Moduli stacks of torsion-free sheaves on K3 surfaces and noncommutaive projective Calabi-Yau schemes

K3曲面上のtorsion-free層のモジュライスタックと非可換射影Calabi-Yauスキーム

July, 2024

Yuki MIZUNO
水野　雄貴

Moduli stacks of torsion-free sheaves on K3 surfaces and noncommutaive projective Calabi-Yau schemes

K3曲面上のtorsion-free層のモジュライスタックと非可換射影Calabi-Yauスキーム

July, 2024

Waseda University Graduate School of Fundamental Science and Engineering

Department of Pure and Applied Mathematics, Research on Algebraic Geometry

Yuki MIZUNO
水野　雄貴

Acknowledgments

I would like to thank his advisor Professor Hajime Kaji for continued encouragement and support. He led me into the study of algebraic geometry. I learned a lot from him.

I am also grateful to Professors Yasunari Nagai and Professor Ryo Ohkawa for helpful comments on Chapter 2, which deepened my understanding on K3 surfaces and stacks.

I would like to thank Professor Atsushi Kanazawa, Professor Izuru Mori, Professor Balázs Szendroi, Professor Shinnosuke Okawa and Professor Kenta Ueyama for helpful comments on Chapter 3.

I also thank the members of the algebraic geometry laboratory of Waseda University. A large part of my research is influenced by discussions with them.

Finally, I would like to express my gratitude to my family for their support and encouragement.

Yuki MIZUNO
Department of Pure and Applied Mathematics
Graduate School of Fundamental Science and Engineering
Waseda University
3-4-1 Ohkubo Shinjuku
Tokyo 169-8555
Japan
E-mail: m7d5932a72xxgxo@fuji.waseda.jp

Contents

Chapter 1

Introduction

In this thesis, we study moduli stacks of torsion-free sheaves on K3 surfaces and noncommutative Projective Calabi-Yau schemes. This thesis consists of two main parts.

In Chapter 2, we study moduli stacks of torsion-free sheaves on K3 surfaces. We also give an application to Brill-Noether theory.

When we consider moduli spaces of coherent sheaves, we have a serious problem. The problem is that we cannot construct moduli spaces parametrizing all coherent sheaves. We have two solutions to this problem. One is restricting coherent sheaves we consider to semistable sheaves. Then, we can construct moduli spaces of semistable sheaves by using geometric invariant theory. The other is constructing the moduli spaces as stacks, which are generalizations of schemes. In this case, moduli spaces we obtain are not generally schemes, but they parametrize all coherent sheaves. We often call moduli spaces as stacks moduli stacks.

K3 surfaces are not only Calabi-Yau manifolds but also (holomorphic) symplectic manifolds. Many examples of Calabi-Yau manifolds are known, while known examples of irreducible symplectic manifolds are only 4 types up to deformations. Many of them are constructed by using moduli spaces of semistable sheaves on K3 surfaces. From this point of view, studying moduli spaces of coherent sheaves is important.

It seems that dealing with moduli stacks is more difficult than moduli schemes because stacks are defined as fibered categories or pseudo functors although schemes are defined as locally ringed spaces. Actually, moduli spaces of semistable sheaves are more studied than moduli stacks of coherent sheaves.

On the other hand, different geometric properties of moduli stacks of coherent sheaves from moduli spaces of semistable sheaves are observed and applications to studying schemes are given. Walter studied moduli stacks of torsion-free sheaves of rank 2 on a ruled surface and gave an application to Brill-Noether theory ([56]). In detail, he gave the irreducible decompositions of moduli stacks of torsion-free sheaves of rank 2 and those of Brill-Noether loci of a Hilbert scheme of points on a ruled surface. As related results, various types of stratifications of stacks are studied by Gómez, Sols and Zamora [14] and Hoskins [16] and others.

