisor D on X, and then apply this to advance toward a Kawamata–Viehweg-type vanishing for nef and big divisors in positive characteristic.

In [Eke84], Ekedahl introduces a duality functor D, and eventually constructs an isomorphism ([Eke84, Theorem III: 2.9])

$$D(R\Gamma(W\Omega_X^\bullet))(-N)[-N] \cong R\Gamma(W\Omega_X^\bullet),$$

where $(-N)$ and $[-N]$ denote shifts in module and complex degree, respectively. He then shows that

$$D(R\Gamma(W\Omega_X^\bullet)) \cong R\operatorname{Hom}_R(R\Gamma(W\Omega_X^\bullet), \check{R}),$$

in $D(R)$, where R is the Raynaud Ring (a non-commutative W-algebra), and $\check{R}$ is a certain R-module. We will construct an isomorphism for Witt divisorial sheaves by adjusting and building on Ekedahl's methods. In particular, where Ekedahl uses the Raynaud ring R, we use the similar Cartier–Dieudonné ring $W[F, V] =: \omega$. The main result of chapter 3 is the following

Theorem 1.0.2 (cf. Theorem 3.2.11)**.** *Let k be a perfect field of characteristic $0 < p$, and $X \xrightarrow{\phi} \operatorname{Spec} k$ be an N-dimensional smooth projective variety. If D is a $\mathbb{Q}$-divisor with simple normal crossing support, then*

$$R\phi_* \mathscr{H}om_{W\mathscr{O}_X}(\omega(D), W\Omega_X^N) \cong R\operatorname{Hom}_\omega(R\phi_*\omega(D), \check{\omega}[-N]).$$

where we write ω for the Cartier–Dieudonné ring $W_\sigma[F, V]$, $\check{\omega}$ for a certain left-ω-module and $\omega(D) := \bigoplus_{t\in\mathbb{Z}} W\mathscr{O}_X(p^t D)$.

We consider $\omega(D)$ instead of $W\mathscr{O}_X(D)$ since, unlike $W\Omega_X^\bullet$ in [Eke84], the latter does not allow for an appropriate ω-module structure.

In chapter 4 we then apply Theorem 1.0.2 to prove a slightly improved vanishing theorem for ample $\mathbb{Q}$-divisors.

Corollary 1.0.3 (cf. Corollary 4.1.1)**.** *Let X and D be as in Theorem 1.0.2. Suppose that $R^j\Gamma_X(W\mathscr{O}_X(p^tD))$ is V-torsion free for $j \leq N$ and large enough t. Then*

$$R\operatorname{Hom}_\omega(R\phi_*\omega(D), \breve{\omega}[-N]) \cong \operatorname{Hom}_\omega(R\phi_*\omega(D), \breve{\omega}[-N]).$$

If furthermore $R^j\phi_*\omega(D) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $j < N$, then

$$R^i\phi_*(\mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(D), W\Omega^N_X)) = 0$$

for any $0 < i$. In particular, by Theorem 1.0.1(i) this is the case when $-D$ is ample.

Theorem 1.0.3 makes a notable distinction between V-torsion and p-torsion. The proof of Theorem 1.0.1 relies on Serre vanishing, which for ample divisors A and sufficiently large t guarantees the vanishing of

$$H^{j<N}(X, W\mathscr{O}_X(-p^tA)).$$

Corollary 1.0.3 shows that, in order for the vanishing of Theorem 1.0.1(ii) to hold for an SNC $\mathbb{Q}$-divisor D, such vanishing for large multiples of D is not necessary (cf. Remark 4.1.2).

We would like to determine the type of settings in which the vanishing of Witt divisorial sheaves may hold. In order to do so, section 4.2 and chapter 5 focus on the simpler special case of surfaces. Using a Serre-like vanishing for nef and big divisors on surfaces (cf. Lemma 4.2.1) and our vanishing theorem, Theorem 4.2.5 proves that vanishing holds for nef and big $\mathbb{Q}$-divisors on a surface.

Theorem 1.0.4 (cf. Theorem 4.2.5)**.** *For X a smooth surface over a field k of positive characteristic and D a nef and big $\mathbb{Q}$-divisor with SNC support on X,*

(i) $H^1(X, W\mathscr{O}_X(-D))_{\mathbb{Q}} = 0$*,*

(ii) $H^1(X, \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(-D), W\Omega^N_X)) = 0$*.*

In chapter 5 we show that the counterexample to Kawamata–Viehweg Vanishing of [CT18] also fails to vanish in the Witt divisoral case.

Theorem 1.0.5 (cf. Theorem 5.2.1)**.** *There exists a klt surface pair (X, Δ) and an integral divisor B on X, with X smooth and $B - \Delta$ nef and big, such that*

$$H^1(X, W\mathscr{O}_X(-B))_{\mathbb{Q}} \neq 0.$$

2 Preliminaries

2.1 Notation

Fix the following notations and conventions:

- A variety over k is a separated integral scheme of finite type over k.
- Throughout chapters 3 and 4 we define $X \xrightarrow{\phi} S = \operatorname{Spec} k$, where k is a perfect field of characteristic $p > 0$, and X is assumed to be a smooth projective variety.
- If not specified otherwise, the dimension of X is denoted by N.
- If C is a complex, $C[i]$ denotes C shifted by i in complex degree.
- If M_n is an inverse system, then $\lim_n M_n$ denotes the inverse limit.
- For a module M, we write $M_{\mathbb{Q}} := M \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.
- Given a ring A, $W(A)$ are the Witt vectors of A. If k is the base field, we may write W instead of $W(k)$ for the sake of brevity. The truncated Witt vectors are denoted by $W_n(A)$ or W_n respectively, for $n \in \mathbb{N}$. F and V denote the Frobenius and *Verschiebungs* maps on $W(A)$, respectively.
- The notion of Witt vectors can be sheafified to yield sheaves $W\mathscr{O}_X$, $W_n\mathscr{O}_X$, $W\mathscr{O}_X(D)$ and $W_n\mathscr{O}_X(D)$ for $n \in \mathbb{N}$ and a $\mathbb{Q}$-divisor D on X. For more details the reader may refer to [Tan22, Sect. 2, 3] or Definitions 2.3.1, 2.3.2. The ringed space $(X, W_n\mathscr{O}_X)$ is known to be a noetherian scheme (cf. [Ill79]).
- $W_\bullet\Omega_X^\bullet$ denotes the De Rham–Witt complex of X (cf. [Ill79]).
- Beginning in chapter 3, we write ω for the (non-commutative) *Cartier–Dieudonné ring* $W_\sigma[F, V]$, that is the W-algebra generated by V and F, subject to the relations

$$aV = V\sigma(a), Fa = \sigma(a)F \text{ for any } a \in W; VF = FV = p,$$

where σ is the Frobenius map on W, induced by that on k. Its truncation at n is denoted by ω_n.

2.2 Witt vectors

The original Kodaira Vanishing is closely related to Hodge decomposition. Hodge decomposition in turn resembles the slope decomposition of crystalline cohomology in terms of the de Rham–Witt complex. This motivates the attempt at finding a useful vanishing theorem in the context of de Rham–Witt.

Let A be a commutative unitary ring of positive characteristic p. The Witt vectors $W(A)$ over A are defined as the set $\prod_{i=0}^{\infty} A$ and endowed with a non-trivial ring structure.

Definition 2.2.1. *The* Witt polynomials *are defined as*

$$W_n(X) := \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \in \mathbb{Z}[X_0, \cdots, X_n] \textit{ for any } n \in \mathbb{N}_0$$

For any $\mathfrak{a} = (a_0, a_1, \cdots) \in W(A)$*, the* $W_n(\mathfrak{a})$ *are called the* ghost components *of* X*.*

The addition and multiplication of elements in the Witt ring $W(A)$ is defined by that of the ghost components as follows.

Theorem 2.2.2 (cf. [Ser79, Theorem II.7])**.** *There exist elements*

$$S_i, P_i \in \mathbb{Z}[X_0, \cdots, X_i, Y_0, \cdots, Y_i], \textit{ for any } i \in \mathbb{N},$$

such that

$$\begin{aligned} W_n(S_0, S_1, \cdots) &= W_n(X) + W_n(Y), \\ W_n(P_0, P_1, \cdots) &= W_n(X) W_n(Y) \end{aligned}$$

for all $0 \leq n$*. For* $\mathfrak{a}, \mathfrak{b} \in W(A)$*, set*

$$\begin{aligned} \mathfrak{a} + \mathfrak{b} &:= (S_0(\mathfrak{a}, \mathfrak{b}), S_1(\mathfrak{a}, \mathfrak{b}), \cdots) \\ \mathfrak{a} \cdot \mathfrak{b} &:= (P_0(\mathfrak{a}, \mathfrak{b}), P_1(\mathfrak{a}, \mathfrak{b}), \cdots). \end{aligned}$$

These laws of addition and multiplication make $W(A)$ *into a commutative unitary ring.*

Example 2.2.3 (cf. [Ser79, Section II.6])**.** *For example,*

$$\begin{aligned} S_0(X, Y) &= X_0 + Y_0, \quad & S_1(X, Y) &= X_1 + Y_1 + C_p(X_0, Y_0), \\ P_0(X, Y) &= X_0 Y_0, & P_1(X, Y) &= X_1 Y_0^p + X_0^p Y_1. \end{aligned}$$

where $C_p(X_0, Y_0) \in \mathbb{F}_p[X_0, Y_0]$ *is the polynomial uniquely determined by*

$$p\, C_p(X_0, Y_0) \equiv X_0^p + Y_0^p - (X_0 + Y_0)^p \pmod{p^2}.$$

Definition 2.2.4. *There are two additive maps on $W(A)$,*

$$V : (a_0, a_1, \cdots) \mapsto (0, a_0, a_1, \cdots),$$
$$F : (a_0, a_1, \cdots) \mapsto (a_0^p, a_1^p, \cdots).$$

called the Verschiebuns *map and the* Frobenius *map, respectively.*

Proposition 2.2.5 (cf. [Ser79, Section II.6])**.** *We have the identity*

$$VF = p = FV.$$

V is F^{-1}-linear while F is F-linear. That is for $\mathfrak{a}, \mathfrak{b} \in W(A)$,

$$\mathfrak{a}V(\mathfrak{b}) = V(F(\mathfrak{a})\mathfrak{b}),$$
$$F(\mathfrak{a})F(\mathfrak{b}) = F(\mathfrak{a}\mathfrak{b}).$$

Example 2.2.6. *Observe that $W_n(1, 0, 0, \cdots) = 1$ for all $n \in \mathbb{N}_0$. It follows that the element $(1, 0, 0, \cdots)$ is the identity element $\underline{1} \in W(A)$. Furthermore, we see that*

$$p\,\underline{1} = VF(\underline{1}) = (0, 1, 0, 0, \cdots) = p \in W(A).$$

Remark 2.2.7. *We see from the above example that $W(A)$ is a ring of characteristic zero (or of mixed characteristic to be precise). This provides yet another motivation to use Witt vectors for vanishing theorems in positive characteristic.*

Definition 2.2.8. *There is a multiplicative injection*

$$A \to W(A)$$
$$a \mapsto \underline{a} := (a, 0, 0, \cdots),$$

called the Teichmüller character*. The element $\underline{a} \in W(A)$ is called the* Teichmüller representative *of a.*

Example 2.2.9. *Using the formulas from example 2.2.3 we can compute that over $\mathbb{F}_3$,*

$$\underline{1} + \underline{1} = (2, 1, \cdots).$$

It follows that the Teichmüller character is not additive.