Let X be a K3 surface over $\mathbb{C}$ of Picard rank $\rho(X) = 1$. Main results of Chapter 2 are giving the irreducible decomposition of the moduli stack $\mathscr{M}^{\mathrm{tf}}(v)$ of torsion-free sheaves on X with Mukai vector v and of Brill-Noether loci of the Hilbert scheme $\mathrm{Hilb}^N(X)$ of N points on X. Since the properties of K3 surfaces are different from those of ruled surfaces, we need different ideas from Walter to get the results and mainly use the theories of sheaves on K3 surfaces by Yoshioka ([23], [24], [60], [63], [62]).

We denote by $\mathscr{M}^{\mathrm{ss}}(v)$ the moduli stack of semistable sheaves with Mukai vector v. Our first result in Chapter 2 is the following.

Theorem 1.0.1. *Let X be a K3 surface of $\rho(X) = 1$ over $\mathbb{C}$, let v_0 be a primitive Mukai vector and, let $v = ([v]_0, [v]_1, [v]_2) := mv_0$ $(m \in \mathbb{Z})$. We assume $[v]_0 = 2$. Then, we have the irreducible decomposition of $\mathscr{M}^{\mathrm{tf}}(v)$ as follows.*

$$\mathscr{M}^{\mathrm{tf}}(v) = \begin{cases} \overline{\mathscr{M}^{\mathrm{ss}}(v)} \cup \bigcup_{(v_1,v_2)\leq 1} \overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} & \text{if } \langle v_0, v_0\rangle \geq -2 \\ \bigcup \overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} & \text{otherwise} \end{cases}$$

,where the stack $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ is defined as

$$\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) := \left\{ E \in \mathscr{M}^{\mathrm{tf}}(v) \;\middle|\; \begin{array}{c} {}^{\exists}(0 \subset E_1 \subset E) : \textit{Harder-Narasimhan filtration} \\ \textit{such that } v(E_1) = v_1, v(E/E_1) = v_2 \end{array} \right\}.$$

Remark 1.0.2. Note that $\mathscr{M}^{\mathrm{ss}}(v) \neq \emptyset$ if and only if $\langle v_0, v_0\rangle \geq -2$ ([63, Corollary 0.3]). And, we can compute the dimensions of $\mathscr{M}^{\mathrm{tf}}(v)$ at each point by using Theorem 2.1.1 and Lemma 2.3.11.

(Classical) Brill-Noether theory is study of special divsors on a curve. To be accurate, a Brill-Noether locus is defined as a special locus of the Picard variety $\mathrm{Pic}(C)$ of a smooth projective curve C, which is well-studied for a long time ([1],[2]). On the other hand, a Brill-Noether locus of a moduli space of sheaves on a smooth projective variety is a special locus defined by cohomologies of coherent sheaves in the space, which is naturally a generalization of a classical Brill-Noether locus on a curve. We think of Hilbert schemes of points as moduli spaces of ideal sheaves of finite schemes and define Brill-Noether loci of Hilbert schemes of points by using the identifications.

In our case, the Brill-Noether locus $W^i_N(nH)$ is defined as

$$W^i_N(nH) = \{[Z] \in \mathrm{Hilb}^N(X) \mid h^1(\mathscr{I}_Z(nH)) > i\},$$

where $i, n, N \in \mathbb{Z}_{\geq 0}$ and H is ample generator of $\mathrm{Pic}(X)$. In particular, A Brill-Noether locus of Hilbert schemes of points parametrizes tuples of points in special positional relationships. Because we have a bijection between the set of irreducible components of $W_N^i(nH)$ of $\mathrm{Hilb}^N(X)$ and a set of irreducible components of $\mathscr{M}^{\mathrm{tf}}(v)$ for some Mukai vector v whose general member satisfies certain conditions, we can classify the irreducible components of $W_N^i(nH)$ of $\mathrm{Hilb}^N(X)$ as an application of the first result if we classify such special irreducible components of $\mathscr{M}^{\mathrm{tf}}(v)$. The second result of Chapter 2 is classifying $W_N^i(nH)$ when $i = 0$.