Definition 2.2.10. *We define the* truncated Witt vectors of length n *as*

$$W_n(A) := W(A)/V^n W(A).$$

The quotient map $W(A) \to W_n(A)$ is denoted by R_n. We will frequently omit the subscript for brevity, and use the same notation for the quotient map $W_m(A) \to W_n(A)$ (where $n < m$).

2.3 Tanaka's Vanishing

Definition 2.3.1 (Teichmüller lifts of line bundles, cf. [Tan22])**.** *For an invertible sheaf $\mathscr{F}$ on X defined by local transition functions (f_{ij}), Tanaka defines the* Teichmüller lift $\underline{\mathscr{F}}$ *of an invertible $\mathscr{O}_X$-module to be the invertible $W\mathscr{O}_X$-module given by the Teichmüller representatives of the transition functions $(\underline{f_{ij}})$. The truncated Teichmüller lift is defined by*

$$\underline{\mathscr{F}}_{\leq n} := W_n\mathscr{O}_X \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}.$$

Definition 2.3.2 (Witt divisorial sheaves, cf. [Tan22])**.** *Alternatively, the* Witt divisorial sheaf *associated to an $\mathbb{R}$-divisor D is defined by*

$$\Gamma_V(W\mathscr{O}_X(D)) := \{(\phi_0, \phi_1, \cdots) \in W(K(X)); div(\phi_n) + p^n D|_V \geq 0\}.$$

As Tanaka shows, for a Cartier divisor on a reasonably nice scheme, these two notions are equivalent, since $W\mathscr{O}(D)|_U = \underline{f}W\mathscr{O}_X|_U$ for U affine open in X and f a local equation for D on U (cf. [Tan22, Proposition 3.12]*). $W_n\mathscr{O}_X(D)$ is a coherent $W_n\mathscr{O}_X$-module (cf.* [Tan22, Proposition 3.8]*).*

The following two propositions, due to Tanaka [Tan22], will be used frequently throughout this paper, often without explicit reference.

Proposition 2.3.3 (Cf. [Tan22, Proposition 3.15])**.** *Let D be an $\mathbb{R}$-divisor on X. Then, for any $0 \leq e, 0 < m \leq n$, there is an isomorphism*

$$R\mathcal{H}om_{W_n\mathscr{O}_X}((F^e)_* W_m\mathscr{O}_X(D), W_n\Omega_X^N) \cong (F^e)_* \mathcal{H}om_{W_m\mathscr{O}_X}(W_m\mathscr{O}_X(D), W_m\Omega_X^N)$$

in $D(W\mathscr{O}_X - \mathfrak{mod})$.

Proposition 2.3.4 (Cf. [Tan22, Proposition 4.9 and Lemma 2.10])**.** *Let D be an $\mathbb{R}$-divisor on X. Let M be a coherent $W_n\mathscr{O}_X$-module such that the induced map $M(U) \to M_\xi$ is injective for any non-empty open subset $U \subset X$, where M_ξ denotes the stalk of M at the generic point ξ of X. Then the induced $W\mathscr{O}_X$-module homomorphism*

$$\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(D), M) \xrightarrow{\theta} \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(D), M)$$

is an isomorphism.

$W_n\Omega_X^N$ and $gr^n W\Omega_X^N$ are two such $W_n\mathscr{O}_X$-modules.

Theorem 2.3.5 (Tanaka, cf. [Tan22, Theorem 1.1])**.** *Let k be a perfect field of characteristic $p > 0$, and let X be an N-dimensional smooth projective variety over k. If A is an ample $\mathbb{Q}$-divisor on X, then there exists s_0 such that for all $s_0 < s$,*

(i)
- $H^j(X, W_n\mathscr{O}_X(-sA)) = 0$ *for any* $j < N, n \in \mathbb{N}$,
- $H^j(X, W\mathscr{O}_X(-sA)) = 0$ *for any* $j < N$,
- $H^j(X, W\mathscr{O}_X(-A)) = p^t$*-torsion, for some* t*, for any* $j < N$.

(ii)
- $R^i\phi_* \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(-A), W\Omega^N_X)_{\mathbb{Q}} = 0$ *for any* $0 < i$,
- $R^i\phi_*(W_n\mathscr{O}_X(sA) \otimes W_n\Omega^N_X) = 0$ *for any* $0 < i, n \in \mathbb{N}, A$ *integral,*
- $R^i\phi_*(W\mathscr{O}_X(sA) \otimes W\Omega^N_X) = 0$ *for any* $0 < i, A$ *integral,*
- $R^i\phi_*(W\mathscr{O}_X(A) \otimes W\Omega^N_X) = 0$ *for any* $0 < i, A$ *integral.*

Remark 2.3.6. *The theorem appears to suggest a Serre-type duality. This duality would be asymmetric in the sense that torsion from (i) does not appear in (ii). Note that the proof of (i) is simple relative to that of (ii). So ideally duality would recover (ii) from (i), potentially facilitating the proof of the theorem for nef and big D.*

2.4 Ekedahl's dualizing complex

Chapter 3 adapts some of its key ideas from [Eke84], where Ekedahl first proved a duality theorem for the de Rham–Witt complex $W\Omega^\bullet_X$. He begins with the truncated case, and then passes to the limit to establish the non-truncated duality theorem. Key to Ekedahl's truncated argument is

Theorem 2.4.1 ([Eke84, Theorem I.4.1])**.** *Let* X *denote a smooth and proper* N*-dimensional scheme over* $S = Spec\ k$*, with* k *a perfect field of positive characteristic* p*. Then there exists a natural isomorphism*

$$W_n\Omega^N_X[N] \xrightarrow{\sim} f^!_n W_n,$$

where $f^!_n$ *is the shriek functor of* [Har66] *for* $f_n : W_nX \to W_nS$.

This explicit knowledge of $f^!_nW_n$ allows Ekedahl to apply coherent duality

$$Rf_*R\mathcal{H}om_{W_n\mathscr{O}_X}(\mathcal{F}, f^!_n\mathcal{G}) \cong R\mathcal{H}om_{W_n\mathscr{O}_S}(Rf_{n*}\mathcal{F}, \mathcal{G})$$

to the right hand side of his isomorphism (cf. [Eke84, Theorem II.2.2])

$$Rf_{n*}W_n\Omega^i_X \xrightarrow{\sim} Rf_{n*}R\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\Omega^{N-i}_X, W_n\Omega^N_X).$$

We will use Theorem 2.4.1 to apply coherent duality in a similar fashion in Proposition 3.1.1 and Lemma 3.2.6.

3 Duality

Generally, working directly with Witt vectors can be challenging, since the inverse limit which defines them robs them of most of their desirable finiteness properties such as coherence, artinianity, and often noetherianity. Therefore, we usually prefer to work with finite truncations before taking the limit.

For instance, for a $\mathbb{Q}$-divisor D, $W_n\mathscr{O}_X(D)$ is a coherent $W_n\mathscr{O}_X$-module, and $W_nX := (X, W_n\mathscr{O}_X)$ is known to be a noetherian scheme if X is noetherian. None of those things can be said in general about $W_n\mathscr{O}(D)$ or the ringed space $WX := (X, W\mathscr{O}_X)$.

When taking the inverse limit of some inverse system of truncated Witt sheaves, we encounter an obstruction in the form of $R^1\lim$. Therefore, it is often required to show that a given inverse system satisfies the Mittag–Leffler condition (cf. Lemma 3.2.1). This is further complicated by the fact that V and F, which are involved in the formation of the inverse systems, are not $W_n\mathscr{O}_X$-linear. These two problems are the focus of section 3.2.

3.1 The truncated case

Proposition 3.1.1. *Let $\mathscr{F}$ be an invertible $\mathscr{O}_X$-module. For any $n > 0$,*

$$W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n} \cong R\,\mathcal{H}om_{W_n\mathscr{O}_X}(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega_X^N) \tag{3.1.1}$$

such that, in particular,

$$H^i(X, W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) \cong \operatorname{Hom}_{W_n}(H^{N-i}(X, \underline{\mathscr{F}}^\vee_{\leq n}), W_n) \text{ for any } i \geq 0, n > 0. \tag{3.1.2}$$

Proof. We have

$$\begin{aligned} W_n\Omega_X^N \otimes \underline{\mathscr{F}}_{\leq n} &\cong \mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X, W_n\Omega_X^N \otimes \underline{\mathscr{F}}_{\leq n}) \\ &\cong \mathcal{H}om_{W_n\mathscr{O}_X}(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega_X^N), \end{aligned}$$

where the second isomorphism holds because $\underline{\mathscr{F}}_{\leq n}$ is locally free, and so $-\otimes \underline{\mathscr{F}}^\vee_{\leq n}$ is fully faithful. Since $R^i\,\mathcal{H}om(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega_X^N) = 0$ for any $0 < i$ (by local freeness), Equation 3.1.1 holds. To show Equation 3.1.2, take global sections of the derived push-forward.

$$\begin{aligned}\Gamma_S(R\phi_*(W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n})) &\cong \Gamma_S(R\phi_* R\,\mathcal{H}om_{W_n\mathscr{O}_X}(\underline{\mathscr{F}}^\vee_{\leq n}, W_n\Omega_X^N)) \\ &\cong \Gamma_S(R\,\mathcal{H}om_{W_n\mathscr{O}_S\cong W_n}(R\phi_*(\underline{\mathscr{F}}^\vee_{\leq n}), W_n[-N])) \\ &\cong \mathrm{Hom}_{W_n}((R\phi_*\underline{\mathscr{F}}^\vee_{\leq n})[N], W_n),\end{aligned}$$

where W_n is the constant sheaf, the second isomorphism is due to Coherent Duality and [Eke84, I, Theorem 4.1], and the third isomorphism is due to W_n being an injective W_n-module. In particular for all i there are isomorphisms

$$H^i(X, W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) \cong \mathrm{Hom}_{W_n}(H^{N-i}(X, \underline{\mathscr{F}}^\vee_{\leq n}), W_n).$$

□

From section 3.2 onwards we will generally be discussing the right hand Hom expression, since for $\mathbb{Q}$-divisors this is not equivalent to the left hand tensor expression.

3.2 Taking the limit

We now attempt passing to the limit. First, recall the following result that shall allow us to control the higher derived limit via the Mittag–Leffler condition.

Lemma 3.2.1 (Chatzistamatiou, Rülling, cf. [CR11, Lemma 1.5.1])**.** *Let $(X, \mathscr{O}_X)$ be a ringed space and $E = (E_n)$ a projective system of $\mathscr{O}_X$-modules (indexed by integers $1 \leq n$). Let $\mathscr{B}$ be a basis of the topology of X. We consider the following two conditions:*

(i) For all $U \in \mathscr{B}$, $H^i(U, E_n) = 0$ for any $i, 1 \leq n$.

(ii) For all $U \in \mathscr{B}$, the projective system $(H^0(U, E_n))_{n\geq 1}$ satisfies the Mittag–Leffler condition.

Then

- *If E satisfies condition (1), then $R^i \lim_n E_n = 0$ for any $2 \leq i$.*
- *If E satisfies conditions (1) and (2), then $R^i \lim_n E_n = 0$ for any $1 \leq i$, i.e. E is* lim*-acyclic.*

3.2.1 The two derived limits

We will encounter the derived limit in two types of situations. One outside of the derived pushforward — which we here call the *outer* derived limit, and one inside of it — the *inner* derived limit. At first glance, the outer derived limit appears to be easily removed using the following proposition.

Proposition 3.2.2. *Let D be a $\mathbb{Q}$-divisor on X. Then*

$$R^i \lim_n R^j \phi_* \mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(D), W_n\Omega_X^N) = 0 \text{ for any } 0 < i, \text{ any } j \in \mathbb{N}.$$

Proof. Set $E_n := \mathcal{H}om_{W_n\mathscr{O}_S}(R^j\phi_* W_n\mathscr{O}_X(D), W_n)$. Since $S = \operatorname{Spec} k$, we have $H^i(S, E_n) = 0$ for any $0 < i$. By Lemma 3.2.1 then $R^i \lim_n E_n = 0$ for any $1 < i$. W_nX is a proper scheme, so by coherence $H^j(X, W_n\mathscr{O}_X(D))$ and E_n are finite, hence Artinian W_n-modules. But a projective system of Artinian W_n-modules satisfies the Mittag–Leffler condition, and so the first statement holds by Lemma 3.2.1, Coherent Duality and [Eke84, I, Theorem 4.1]. □

And indeed, using it one can derive a form of Tanaka's vanishing (ii) from (i). However, as we will see below, when attempting to establish a more general duality theorem, a derived tensor product arises within the $\mathcal{H}om$ functor. Therefore, we actually need to keep working with derived functors, and use some more complex constructions to accommodate the derived tensor product.

3.2.2 The Cartier–Dieudonné ring

We now attempt to establish a more general duality in the spirit of Ekedahl [Eke84]. A crucial ingredient to Ekedahl's result was the isomorphism in $D(W[d])$:

$$R_n \overset{L}{\underset{R}{\otimes}} R\Gamma_S(W\Omega_X^\bullet) \cong R\Gamma_S(W_n\Omega_X^\bullet).$$

We want to employ an isomorphism of a similar shape to that of Ekedahl. Say

$$W_n \otimes_W^L R\Gamma_X\left(W\mathscr{O}_X(D)\right) \cong R\Gamma_X\left(W_n\mathscr{O}_X(D)\right).$$

To commute the tensor product with the global section functor we are required like Ekedahl to use the derived tensor product. Unfortunately, the above isomorphism does *not* hold. To compute the derived tensor product, we would need to construct a projective resolution of W_n, such as

$$0 \to W \xrightarrow{V} W \xrightarrow{R} W_n \to 0.$$

However, V is not a W-linear map for most base fields. And even when it is (such as for $k = \mathbb{F}_p$), the map induced on $W\mathscr{O}_X(D)$ via the tensor product would be multiplication by p instead of the required Verschiebung. The quest is therefore to find a ring ω that will allow us to construct such a projective resolution which fulfills these requirements.

Define ω to be the Cartier-Dieudonné ring $W_\sigma[F,V]$, that is the (non-commutative) W-algebra generated by V and F, subject to the relations

$$aV = V\sigma(a), Fa = \sigma(a)F \text{ for any } a \in W; VF = FV = p,$$

where σ is the Frobenius map on W, induced from that on k. While as a set ω is equal to $(\bigoplus_i WV^i) \oplus (\bigoplus_j WF^j)$, it is a non-commutative ring with an evident left-W-module structure. It follows from the definition (and the fact that $k^p = k$) that every element of ω can be uniquely described by a sum

$$\sum_{0<i} a_{-i}V^i + \sum_{0\leq j} b_jF^j, a_i, b_j \in W.$$

Let

$$\omega_n := \omega/V^n\omega,$$

which is a (W,ω)-bimodule, since $V^n\omega$ is a sub-left-W-module of ω and a right-ω-ideal generated by V^n. We then have, as sets,

$$\omega_n = \bigoplus_{0<i<n} a_{-i}V^i \oplus \bigoplus_{0\leq j} b_jF^j, a_{-i} \in W_{n-i}, b_j \in W_n.$$

This yields two sets of left-ω-module homomorphisms: a natural restriction map $\omega_n \xrightarrow{\pi} \omega_{n-1}$, as well as an injective map $\omega_{n-1} \xrightarrow{\varrho} \omega_n$, both induced by the respective maps R and $\varrho = \{\text{multiplication by } p\}$ on $W_\bullet$.

3.2.3 ω-modules

Lemma 3.2.3. *Let A be a k-algebra. Then $W(A)$ has a natural structure of left-ω-modules and there is an isomorphism of left-W-modules*

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} W(A) \cong W_n(A).$$

For a sheaf of left-ω-modules M on X,

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma(M) \cong R\Gamma(M_n),$$

where $M_n := M/V^nM \cong \omega_n \underset{\omega}{\otimes} M$.

Proof. The left-ω-module structure on $W(A)$ is given by

$$\omega \times W(A) \xrightarrow{\quad\cdot\quad} W(A)$$

$$(\Sigma_i a_iV^i + \Sigma_j b_jF^j, w) \longmapsto \Sigma_i a_iV^i(w) + \Sigma_j b_jF^j(w).$$

To compute the derived tensor product

$$D(\omega - \mathfrak{lmod}) \xrightarrow{\omega_n \overset{L}{\underset{\omega}{\otimes}} \cdot} D(\mathfrak{ab}),$$

take a projective resolution $P^\bullet$ of ω_n:

$$0 \to \omega \xrightarrow{V^n \cdot} \omega \to \omega_n \to 0.$$

This complex of right-ω-modules, when tensored with $W(A)$, yields a complex $P^\bullet \otimes_\omega W(A)$:

$$0 \to W(A) \xrightarrow{V^n} W(A) \to 0.$$

To see that this represents $W_n(A)$ simply observe that the map induced by $\omega \xrightarrow{V^n \cdot} \omega$ via the tensor product is precisely the n-fold Verschiebungs map on $W(A)$:

$$\begin{array}{ccccccc} W(A) & \xrightarrow{\sim} & \omega \underset{\omega}{\otimes} W(A) & \xrightarrow{V} & \omega \underset{\omega}{\otimes} W(A) & \xrightarrow{\sim} & W(A) \\ a & \longmapsto & 1 \otimes a & \longmapsto & V \otimes a & \longmapsto & V \cdot a = V(a). \end{array}$$

Analogously, the action $F\cdot$ on $W(A)$ induced via the tensor product is the familiar Frobenius map F.

Moreover, since ω_n is a left-W-module, so is $\omega_n \otimes^L_\omega W(A)$. Lastly, to see that the $D(\mathfrak{ab})$-isomorphism is in fact in $D(W - \mathfrak{lmod})$, simply observe that the left-W-module structures on both sides coincide via the isomorphism.

For the second statement, let $M \in (X, \omega)$, that is M is a sheaf of left-ω-modules on X. Let $P^\bullet$ be the projective resolution of ω_n

$$0 \to \omega \xrightarrow{V^n \cdot} \omega \to \omega_n \to 0.$$

Then, since P^i is flat for all i,

$$\omega_n \otimes^L_\omega R\Gamma(M) \cong P^\bullet \underset{\omega}{\otimes} R\Gamma(M) \cong R\Gamma(P^\bullet \underset{\omega}{\otimes} M) \cong R\Gamma(\omega_n \otimes^L_\omega M) \cong R\Gamma(M_n).$$

□

Lemma 3.2.4. *The ring ω has a natural $\mathbb{Z}$-grading given by F and V:*

$$\omega = \left(\bigoplus_{0<i} WV^i \right) \oplus \left(\bigoplus_{0\le j} WF^j \right).$$

Let D be a $\mathbb{Q}$-divisor on X, and write $\omega(D) := \bigoplus_{t\in\mathbb{Z}} W\mathscr{O}_X(p^t D)$. Then $\omega(D)$ is a sheaf of graded left-ω-modules, and

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} \omega(D) \cong \omega_n(D) := \bigoplus_{t\in\mathbb{Z}} W_n\mathscr{O}_X(p^t D).$$

By Lemma 3.2.3 then

$$\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma_X(\omega(D)) \cong R\Gamma_X(\omega_n(D)).$$

Proof. We have the following maps F and V:

$$W\mathscr{O}_X(D) \xrightarrow{F} F_*W\mathscr{O}_X(pD)$$
$$F_*W\mathscr{O}_X(pD) \xrightarrow{V} W\mathscr{O}_X(D).$$

That is by definition, since

$$V^n(W\mathscr{O}_X(p^tD)) \subset W\mathscr{O}_X(p^{t-n}) \text{ and } F^n(W\mathscr{O}_X(p^tD)) \subset W\mathscr{O}_X(p^{t+n}D),$$

$\omega(D)$ is in fact a sheaf of $\mathbb{Z}$-graded left-ω-modules. The last statement follows from Lemma 3.2.3 and the fact that

$$W_n\mathscr{O}_X(D) \cong W\mathscr{O}_X(D)/V^n((F^n)_*W\mathscr{O}_X(p^nD)).$$

□

Proposition 3.2.5. *As left-W-modules, $\omega \cong (\bigoplus_i W) \oplus (\bigoplus_j W)$. Similarly, as left-$W_n$-modules, $\omega_n \cong (\bigoplus_{i<n} W_{n-i}) \oplus (\bigoplus_j W_n)$. It follows that*

$$\mathrm{Hom}_{W_n}(\omega_n, W_n) \cong \left(\bigoplus_{0<i<n} W_{n-i}\right) \oplus \left(\prod_{0\leq j} W_n\right)$$

as W_n-modules. With the left-ω-module structure induced by the right-structure on ω_n, there is an isomorphism

$$\mathrm{Hom}_{W_n}(\omega_n, W_n) \cong \bigoplus_{0<i<n} F^i W_{n-i} \oplus \prod_{0\leq j} F^{-j} W_n$$

of left-ω-modules.