Theorem 1.0.3. *Let X be a K3 surface of $\rho(X) = 1$ over $\mathbb{C}$, let $v := (2, nH, \frac{n^2}{2}H^2 - N + 2) = mv_0$ (v_0 : primitive Mukai vector, $m \in \mathbb{Z}$) and let nH be an effective divisor on X ($n \in \mathbb{Z}_{\geq 0}, H$: the generator of $\mathrm{Pic(X)}$). We assume $N \leq h^0(\mathscr{O}(nH))$. Then, we classify the irreducible components of*

$$W_N^0(nH) = \{[Z] \in \mathrm{Hilb}^N(X) \mid h^1(\mathscr{I}_Z(nH)) \geq 1\}$$

into one of the following.

(α) : for all (v_1, v_2), if $\langle v_1, v_2\rangle \leq 1$, $[v_1]_1, [v_2]_1 \neq 0$: effective and $-1 < [v_2]_2$, there exists a unique irreducible component of $W_N^0(nH)$ such that, for a general member Z, the torsion-free sheaf E fitting into the extension

$$0 \to \mathscr{O}_X \to E \to \mathscr{I}_Z(nH) \to 0$$

is contained in $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$.

(β) : if $\langle v_0, v_0\rangle \geq -2$ except for the case " $H^2 = 2$ and $v = (2, 3H, 5)$ " , there exists a unique irreducible component of $W_N^0(nH)$ such that for a general member Z, the torsion-free sheaf E fitting into the extension

$$0 \to \mathscr{O}_X \to E \to \mathscr{I}_Z(nH) \to 0$$

is contained in $\mathscr{M}^{ss}(v)$.

Remark 1.0.4. If $N > h^0(\mathscr{O}(nH))$, then $W_N^0(nH) = \mathrm{Hilb}^N(X)$. And by using Theorem 2.1.3, we see not only whether $W_N^0(nH)$ is empty or not but also the dimensions and the number of the irreducible components of $W_N^0(nH)$.

Note that we consider all K3 surfaces of Picard rank 1 although Walter ([56]) only consider a special ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$ in applications of Brill-Noether theory.

In Chapter3, we study noncommutative projective Calabi-Yau schemes.

Two Noetherian schemes S, T are reconstructed by the categories $\mathrm{Coh}(S)$ and $\mathrm{Coh}(T)$ of coherent sheaves on S, T, respectively. This is proved by Gabriel in 1962 ([12]). On the other hand, it is well-known that the category $\mathrm{Coh}(S)$ of coherent sheaves on a projective scheme S over a field k is equivalent to the Serre quotient category $\mathrm{qgr}(R)$ of a the category of finitely R-graded modules by the subcategory of torsion R-modules, where R is a graded k-algebra ([44]). From this point of view, a noncommutative projective scheme $\mathrm{proj}(R)$ associated to a (not nececessarily commutative) graded k-algebra R is defined by the pair $(\mathrm{qgr}(R), \mathcal{R})$, where $\mathcal{R}$ is the object in $\mathrm{qgr}(R)$ naturally given by R.

As mentioned above, to find an essentially new example of holomorphic symplectic manifold is very difficult. On the other hand, a triangulated subcategory of the derived category of a cubic fourfold in $\mathbb{P}^5$, which is obtained by some semiorthogonal decompositions, has the same property as the derived category of coherent sheaves on a projective K3 surface such as the Serre functor exists and is the 2-shift functor [2] ([25]). Moreover, some such categories are not obtained as the derived categories of coherent sheaves of projective K3 surfaces and called noncommutative K3 surfaces.

A noncommutative projective scheme $\mathrm{proj}(R)$ is Calabi-Yau n scheme if $\mathrm{qgr}(R)$ is smooth and the Serre functor of $D^b(\mathrm{qgr}(R))$ is the n-shift functor. It seems that describing a noncommutative K3 surface as the derived category of a noncommutative projective scheme is interesting. However, there are few examples of known noncommutative projective Calabi-Yau schemes. The only example of a non-commutative projective Calabi-Yau scheme known to the author is the one constructed by Kanazawa ([22]).

In Chapter 3, we construct new examples noncommutative projective Calabi-Yau schemes by using noncommutative Segre products and weighted hypersurfaces.