Proof. Since $k = k^p$, elements $a \in \omega$ can be uniquely written as

$$a = \sum_i a_i V^i + \sum_j b_j F^j, a_i, b_j \in W.$$

The natural identification is clearly additive and bijective:

$$\begin{array}{ccc} & W & \\ \swarrow & & \searrow \\ \omega & \xrightarrow{\sim} & (\bigoplus_i W) \oplus (\bigoplus_j W) \end{array}$$

$$\textstyle\sum_i a_i V^i + \sum_j b_j F^j \longmapsto \sum_i a_i + \sum_j b_j.$$

It is W-linear (on the left), since the left-W-module structure of ω is simply multiplication on the left. Analogously, $\omega_n \cong (\bigoplus_{i<n} W_{n-i}) \oplus (\bigoplus_j W_n)$ as left-W-modules. Therefore, in $(W - \mathfrak{mod})$,

$$\begin{aligned}
\mathrm{Hom}_{W_n}(\omega_n, W_n) &\cong \left(\bigoplus_{0<i<n} \mathrm{Hom}_{W_n}(W_{n-i}, W_n) \right) \\
&\oplus \left(\prod_{0 \le j} \mathrm{Hom}_{W_n}(W_n, W_n) \right) \\
&\cong \left(\bigoplus_{0<i<n} W_{n-i} \right) \oplus \left(\prod_{0 \le j} W_n \right) =: \breve{\omega}_n.
\end{aligned} \tag{3.2.1}$$

The right-ω-module structure on ω_n induces a structure of (graded) left-ω-modules on $\mathrm{Hom}_{W_n}(\omega_n, W_n)$. For any $\alpha \in \breve{\omega}_n$,

$$\begin{aligned}
V \cdot (\omega_n \xrightarrow{\alpha} W_n) &= \omega_n \xrightarrow{\alpha \circ (\cdot V)} W_n, \\
F \cdot (\omega_n \xrightarrow{\alpha} W_n) &= \omega_n \xrightarrow{\alpha \circ (\cdot F)} W_n.
\end{aligned}$$

Let

$$\begin{aligned}
\alpha &= \bigoplus_i \alpha_i \in \bigoplus_{0 \le i < n} \mathrm{Hom}_{W_n}(W_{n-i} V^i, W_n) \cong \bigoplus_{0 \le i < n} W_{n-i}, \\
\beta &= \prod_j \beta_j \in \prod_{0 \le j} \mathrm{Hom}_{W_n}(W_n F^j, W_n) \cong \prod_{0 \le j} W_n.
\end{aligned}$$

Under the above isomorphisms, $\alpha_i \in W_{n-i}$ corresponds to the map in $\mathrm{Hom}(W_{n-i}, W_n)$ which takes 1 to that element in W_n which corresponds to α_i under the isomorphism $W_{n-i} \xrightarrow{p^i} \mathrm{im}(p^i) \subset W_n$. That is, it takes 1 to $p^i \alpha_i \in W_n$. Similarly, $\beta_j \in W_n$ corresponds to the map in $\mathrm{Hom}(W_n, W_n)$ which takes 1 to β_j.

Let $\alpha_k = 0$ for any $k \neq i, \beta_l = 0$ for any $l \neq j$. That is α is zero outside of $W_{n-i} V^i \subset \omega_n$, and β is zero outside of $W_n F^j \subset \omega_n$. Then

$$\begin{aligned}
V \cdot \alpha &= p^{-i+1}((V \cdot \alpha_i)(1)) = p^{-i+1}((\alpha_i \circ (\cdot V))(1)) \\
&= p^{-i+1} p^i \alpha_i = p \alpha_i \in W_{n-i+1} \text{ for any } 0 \le i, \\
V \cdot \beta &= (V \cdot \beta_j)(1) = (\beta_j \circ (\cdot V))(1) \\
&= \beta_j(p) = p \beta_j \in W_n \text{ for any } 0 < j; \\
F \cdot \alpha &= p^{-i-1}((F \cdot \alpha_i)(1)) = p^{-i-1}((\alpha_i \circ (\cdot F))(1)) \\
&= p^{-i-1}(\alpha_i(p)) = p^{-i-1}(p p^i \alpha_i) = R(\alpha_i) \in W_{n-i-1} \text{ for any } 0 \le i \\
F \cdot \beta &= (F \cdot \beta_j)(1) = (\beta_j \circ (\cdot F))(1) \\
&= \beta_j(1) = \beta_j \in W_n \text{ for any } 0 < j,
\end{aligned}$$

where R is the natural restriction map $W_k \xrightarrow{R} W_{k-1}$. There is thus an isomorphism of left-ω-modules

$$\begin{aligned}\breve{\omega}_n &\cong \bigoplus_{0<i<n} F^i W_{n-i} \oplus \prod_{0\leq j} F^{-j} W_n \\ &\cong \bigoplus_{0<i<n} F^i W_{n-i} \oplus \prod_{0\leq j} V^j p^{-j} W_n,\end{aligned}$$

where the graded left-ω-module structure on the latter is the natural one, that is multiplication on the left.

□

Note that the injective left-W-linear maps $\omega_{n-1} \xrightarrow{\varrho} \omega_n$ form a direct system. $\mathrm{Hom}_{W_n}(\omega_n, W_n[-N])$ then form an inverse system (cf. [Eke84, III.2.3.*]) with boundary maps π defined by the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{H}om_{W_n}(\omega_n, W_n[-N]) & \xrightarrow{\varrho^*} & \mathcal{H}om_{W_n}(j_{n,*}\omega_{n-1}, W_n[-N]) \\ \downarrow \pi & \nearrow_{\varrho_*} & \\ j_{n,*}\,\mathcal{H}om_{W_{n-1}}(\omega_{n-1}, W_{n-1}[-N]). & & \end{array} \tag{3.2.2}$$

Here $W_{n-1}S \xrightarrow{j_n} W_nS$ is the natural immersion. There exist unique such maps π because, by coherent duality and the fact that $W_{n-1} \cong j_n^! W_n$, ϱ_* is an isomorphism.

3.2.4 The outer derived limit

With ω-modules introduced, we can finally take care of the outer derived limit as planned. This will bring us close to establishing a duality theorem for any $\mathbb{Q}$-divisor.

Lemma 3.2.6. *Let D be a $\mathbb{Q}$-divisor on X. Assume that*

$$R\lim_n R\,\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(p^tD), W_n\Omega_X^N) \cong \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(p^tD), W\Omega_X^N)$$

for all $t \in \mathbb{Z}$ — that is the (inner) higher derived limits on the left hand side vanish. Then

$$R\phi_*\,\mathcal{H}om_{W\mathscr{O}_X}(\omega(D), W\Omega_X^N) \cong R\,\mathrm{Hom}_\omega(R\phi_*\omega(D), \breve{\omega}[-N]),$$

where

$$\breve{\omega} := \prod_{j\in\mathbb{Z}} F^j W$$

Proof. By assumption,

$$\begin{aligned}
&R\phi_* \mathcal{H}om_{W\mathscr{O}_X}\left(\omega(D), W\Omega_X^N\right)\\
\cong &\prod_{t\in\mathbb{Z}} R\phi_* \mathcal{H}om_{W\mathscr{O}_X}\left(W\mathscr{O}_X(p^tD), W\Omega_X^N\right)\\
\cong &\prod_{t\in\mathbb{Z}} R\phi_* R\lim_n R\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(p^tD), W_n\Omega_X^N)\\
\cong &R\lim_n \prod_{t\in\mathbb{Z}} R\phi_* R\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(p^tD), W_n\Omega_X^N) \qquad \text{(e.g. [Stacks, 0BKP])}\\
\cong &R\lim_n R\mathcal{H}om_{W_n\mathscr{O}_S}\left(\bigoplus_{t\in\mathbb{Z}} R\phi_* W_n\mathscr{O}_X(p^tD), W_n[-N]\right) \quad \text{(by [Eke84, Thm. 4.1])}
\end{aligned}$$

By derived tensor-hom adjunction (see for instance [Yek19, Proposition 14.3.18]) we have an isomorphism

$$\begin{aligned}
&R\lim_n R\operatorname{Hom}_{W_n}\left(\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma_X(\omega(D)), W_n[-N]\right)\\
&\cong R\lim_n R\operatorname{Hom}_{\omega}\left(R\Gamma_X(\omega(D)), R\operatorname{Hom}_{W_n}(\omega_n, W_n[-N])\right)
\end{aligned} \tag{3.2.3}$$

in $D(W - \mathfrak{bimod})$.

Let $0 \neq w \in \operatorname{Hom}(W_{n-i}, W_n) \cong W_{n-i}$. Since π is induced by the commutative Diagram 3.2.5, $\pi(w)$ is the unique map such that the following diagram commutes:

$$\begin{array}{ccc}
W_{n-i} & \xrightarrow{w} & W_n\\
\varrho\uparrow & & \varrho\uparrow\\
W_{n-i-1} & \xrightarrow{\pi(w)} & W_{n-1}
\end{array}.$$

This unique map is $R(w)$ (since for $\tau \in W_{n-i-1}$, $\varrho(R(w)(\tau)) = w\varrho(\tau) \in W_n$). The induced maps $W_n \xrightarrow{\pi} W_{n-1}$ are therefore precisely the term-wise restriction maps R. Taking the limit yields an isomorphism of right-W_n-modules

$$\begin{aligned}
\lim_n \operatorname{Hom}_{W_n}(\omega_n, W_n) &\cong \lim_n \left\{\left(\bigoplus_{0<i<n} W_{n-i}\right) \oplus \left(\prod_{0\leq j} W_n\right)\right\}\\
&\cong \prod_{j\in\mathbb{Z}} W,
\end{aligned}$$

whose ω-module structure is given by that on the $\breve{\omega}_n$ (cf. Proposition 3.2.5). That is as sets

$$\lim_n \operatorname{Hom}_{W_n}(\omega_n, W_n) \cong \prod_{j\in\mathbb{Z}} F^{-j}W =: \breve{\omega}.$$

with the obvious structure of graded left-ω-modules. The result then follows from Equation 3.2.3. □

It remains to prove the vanishing of the inner higher derived limit.

3.2.5 Duality for $\mathbb{Z}$-divisors

For integral divisors, projectivity makes vanishing of the (inner) higher derived limit straightforward.

Lemma 3.2.7. *For $\mathscr{F}$ an invertible sheaf of $\mathscr{O}_X$-modules,*

$$W\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}} \cong \lim_n (W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) \cong R\lim_n (W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}). \tag{3.2.4}$$

Proof.

$$\begin{aligned} W\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}} &\cong \lim_n W_n\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}} \\ &\cong \lim_n (W_n\Omega_X^N \underset{W\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}) \\ &\cong \lim_n (W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}). \end{aligned}$$

Take the exact sequence (cf. [Ill79]) of $W_{n+1}\mathscr{O}_X$-modules

$$0 \to \mathrm{gr}^n W\Omega_X^N \to W_{n+1}\Omega_X^N \to W_n\Omega_X^N \to 0,$$

where $\mathrm{gr}^n W\Omega_X^N$ is coherent. Tensoring with $\underline{\mathscr{F}}$ over $W\mathscr{O}_X$ yields an exact sequence

$$\begin{aligned} 0 &\to \mathrm{gr}^n W\Omega_X^N \underset{W_{n+1}\mathscr{O}_X/p}{\otimes} \underline{\mathscr{F}}_{\leq n+1}/p \\ &\to W_{n+1}\Omega_X^N \underset{W_{n+1}\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n+1} \to W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n} \to 0. \end{aligned}$$

For any $x \in X$, let U_x be an affine open neighborhood of x. Then

$$H^1(U_x, \mathrm{gr}^n W\Omega_X^N \underset{W_{n+1}\mathscr{O}_X/p}{\otimes} \underline{\mathscr{F}}_{\leq n+1}/p) = 0$$

by coherence, and therefore

(i) $H^i(U_x, W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n}) = 0$ for any $i > 0$, again by coherence,

(ii) $H^0(U_x, W_{n+1}\Omega_X^N \underset{W_{n+1}\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n+1}) \to H^0(U_x, W_n\Omega_X^N \underset{W_n\mathscr{O}_X}{\otimes} \underline{\mathscr{F}}_{\leq n})$ is surjective for all $n > 0$.

In this fashion a basis $\mathscr{U}$ for the topology of X can be found, such that the above two properties hold for all $U \in \mathscr{U}$, and so by Lemma 3.2.1, Equation 3.2.4 holds. □

It immediately follows the duality theorem for $\mathbb{Z}$-divisors.

Proposition 3.2.8. *Let D be a $\mathbb{Z}$-divisor on X. Then*

$$R\phi_* \mathcal{H}om_{W\mathscr{O}_X}(\omega(D), W\Omega_X^N) \cong R\,\mathrm{Hom}_\omega(R\phi_*\omega(D), \check{\omega}[-N]).$$

Proof. By Lemma 3.2.7 the assumptions of Lemma 3.2.6 are satisfied, and therefore the statement follows. □

3.2.6 Duality for $\mathbb{Q}$-divisors

In light of Proposition 3.2.8, we would like to show that the (inner) derived limit vanishes for $\mathbb{Q}$-divisors as well. We divide this process into two steps. First, we consider only $\mathbb{Q}$-divisors such that p^tD is integral for large enough t. In this case, vanishing of $R^1\lim$ follows from a reduction to $\mathbb{Z}$-divisors via the Frobenius map (cf. Lemma 3.2.9). Then we consider $\mathbb{Z}_{(p)}$-divisors, that is $\mathbb{Q}$-divisors such that ℓD is integral for $\ell \in \mathbb{Z}\backslash p\mathbb{Z}$. Tanaka's results on weakly ℓ-cyclic morphisms (cf. [Tan22]) allow us to show vanishing of $R^1\lim$ in this case. General $\mathbb{Q}$-divisors can then be reduced to $\mathbb{Z}_{(p)}$-divisors by the same Frobenius method of Lemma 3.2.9.

Lemma 3.2.9. *Let D be a $\mathbb{Q}$-divisor on X. Suppose there exists a basis $\mathscr{B}$ for the topology of X such that, for some $t_0 \in \mathbb{N}, t_0 \leq t$ and any $U \in \mathscr{B}$, the map*

$$\Gamma_U(\mathcal{H}om(W\mathscr{O}_X(p^tD), W_{n+1}\Omega^N_X)) \to \Gamma_U(\mathcal{H}om(W\mathscr{O}_X(p^tD), W_n\Omega^N_X))$$

induced by $W_{n+1}\Omega^N_X \xrightarrow{R} W_n\Omega^N_X$ is surjective.

Then

$$\mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(D), W\Omega^N_X) \cong R\lim_n R\,\mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(D), W_n\Omega^N_X).$$

Proof. The statement follows from

$$R^i\lim_n R\,\mathcal{H}om(W_n\mathscr{O}_X(D), W_n\Omega^N_X) = 0 \text{ for any } 1 \leq i.$$

Let $E_n := \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(D), W_n\Omega^N_X)$. By [CR11, Lemma 1.5.1] we need to show that for $\mathscr{B}$ a basis for the topology of X, $U \in \mathscr{B}$, the projective system $((H^0(U, E_n))_{1\leq n}, \psi_{ij})$ satisfies the Mittag–Leffler condition (ML), i.e. that the projective system is *eventually stable*. That is, for any $k \in \mathbb{N}$, $\exists N_k \in \mathbb{N}$ such that $\psi_{kn}(H^0(U, E_n)) = \psi_{kl}(H^0(U, E_l))$, for $N_k \leq n, l$. (Condition (1) of the above lemma is satisfied by coherence.)

For any $x \in X$ fix an affine open subset $U_x \subset X$, and for ease of notation write $\mathcal{H}_j(D) := \Gamma_U\left(\mathcal{H}om(W_j\mathscr{O}(D), W_n\Omega^N_X)\right)$. Note that, thanks to Proposition 2.3.3 and 2.3.4, for $j \leq n$ we know that

$$\mathcal{H}_j(D) \cong \Gamma_U\left(\mathcal{H}om(W_j\mathscr{O}(D), W_j\Omega^N_X)\right) \cong H^0(U, E_j).$$

Suppose now that $t_0 \leq t$. Then by assumption

$$\mathcal{H}_j(p^tD) \xrightarrow{\psi_{ij}} \mathcal{H}_i(p^tD)$$

is surjective for $i \leq j$. For D we have an exact sequence

$$0 \to F_*W_{j-1}\mathscr{O}(pD) \xrightarrow{V} W_j\mathscr{O}(D) \to \mathscr{O}(D) \to 0.$$

Because $H^1(U, \mathcal{H}om(\mathscr{O}_X(D), W_n\Omega_X^N)) \cong H^1(U, E_1) = 0$ due to coherence, this sequence yields a surjective map

$$\mathcal{H}_j(D) \xrightarrow{V^*} \mathcal{H}_{j-1}(pD).$$

Fix k and set $n := k+t+1$. Since $FV = VF = p$, the below diagram is commutative.

$$\begin{array}{ccccccc}
 & & & & & & \mathcal{H}_n(D) \\
 & & & & & & \big\downarrow{\scriptstyle V^*} \\
 & & & & \mathcal{H}_{n-1}(D) & \xleftarrow[F^*]{} & \mathcal{H}_{n-1}(pD) \\
 & & & & \big\downarrow{\scriptstyle V^*} & & \big\downarrow{\scriptstyle V^*} \\
 & & & & \vdots & & \vdots \\
 & & & & \big\downarrow{\scriptstyle V^*} & & \big\downarrow{\scriptstyle V^*} \\
 & & \dots & \xleftarrow[F^*]{} & \mathcal{H}_{k+1}(p^{t-1}D) & \xleftarrow[F^*]{} & \mathcal{H}_{k+1}(p^tD) \\
 & & & & \big\downarrow{\scriptstyle V^*} & \swarrow{\scriptstyle p^*} & \\
\mathcal{H}_k(D) & \xleftarrow[F^*]{} & \dots & \xleftarrow[F^*]{} & \mathcal{H}_k(p^tD) & &
\end{array} \tag{3.2.5}$$

The diagonal maps $F^* \circ V^*$ are exactly the transition maps ψ_{ij} of the projective system $(H^0(U, E_n))_{1 \leq n}$. We see this once we compose R with the isomorphisms of Proposition 2.3.3 and 2.3.4 appropriately. The ψ_{ij} are given via the chain of isomorphisms

$$\begin{aligned}
\mathcal{H}om(W_{n+1}\mathscr{O}_X(D), W_{n+1}\Omega_X^N) &\xrightarrow{R_{n+1}^*} \mathcal{H}om(W\mathscr{O}_X(D), W_{n+1}\Omega_X^N) \\
&\xrightarrow{R\circ} \mathcal{H}om(W\mathscr{O}_X(D), W_n\Omega_X^N) \\
&\xrightarrow{(R_n^*)^{-1}} \mathcal{H}om(W_n\mathscr{O}_X(D), W_n\Omega_X^N) \\
&\xrightarrow{p\circ} \mathcal{H}om(W_n\mathscr{O}_X(D), W_{n+1}\Omega_X^N).
\end{aligned}$$

So ψ_{nn+1} sends f to

$$\begin{aligned}
g &= p \circ R_n^{n+1} \circ f \circ (R_n^{n+1})^{-1} \\
&= p \cdot f \circ (R_n^{n+1})^{-1} \\
&= p \cdot (f \circ (R_n^{n+1})^{-1}) \\
&= f \circ p
\end{aligned}$$

That is $\psi_{nn+1} = p^*$. Here we differentiate between the notations p and $p\cdot$, which stand for either of the maps $W_n\mathscr{O}_X(D) \xrightarrow{p} W_{n+1}\mathscr{O}_X(D)$, $W_n\Omega_X^N \xrightarrow{p} W_{n+1}\Omega_X^N$, or the module operation of multiplication by p, respectively.

Set $N_k := n - 1$. Then for $N_k \leq n, l$, $\psi_{kn}(H^0(U, E_k)) = \psi_{kl}(H^0(U, E_l))$, i.e. ψ is eventually stable, hence ML is satisfied and the result follows. □

Proposition 3.2.10. *Let D be a $\mathbb{Q}$-divisor on X with simple normal crossing support, such that ℓD is a $\mathbb{Z}$-divisor for some $\ell \in \mathbb{Z}\backslash p\mathbb{Z}$. Let $\mathscr{B}$ be an affine basis for the topology of X, and $U \in \mathscr{B}$. Then the map*

$$\Gamma_U(\mathcal{H}om(W\mathscr{O}_X(D), W_{n+1}\Omega_X^N)) \to \Gamma_U(\mathcal{H}om(W\mathscr{O}_X(D), W_n\Omega_X^N))$$

induced by $W_{n+1}\Omega_X^N \xrightarrow{R} W_n\Omega_X^N$ is surjective.

Proof. First suppose that $\ell = 1$. Apply $\mathcal{H}om(W_{n+1}\mathscr{O}_X(D), \cdot)$ to the short exact sequence of $W_{n+1}\mathscr{O}_X$-modules (cf. [Ill79])

$$0 \to \mathrm{gr}^n W\Omega_X^N \to W_{n+1}\Omega_X^N \to W_n\Omega_X^N \to 0,$$

where $\mathrm{gr}^n W\Omega_X^N$ is coherent (cf. [Tan22, 2.6]). Since $W_{n+1}\mathscr{O}_X(D)$ is locally free, the result is another short exact sequence

$$\begin{aligned} 0 &\to \mathcal{H}om_{W_{n+1}\mathscr{O}_X}(W_{n+1}\mathscr{O}_X(D), \mathrm{gr}^n W\Omega_X^N) \\ &\to \mathcal{H}om_{W_{n+1}\mathscr{O}_X}(W_{n+1}\mathscr{O}_X(D), W_{n+1}\Omega_X^N) \\ &\to \mathcal{H}om_{W_n\mathscr{O}_X}(W_n\mathscr{O}_X(D), W_n\Omega_X^N) \to 0 \end{aligned}$$

Applying Γ_U to that sequence yields yet another short exact sequence (thanks to coherence), and therefore the desired surjection.