In order to construct noncommutative projective Calabi-Yau schemes as noncommutative analogues of complete intersections in Segre products, we perform a more detailed analysis of noncommutative projective schemes defined by $\mathbb{Z}^2$-graded algebras, which were studied by Van Rompay ([53]). A different approach to noncommutative Segre products is also studied in [15]. The result regarding noncommutative Segre products is the following.

Theorem 1.0.5. *Let* $A := k\langle x_0, \cdots, x_n\rangle/(x_j x_i - q_{ji} x_i x_j)_{i,j}$, $B := k\langle y_0, \cdots, y_m\rangle/(y_j y_i - q'_{ji} y_i y_j)_{i,j}$ *and* $C := A \otimes_k B$, *where* $q_{ji}, q'_{ji} \in k^\times$ *for all* i, j. *We regard* C *as an* $\mathbb{N}^2$*-graded algebra with* $\mathrm{bideg}(x_i) = (1, 0)$ *and* $\mathrm{bideg}(y_i) = (0, 1)$ *for all* i.

1. *Let* $f := \sum_{i=0}^{n} x_i^{n+1}$ *and* $g := \sum_{i=0}^{m} y_i^{m+1}$. *We assume that (i)* $q_{ii} = q_{ij} q_{ji} = q_{ij}^{n+1} = 1$ *for all* i, j, *(ii)* $q'_{ii} = q'_{ij} q'_{ji} = q'^{m+1}_{ij} = 1$ *for all* i, j.

   Then, $\mathrm{proj}(C/(f, g))$ *is a noncommutatative projective Calabi-Yau scheme of dimension* $(n + m - 2)$ *if and only if* $\prod_{i=0}^{n} q_{ij}$ *and* $\prod_{i=0}^{m} q'_{ij}$ *are independent of* j, *respectively.*

2. *Suppose that* $m = n + 1$ *(resp.* $m = n$*) and* $q'_{ij} = 1$ *for all* i, j. *Let* $f := \sum_{i=0}^{n} x_i^{n+1} y_i$ *and* $g := \sum_{i=0}^{n+1} y_i^{n+1}$ *(resp.* $\sum_{i=0}^{n} y_i^{n}$*). We assume that* $q_{ii} = q_{ij} q_{ji} = q_{ij}^{n+1} = 1$ *for all* i, j.

   Then, $\mathrm{proj}(C/(f, g))$ *is a noncommutatative projective Calabi-Yau scheme of dimension* $(2n - 1)$*(resp.* $(2n - 2)$*) if and only if* $\prod_{i=0}^{n} q_{ij}$ *is independent of* j.

In order to construct noncommutative projective Calabi-Yau schemes as noncommutative analogues of weighted hypersurfaces, we consider quotients of weighted quantum polynomial rings. Local structures of noncommutative projective schemes of quotients of weighted quantum polynomial rings are somewhat complicated. This makes studying the smooothness of noncommutative weighte hypersurfaces difficult. We overcome this by using the notion of quasi-veronese algebras introduced by Mori ([33]). The result regarding noncommutative weighted hypersurfaces is the following.

Theorem 1.0.6. *Let* $(d_0, \cdots, d_n) \in \mathbb{Z}_{>0}^{n+1}$ *and* $d := \sum_{i=0}^{n} d_i$ *such that* d *is divisible by* d_i *for all* i*. Let* $C := k\langle x_0, \cdots, x_n\rangle/(x_jx_i - q_{ji}x_ix_j)_{i,j}$*, where* $q_{ji} \in k^{\times}, \deg(x_i) = d_i$ *for all* i, j*. Let* $f := \sum_{i=0}^{n} x_i^{h_i}$*, where* $h_i := d/d_i$*.*

We assume that $q_{ii} = q_{ij}q_{ji} = q_{ij}^{h_i} = q_{ij}^{h_j} = 1$ *for all* i, j*. Then,* $\mathrm{proj}(C/(f))$ *is a noncommutative projective Calabi-Yau scheme of dimension* $(n-1)$ *if and only if there exists* $c \in k$ *such that* $c^{d_j} = \prod_{i=0}^{n} q_{ij}$ *for all* j*.*

Moreover, we prove that some our constructions in Theorem 1.0.6 give essentially new examples of noncommutative projective Calabi-Yau schemes which are not isomorphic to commutative Calabi-Yau manifolds.