Now assume $1 \leq \ell$. Let $\mathcal{H}_n(D)$ be as in 3.2.9. We use the proof of [Tan22, Theorem 4.15] to show surjectivity of

$$\mathcal{H}_n(D) \xrightarrow{p^*} \mathcal{H}_{n-1}(D)$$

for $\mathbb{Z}_{(p)}$-divisors, that is $\mathbb{Q}$-divisors D such that ℓD is a $\mathbb{Z}$-divisor for $\ell \in \mathbb{Z}\backslash p\mathbb{Z}$. Then we use Lemma 3.2.9 to reduce the case of arbitrary $\mathbb{Q}$-divisors to that of $\mathbb{Z}_{(p)}$-divisors.

First assume that D is a $\mathbb{Z}_{(p)}$-divisor. According to [Tan22, Proposition 4.14] there exists a finite surjective k-morphism $X'' \xrightarrow{h} X$, such that h^*D is a $\mathbb{Z}$-divisor. Analogue to step 1 of [Tan22, Theorem 4.15], there is a split surjection

$$\mathcal{H}om_{W_n\mathscr{O}_X}\left(h_*W_n\mathscr{O}_{X''}(D_{X''}), W_n\Omega_X^N\right) \to \mathcal{H}om_{W_n\mathscr{O}_X}\left(W_n\mathscr{O}_X(D), W_n\Omega_X^N\right),$$

of $W_n\mathscr{O}_X$-modules (push forward the split surjection on X' along f). Because $D_{X''}$ is a $\mathbb{Z}$-divisor,

$$\mathcal{H}om_{W\mathscr{O}_{X''}}\left(W\mathscr{O}_{X''}(D_{X''}), W_n\Omega_{X''}^N\right) \xrightarrow{R''} \mathcal{H}om_{W\mathscr{O}_{X''}}\left(W\mathscr{O}_{X''}(D_{X''}), W_{n-1}\Omega_{X''}^N\right)$$

is surjective as well. Pushing forward along h, applying coherent duality, [Eke84, Theorem 4.1], and the fact that h is affine and therefore $R^ih_*\mathscr{F} = 0$ for $0 < i$ and quasi-coherent $\mathscr{F}$, we obtain another surjection

$$\mathcal{H}om\left(h_*W_n\mathscr{O}_{X''}(D_{X''}), W_n\Omega_X^N\right) \xrightarrow{h_*R''} \mathcal{H}om\left(h_*W_{n-1}\mathscr{O}_{X''}(D_{X''}), W_{n-1}\Omega_X^N\right).$$

This yields the following commutative diagram, where h_*R'' is still surjective due to coherence.

$$\begin{array}{ccc} \Gamma_U\left(\mathcal{H}om(h_*W_n\mathscr{O}_{X''}(D_{X''}), W_n\Omega^N_X)\right) & \twoheadrightarrow & \Gamma_U\left(\mathcal{H}om(W\mathscr{O}_X(D), W_n\Omega^N_X)\right) \\ \Big\downarrow h_*R'' & & \Big\downarrow R \\ \Gamma_U\left(\mathcal{H}om(h_*W_{n-1}\mathscr{O}_{X''}(D_{X''}), W_{n-1}\Omega^N_X)\right) & \twoheadrightarrow & \Gamma_U\left(\mathcal{H}om(W\mathscr{O}_X(D), W_{n-1}\Omega^N_X)\right), \end{array}$$

Because the diagram is commutative, R is surjective. □

It immediately follows the main duality

Theorem 3.2.11. *Let X be a smooth projective variety of dimension N over a perfect field of positive characteristic p, and let D be a $\mathbb{Q}$-divisor with simple normal crossing support. Then there is an isomorphism*

$$R\phi_*\,\mathcal{H}om_{W\mathscr{O}_X}\left(\omega(D), W\Omega^N_X\right) \cong R\operatorname{Hom}_\omega\left(R\phi_*\omega(D), \breve{\omega}[-N]\right).$$

Proof. Choose $t_0 \in \mathbb{N}$ such tat $p^{t_0}D$ is a $\mathbb{Z}_{(p)}$-divisor. Then, by Proposition 3.2.10, Lemma 3.2.9 applies to $p^{t_0}D$ and hence also to D. The statement then immediately follows from Proposition 3.2.8. □

4 Vanishing

4.1 General vanishing

The following corollary shows that the vanishing of the dual cohomology depends on the V-torsion of the cohomology.

Corollary 4.1.1. *Let X and D be as in Theorem 3.2.11. Suppose that $R^j\Gamma_X(W\mathscr{O}_X(p^tD))$ is V-torsion free for $j \leq N$ and large enough t. Then*

$$R\operatorname{Hom}_\omega\left(R\phi_*\omega(D), \breve{\omega}[-N]\right) \cong \operatorname{Hom}_\omega\left(R\phi_*\omega(D), \breve{\omega}[-N]\right).$$

If furthermore $R^j\phi_*\omega(D) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $j < N$, then

$$R^i\phi_*\left(\mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(D), W\Omega_X^N)\right) = 0$$

for any $0 < i$. In particular, by Theorem 2.3.5(i) this is the case when $-D$ is ample.

Proof. From Proposition 3.2.8, we know that

$$\begin{aligned} R\operatorname{Hom}_\omega(R\phi_*\omega(D), \breve{\omega}[-N]) &\cong R\lim_n R\operatorname{Hom}_\omega(R\Gamma_X(\omega(D)), \breve{\omega}_n[-N]) \\ &\cong \lim_n \operatorname{Hom}_{W_n}(\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma_X(\omega(D)), W_n[-N]). \end{aligned}$$

There is a spectral sequence

$$E_{n,2}^{p,q} := \omega_n \overset{L^p}{\underset{\omega}{\otimes}} R^q\Gamma_X(\omega(D)) \Rightarrow \omega_n \overset{L^{p+q}}{\underset{\omega}{\otimes}} R\Gamma_X(\omega(D)) =: E_n^{p+q},$$

which degenerates at the second sheet, yielding the exact sequence

$$0 \to E_{n,2}^{0,j} \to E_n^j \to E_{n,2}^{-1,j+1} \to 0.$$

Applying $\lim_n \operatorname{Hom}_{W_n}(\cdot, W_n[-N])$ we obtain a left-exact sequence. The right term of this exact sequence vanishes for any j. To see this observe that $E_{n,2}^{-1,j+1}$ is in fact the V^n-torsion of $R^{j+1}\Gamma_X(\omega(D))$. Recall that V^n maps

$$R^k\Gamma_X(W\mathscr{O}_X(p^{t+n}D)) \to R^k\Gamma_X(W\mathscr{O}_X(p^tD)).$$

For large enough $n, j \leq N$, $R^j\Gamma_X(W\mathscr{O}_X(p^nD))$ is torsion-free by assumption, and therefore

$$\mathrm{Hom}_{W_n}(R^j\Gamma_X(W\mathscr{O}_X(p^nD))[V^n], W_n) = 0.$$

Since all but finitely many elements of each inverse system are zero, then,

$$\begin{aligned}\lim_n \mathrm{Hom}_{W_n}(E_{n,2}^{-1,j}, W_n) &\cong \lim_n \mathrm{Hom}_{W_n}(R^j\Gamma_X(\omega(D))[V^n], W_n)\\ &\cong \prod_{t\in\mathbb{Z}} \lim_n \mathrm{Hom}_{W_n}(R^j\Gamma_X(W\mathscr{O}_X(p^{t+n}D))[V^n], W_n)\\ &= 0.\end{aligned}$$

We conclude that $E_{n,2}^{0,j} \cong E_n^j$, and therefore

$$\begin{aligned}R\,\mathrm{Hom}_\omega(R\phi_*\omega(D), \breve{\omega}[-N]) &\cong \lim_n \mathrm{Hom}_{W_n}(\omega_n \overset{L}{\underset{\omega}{\otimes}} R\Gamma_X(\omega(D)), W_n[-N])\\ &\cong \lim_n \mathrm{Hom}_{W_n}(\omega_n \underset{\omega}{\otimes} R\Gamma_X(\omega(D)), W_n[-N])\\ &\cong \lim_n \mathrm{Hom}_\omega(R\Gamma_X(\omega(D)), \breve{\omega}_n[-N])\\ &\cong \mathrm{Hom}_\omega(R\Gamma_X(\omega(D)), \breve{\omega}[-N]).\end{aligned}$$

This proves the first statement.

Now suppose that $R^j\Gamma_X(\omega(D))$ is torsion for $j < N$. Then

$$\mathrm{Hom}_\omega(R^{N-i}\Gamma_X(\omega(D)), \breve{\omega}) = 0 \text{ for any } 0 < i,$$

which concludes the proof. □

Remark 4.1.2. *Corollary 4.1.1 gives some additional insight into the composition of $R\phi_* \mathcal{H}om(W\mathscr{O}_X(D), W\Omega_X^N)$. Of particular interest is the distinction between V-torsion and p-torsion. The proof of vanishing 2.3.5(ii) for ample divisors A in* [Tan22] *relies on Serre vanishing, which guarantees that the cohomology groups $H^{j<N}(X, W\mathscr{O}_X(-p^tA))$ vanish for large enough t. Similarly, Theorem 4.2.5(ii) relies on the vanishing of Lemma 4.2.1. Corollary 4.1.1 however does not require the cohomology of $-p^tA$ to vanish for any t — only to be torsion and to be V-torsion free for large enough t. There are examples which show that this distinction is relevant. For example, for a supersingular abelian surface X, $H^2(X, W\mathscr{O}_X) \cong k[[V]]$ (cf.* [Ill79, 7.1 (b)]*) is V-torsion free p-torsion.*

In fact, by setting $D = 0$, Corollary 4.1.1 allows us to recover the $W\Omega_X^2$ column of the table in [Ill79, 7.1 (b)] *from the $W\mathscr{O}_X$ column.*

4.2 Vanishing on surfaces

Throughout this section we assume X to be of dimension $N = 2$, and D to be a nef and big $\mathbb{Q}$-divisor with simple normal crossing support on X.

Lemma 4.2.1. *There exists $n_0 \in \mathbb{N}$ such that*

$$H^1(X, \mathscr{O}_X(-p^n D)) = 0$$

for all $n_0 \leq n$.

Proof. Let t_1 be large enough such that $p^{t_1} D$ is a $\mathbb{Z}_{(p)}$-divisor, i.e. such that there exists $\ell \in \mathbb{Z} \backslash p\mathbb{Z}$ such that $\ell p^{t_1} D$ is a $\mathbb{Z}$-divisor. Write $D_1 := p^{t_1} D$ for brevity. Then by [Tan22, Theorem 4.14], there exists a finite surjective morphism of smooth surfaces

$$X'' \xrightarrow{h} X$$

such that $h^* D_1$ is a (nef and big) $\mathbb{Z}$-divisor. By [LM81, Proposition 2], there exists $t_0 \in \mathbb{N}$ such that

$$\begin{aligned} H^1(X'', \mathscr{O}_{X''}(-p^t h^* D_1)) = 0 \text{ for any } t_0 \leq t, \text{ or equivalently} \\ H^1(X'', \mathscr{O}_{X''}(-p^t h^* D)) = 0 \text{ for any } t_0 + t_1 \leq t. \end{aligned}$$

By [Tan22, Theorem. 4.15, step 1], the natural map

$$\mathscr{O}_X(-p^t D) \xrightarrow{\alpha} h_* \mathscr{O}_{X''}(-p^t h^* D)$$

splits. (Cf. for example [Tan22, Prop. 4.7 (1); Lem. 4.4 (1)], or just tensor by $\mathscr{O}_X$, to see that Tanaka's statement applies to the truncated case as well — in particular to $n = 1$). Since h is finite, all $R^i h_*$ vanish for $0 < i$, and we have an injection

$$H^1(X, \mathscr{O}_X(-p^t D)) \xrightarrow{\alpha} H^1(X'', \mathscr{O}_{X''}(-p^t h^* D)).$$

Hence

$$H^1(X, \mathscr{O}_X(-p^t D)) = 0 \text{ for any } t_0 + t_1 \leq t.$$

□

Remark 4.2.2. *It was pointed out to the author by Shunsuke Takagi that the above lemma also follows from* [Tan15, Theorem 2.6]. *This is done by choosing $e, l \in \mathbb{N}$ such that $p^e(p^l - 1)D$ is a $\mathbb{Z}$-divisor, and then setting*

$$B_i := p^{e+i} D, N_i := p^{e+i}(p^l - 1)D$$

for each $0 \leq i \leq l - 1$. Applying [Tan15, Theorem 2.6] *to B_i and N_i yields $r_i \in \mathbb{N}$ such that for all $r_i \leq t$,*

$$H^1\left(X, K_X + \lceil B_i \rceil + (p^{l(t-1)} + \cdots + p^l + 1)N_i\right) = 0.$$

Since

$$\lceil B_i \rceil + (p^{l(t-1)} + \cdots + p^l + 1)N_i = \lceil p^{e+lt+i} D \rceil,$$

it follows that after choosing $r := \max_i \{r_i\}$,

$$H^1(X, \mathscr{O}(-p^n D)) \cong H^1(X, K_X + \lceil p^n D \rceil) = 0 \text{ for any } e + lr \leq n.$$

Lemma 4.2.3. *There exists $n_0 \in \mathbb{N}$ such that*

$$H^1(X, W\mathscr{O}_X(-D)) \cong H^1(X, W_n\mathscr{O}(-D))$$

for all $n_0 \leq n$. In particular, $H^1(X, W\mathscr{O}_X(-D))_{\mathbb{Q}} = 0$.

Proof. Let n_0 as in Lemma 4.2.1, and $n_0 \leq t$, and consider the following part of the long exact cohomology sequence:

$$0 = H^1(X, W\mathscr{O}_X(-p^tD)) \xrightarrow{V^t} H^1(X, W\mathscr{O}_X(-D)) \to H^1(X, W_t\mathscr{O}_X(-D))$$

We observe that

$$h^1(X, W\mathscr{O}_X(-D)) \leq h^1(X, W_t\mathscr{O}_X(-D)).$$

Similarly,

$$0 = H^1(X, W_{t-t_0}\mathscr{O}_X(-p^{t_0}D)) \xrightarrow{V^{t_0}} H^1(X, W_t\mathscr{O}_X(-D)) \to H^1(X, W_{t_0}\mathscr{O}_X(-D))$$

is exact. So we have

$$0 \leq h^1(X, W\mathscr{O}_X(-D)) \leq h^1(X, W_t\mathscr{O}_X(-D)) \leq h^1(X, W_{t_0}\mathscr{O}(-D))$$

for all $t_0 \leq t$, and since the latter is finite there exists a t_1 for which $h^1(X, W_t\mathscr{O}_X(-D))$ is smallest. Then

$$h^1(X, W\mathscr{O}_X(-D)) = h^1(X, W_{t_1}\mathscr{O}(-D)).$$

□

Proposition 4.2.4. *For $0 << n$,*

$$ker\Big(H^2(X, W\mathscr{O}_X(-p^nD)) \xrightarrow{V^s} H^2(X, W\mathscr{O}_X(-p^{n-s}D))\Big) = 0$$

for all $0 < s \leq n$.

Proof. Let n_0 be the integer from the above lemma and $n_0 < n$. We have the following exact sequence

$$\begin{aligned} 0 \to H^1(X, W\mathscr{O}_X(-D)) &\xrightarrow{\sim} H^1(X, W_n\mathscr{O}_X(-D)) \\ &\xrightarrow{0} H^2(X, W\mathscr{O}_X(-p^nD)) \xrightarrow{V^n} H^2(X, W\mathscr{O}_X(-D)). \end{aligned}$$

□

From Theorem 3.2.11, Lemma 4.2.3, Proposition 4.2.4, it immediately follows the following Kawamata-Viehweg type vanishing theorem for Witt divisorial sheaves of nef and big divisors.

Theorem 4.2.5. *Let X be a smooth projective surface over a perfect field of positive characteristic p, and D a nef and big $\mathbb{Q}$-divisor with simple normal crossing support on X. Then*

(i) $H^1(X, W\mathscr{O}_X(-D))_{\mathbb{Q}} = 0$,

(ii) $H^1(X, \mathcal{H}om_{W\mathscr{O}_X}(W\mathscr{O}_X(-D), W\Omega^N_X)) = 0$.

□

5 Counterexample

There are examples of surfaces for which Ramanujam vanishing, i.e. the vanishing of $H^1(X, L^{-1})$ for nef and big L, fails (cf. [Muk13]). Yet for Witt divisorial sheaves, the vanishing from section 4.2 holds. Similarly, there are counterexamples of a Kawamata–Viehweg type vanishing for surface pairs (cf. [CT18]). It is then a natural question whether such a Kawamata–Viehweg type vanishing holds for Witt divisorial sheaves. Below we compute an example to see that the answer is no.

In [CT18] Cascini and Tanaka show that a number of rational surfaces, originally constructed by Langer [Lan16], paired with certain boundary divisors, violate Kawamata-Viehweg vanishing. In particular this includes a weak Del Pezzo surface in characteristic 2.

5.1 Construction

Let $q = p^e$ and k be a field containing $\mathbb{F}_q$. Cascini and Tanaka [CT18, Notation 2.1] define the surface S to be the base change to k of the blowup of $\mathbb{P}^2_{\mathbb{F}_p}$ at its $q^2 + q + 1$ $\mathbb{F}_q$-rational points. B and Δ are then defined as follows:

- $B := (q^2 + 1)f^*H - q\Sigma_{i=1}^{q^2+q+1} E_i$,
- $\Delta := \frac{q}{q+1}\Sigma_{i=1}^{q^2+q+1} L'_i$,

where H is the hyperplane divisor on $\mathbb{P}^2_k$, and the L'_i are the proper transforms of the $\mathbb{F}_q$-lines on $\mathbb{P}^2_{\mathbb{F}_q}$, pulled back to S along the base change map. Then (cf. [CT18, Theorem 3.1])

- $E_i^2 = -1$,
- $(L'_i)^2 = -q$,
- (S, Δ) is klt,
- $B - \Delta$ and $-K_S$ are nef and big,
- $h^1(S, \mathscr{O}_S(-B)) \geq \frac{1}{2}(q^2 + q)$.

5.2 Witt cohomology computation

We shall attempt to calculate $H^1(S, W\mathscr{O}_S(-B))$ in the case of $q = p = 2$ by calculating the cohomology of the truncated sheaves $W_n\mathscr{O}_S(-B)$ for all n and their transition maps, to then finally calculate the limit.

Theorem 5.2.1. *Let $q = p = 2$. Then*

$$H^1(S, W\mathscr{O}_S(-B))_{\mathbb{Q}} \neq 0.$$

Proof. Push forward along f the following exact sequence:

$$0 \to F_*\mathscr{O}_S(-p^{n-1}B) \xrightarrow{V^{n-1}} W_n\mathscr{O}_S(-B) \xrightarrow{R} W_{n-1}\mathscr{O}_S(-B) \to 0. \tag{5.2.1}$$

Since $R^2f_*\mathscr{O}(-p^tB) = 0$, for each term we obtain a Leray spectral sequence which degenerates at page three. Thanks to functoriality then we have a three term complex of five term exact sequences. For each term, the five term exact sequence is

$$\begin{aligned} 0 &\to H^1(\mathbb{P}^2, f_*W_k\mathscr{O}_S(-p^lB)) \to H^1(S, W_k\mathscr{O}_S(-p^lB)) \\ &\to H^0(\mathbb{P}^2, R^1f_*W_k\mathscr{O}_S(-p^lB)) \to H^2(\mathbb{P}^2, f_*W_k\mathscr{O}_S(-p^lB)) \\ &\to H^2(S, W_k\mathscr{O}_S(-p^lB)). \end{aligned} \tag{5.2.2}$$

Remark 5.2.2. *It follows from the seven term exact sequence for spectral sequences that the final arrow is in fact a surjection. However, we don't use this fact in the rest of the proof.*

We calculate the first, third, fourth and fifth terms to obtain the second. To calculate the limit we then need to determine the transition maps

$$H^1(S, W_k\mathscr{O}_S(-p^lB)) \xrightarrow{R_n} H^1(S, W_{k-1}\mathscr{O}_S(-p^lB)).$$

First term: We have $f_*W_n\mathscr{O}_S(-B) = W_n\mathscr{O}_{\mathbb{P}^2}(-5)$, and hence

$$h^1(\mathbb{P}^2, f_*W_n\mathscr{O}_S(-p^tB)) = 0 \text{ for any } 0 \leq t.$$

Third term: Since $R^1f_*\mathscr{O}_S(-B)$ is supported on the seven k-points and the E_i have multiplicity 2 in $-B$, it is a skyscraper sheaf with global sections k^7 and zero higher cohomology. Similarly, $H^0(\mathbb{P}^2, R^1f_*\mathscr{O}_S(-p^nB)) = k^{7t_{2p^n}}$ where $t_l = \frac{l(l-1)}{2}$. We chase the derived push forward along f of the exact sequence

$$0 \to \mathscr{O}_S((n-1)E_i) \to \mathscr{O}_S(nE_i) \to \mathscr{O}_{E_i}(-n) \to 0$$

to see this. Pushing forward along f we obtain

$$0 \to R^1f_*\mathscr{O}_S((n-1)E_i) \to R^1f_*\mathscr{O}_S(nE_i) \to R^1f_*\mathscr{O}_{E_i}(-n) \to 0.$$

Away from its corresponding $\mathbb{F}_q$-rational point P_i, $R^1 f_* \mathscr{O}_{E_i}(-n)$ is zero. On any open subset containing P_i, its sections are isomorphic to $H^0(\mathbb{P}^1, \mathscr{O}(-n))$, which is of dimension $n-1$ for $0 < n$. Therefore $R^1 f_* \mathscr{O}_S(nE_i)$ is a skyscraper sheaf supported on P_i, with

$$h^0(\mathbb{P}^2, R^1 f_* \mathscr{O}_S(E_i^n)) = h^0(\mathbb{P}^2, R^1 f_* \mathscr{O}_S(E_i^{n-1})) + n - 1.$$

I.e. its dimension is $\frac{n(n-1)}{2}$. There are seven $\mathbb{F}_q$-rational points P_i, and so seven corresponding E_i in $-B$. The rest of $-B$ does not contribute because it has zero intersection with E_i. Hence the statement

$$h^0(\mathbb{P}^2, R^1 f_* \mathscr{O}_S(-p^n B)) = 7t_{2p^n}$$

follows.

To understand the Witt module structure of the Witt divisorial sheaf cohomology we need to understand how V and F act on $R^1 f_* \mathscr{O}_S(-p^n B)$. First, since

$$f_* W_n \mathscr{O}_S(-p^t B) \xrightarrow{R_n} f_* W_{n-1} \mathscr{O}_S(-p^t B)$$

is naturally surjective,

$$R^1 f_* \mathscr{O}_S(-p^{t+n-1} B) \xrightarrow{V^{n-1}} R^1 f_* W_n \mathscr{O}_S(-p^t B)$$

is injective.

To understand F, take the short exact sequence (cf. [Muk13])

$$0 \to \mathscr{O}_S \xrightarrow{F} F_* \mathscr{O}_S \to \mathscr{B}_S \to 0 \tag{5.2.3}$$

tensor with $\mathscr{O}_S(-p^n B)$, and push forward along f. Outside of the seven k-points of $\mathbb{P}^2_k$, f is an ismorphism, and so the restriction of $f_* \mathscr{B}_S$ to that open subset U is just $(\mathscr{B}_{\mathbb{P}^2} \otimes \mathscr{O}_{\mathbb{P}^2}(-p^n 5))|_U$. Therefore we just need to determine the stalks on the seven k-points. Let $E := E_i$ for any i, and $E_i \xrightarrow{i} S$ the inclusion. Applying i^* to 5.2.3 we obtain

$$\mathscr{O}_E \xrightarrow{F} F_* \mathscr{O}_E \to (\mathscr{B}_S)|_E \to 0.$$

We have $F_* \mathscr{O}_E \cong \mathscr{O}_E \oplus \mathscr{O}_E(-1)$ (cf. [Tho00], for example). Since F is non-zero on global sections, it follows that $(\mathscr{B}_S)|_E \cong \mathscr{O}_E(-1)$.

Because $-B.E = -2$, repeating the entire process after tensoring with $\mathscr{O}_S(-p^n B)$ yields

$$(\mathscr{B}_S \otimes \mathscr{O}_S(-p^n B))|_E \cong \mathscr{O}_E(-1 - 2p^n),$$

which has no global sections. Therefore,

$$R^1 f_* \mathscr{O}_S(-p^n B) \xrightarrow{F} R^1 f_* \mathscr{O}_S(-p^{n+1} B)$$

is injective.

Finally, to compute the limit, we need to determine the transition maps R_n. Since $R^2 f_* \mathscr{O}_S(-p^t B)$ is zero for all t,

$$R^1 f_* W_n \mathscr{O}_S(-p^t B) \xrightarrow{R_n} R^1 f_* W_{n-1} \mathscr{O}_S(-p^t B)$$

is surjective for all t. Therefore, $H^0(\mathbb{P}^2, R^1 f_* W_n \mathscr{O}_S(-B))$ is a direct sum of W_t^T for $1 \le t \le n$, certain T depending on t and n, including $(T, t) = (7, n)$.

Fourth term: Using Serre duality we can compute

$$h^2(\mathbb{P}^2, f_* \mathscr{O}_S(-B)) = h^0(\mathbb{P}^2, \mathscr{O}(2)) = 6,$$
$$h^2(\mathbb{P}^2, f_* \mathscr{O}_S(-p^t B)) = h^0(\mathbb{P}^2, \mathscr{O}(p^t 5 - 3)) = \binom{2 + p^t 5 - 3}{p^t 5 - 3} \text{ for any } 1 \le t.$$

Now, $H^2(\mathbb{P}^2, W_n \mathscr{O}_{\mathbb{P}^2}(-5))$ sits in the following exact sequence of W_n-modules:

$$\cdots \to H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-p^{n-1}5)) \xrightarrow{V^{n-1}} H^2(\mathbb{P}^2, W_n \mathscr{O}_{\mathbb{P}^2}(-5))$$
$$\xrightarrow{R} H^2(\mathbb{P}^2, W_{n-1} \mathscr{O}_{\mathbb{P}^2}(-5)) \to 0.$$

To compute its structure as a W_n-module for $1 < n$ then, we need to know how V and F act on $H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-p^n 5))$ for any n.

Since the H^1 are zero, V is injective on H^2. Since $\mathbb{P}^2$ is F-split (cf. [SZ15, Example 2.2]), F is injective on H^2. It follows that similar to the third term, the fourth term is a direct sum of W_t^T, for certain t, T depending on t and n, including $(T, t) = (6, n)$.

Fifth term: Using Serre duality we can easily compute $H^2(S, \mathscr{O}_S(-B)) = 0$. For $1 < n$, the fifth term then depends on the image of the H^1 term under R, as it is the quotient Q^{n-1} of $H^2(S, \mathscr{O}_S(-p^{n-1}B))$ by the cokernel of R, and so we cannot compute it before knowing the behavior of R on the first cohomology.

Taking the limit: Having computed the relevant terms as far as possible, sequence 5.2.2 becomes a four term exact sequence, and the three term complex of (now four term) exact sequences induced by 5.2.1 yields the commutative diagram of W_n-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(S, \mathscr{O}_S(-p^{n-1}B)) & \longrightarrow & A_1^{n-1} & \longrightarrow & B_1^{n-1} & \longrightarrow & Q_1^{n-1} \\
 & & \downarrow{\scriptstyle V^{n-1}} & & \downarrow{\scriptstyle V^{n-1}} & & \downarrow{\scriptstyle V^{n-1}} & & \downarrow{\scriptstyle V^{n-1}} \\
0 & \longrightarrow & H^1(S, W_n \mathscr{O}_S(-B)) & \longrightarrow & A_n^0 & \xrightarrow{\phi} & B_n^0 & \longrightarrow & Q_n^0 \\
 & & \downarrow{\scriptstyle R_n} & & \downarrow{\scriptstyle R_n} & & \downarrow{\scriptstyle R_n} & & \downarrow{\scriptstyle R_n} \\
0 & \longrightarrow & H^1(S, W_{n-1} \mathscr{O}_S(-B)) & \xrightarrow{\psi} & A_{n-1}^0 & \longrightarrow & B_{n-1}^0 & \longrightarrow & Q_{n-1}^0.
\end{array}$$

Here we denote

$$\begin{aligned}
A_k^l &:= R^1 f_* W_k \mathscr{O}_S(-p^l B),\\
B_k^l &:= H^2(\mathbb{P}^2, W_k \mathscr{O}_S(-p^l B)),\\
Q_k^l &:= H^2(S, W_k \mathscr{O}_S(-p^l B)).
\end{aligned}$$

for brevity.

Remark 5.2.3. *In the case $n = 2$ the bottom row gives another proof that*

$$H^1(S, \mathscr{O}_S(-B)) \neq 0.$$

In fact its dimension is equal to one, because Q_1^0 is zero.

As shown above, the A_k^l and B_k^l are direct sums of W_k^T with T an integer depending on k and l. Both R_n and ϕ are evaluated on each summand individually. Now let e be a non-zero element in $H^1(S, W_{n-1}\mathscr{O}_S(-B))$. We already know that the second and third columns are in fact short exact sequences, with V^{n-1} injective and R_n surjective. Hence ψ maps e into A_{n-1}^0, where it has a non-zero preimage under R_n in A_n^0. In fact, since A_1^{n-1} is the kernel of R_n, $(R_n)^{-1}(\psi(e))$ has dimension

$$\dim(A_1^{n-1}) = 7\left(\frac{(2p^{n-1}-1)2p^{n-1}}{2}\right).$$

On the other hand, the image of A_1^{n-1} in B_n^0 lies in B_1^{n-1}, which has dimension

$$\dim(B_1^{n-1}) = \binom{2 + p^{n-1}5 - 3}{p^{n-1}5 - 3} = \frac{(2 + p^{n-1}5 - 3)!}{2!(p^{n-1}5 - 3)!}.$$

Comparing the two dimensions, we can see that for $0 < n$,

$$\begin{aligned}
&\dim\left(B_1^{n-1}\right) - \dim\left(A_1^{n-1}\right)\\
&= \frac{(2 + p^{n-1}5 - 3)!}{2!\,(p^{n-1}5 - 3)!} - 7\left(\frac{(2p^{n-1}-1)\,2p^{n-1}}{2}\right)\\
&= \frac{(p^{n-1}5 - 1)!}{2!\,(p^{n-1}5 - 3)!} - 7\left(p^{n-1}\left(2p^{n-1} - 1\right)\right)\\
&= \frac{(p^{n-1}5 - 1)\,(p^{n-1}5 - 2)}{2} - 7\left(p^{n-1}\left(2p^{n-1} - 1\right)\right)\\
&= \frac{\left(25 \cdot p^{2(n-1)} - 15 \cdot p^{n-1} + 2\right)}{2} - 7\left(2 \cdot p^{2(n-1)} - p^{n-1}\right)\\
&= \frac{1}{2}\left((25 - 28)p^{2(n-1)} + (14 - 15)\,p^{n-1} + 2\right)\\
&= \frac{1}{2}\left(-3 \cdot p^{2(n-1)} - p^{n-1} + 2\right) < 0.
\end{aligned}$$

Evidently, for $0 < n$,

$$\dim(B_1^{n-1}) < \dim(A_1^{n-1}).$$

That is, the preimage of $\psi(e) \in A^0_{n-1}$, which lies in A^0_n, has non-zero kernel under ϕ. I.e. it, and so e, has a non-zero preimage in $H^1(S, W_n\mathscr{O}_S(-B))$.

Hence the transition maps R_n on $H^1(S, W_n\mathscr{O}_S(-B))$ are surjective, and

$$H^1(S, W\mathscr{O}_S(-B))_{\mathbb{Q}} \neq 0.$$

In fact, it has infinite dimension over $W_{\mathbb{Q}}$. □

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