Proposition 1.0.7. *There exists a noncommutative projective Calabi-Yau scheme of dimension 2 which is obtained in Theorem 1.0.6 and not isomorphic to either commutative Calabi-Yau surfaces or noncommutative projective Calabi-Yau schemes of dimensions 2 obtained in [22].*

Chapter 2

Classifying the irreducible components of moduli stacks of torsion-free sheaves on K3 surfaces and an application to Brill-Noether theory

2.1 Introduction

Moduli spaces of sheaves is one of the most central areas of algebraic geometry. By considering them, many interesting objects have been found. On K3 surfaces, moduli spaces of sheaves can have symplectic structures, which was first observed by Mukai ([38]). On the other hand, as is well-known, we can construct such moduli spaces by restricting objects to coherent sheaves satisfying stability. However, the moduli spaces do not parametrize unstable sheaves. In this point, stack is important and useful tool to construct moduli spaces which is difficult to construct in the framework of scheme.

Our original motivation of the present paper is studying symplecticity of moduli spaces of sheaves on K3 surfaces. Moreover, in [37] and [63] and others, it was shown that non-emptiness, irreducibility and other properties of moduli schemes depend essentially on Mukai vector. In [24] and [62], properties of the moduli stacks of semistable sheaves on K3 surfaces are studied. Although we can study moduli spaces of unstable sheaves on K3 surfaces by using stack theory, detailed observations are less than studies of moduli schemes.

Various types of stratifications of stacks are studied by Gómez, Sols and Zamora [14] and Hoskins [16] and others. However, it seems that irreducible decomposition of moduli stacks of sheaves is not treated in these papers. In the present article, we first classify the irreducible components of moduli stacks of torsion-free sheaves of rank 2 on K3 surfaces of Picard number $\rho = 1$. Classifying the irreducible components of moduli stacks of torsion-free sheaves on ruled surfaces is discussed in [56]. However, we need new ideas to solve our problem because K3 surfaces have trivial canonical sheaves and may not be fibered surfaces. Important results and methods in this paper are studies of moduli stacks of semistable sheaves and filtered sheaves by Yoshioka ([23], [24], [62], [60]), the classical theory by Shatz ([45]) and generalized Shatz's theory by Nitsure ([39]). By using these theories, we obtain our first result. More precisely, we first take stratification of moduli stacks of torsion-free sheaves by moduli stacks of semistable sheaves and ones of Harder-Narasimhan filtrations. After that, we analyze the strata and describe the irreducible components by using the above theory of Yoshioka.

If $\mathscr{M}^{\mathrm{tf}}(v)$ and $\mathscr{M}^{\mathrm{ss}}(v)$ denote respectively the moduli stacks of torsion-free sheaves and semistable sheaves with Mukai vector v (in detail, see Definition 2.2.1), our first result is the following.

Theorem 2.1.1. *Let X be a K3 surface of $\rho(X) = 1$ over $\mathbb{C}$, let v_0 be a primitive Mukai vector and, let $v = ([v]_0, [v]_1, [v]_2) := mv_0$ $(m \in \mathbb{Z})$. We assume $[v]_0 = 2$. Then, we have the irreducible decomposition of $\mathscr{M}^{\mathrm{tf}}(v)$ as follows.*

$$\mathscr{M}^{\mathrm{tf}}(v) = \begin{cases} \overline{\mathscr{M}^{\mathrm{ss}}(v)} \cup \bigcup_{\langle v_1, v_2\rangle \leq 1} \overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} & \text{if } \langle v_0, v_0\rangle \geq -2 \\ \bigcup \overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} & \text{otherwise} \end{cases}$$

, where the stack $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ is defined as

$$\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) := \left\{ E \in \mathscr{M}^{\mathrm{tf}}(v) \;\middle|\; \begin{array}{c} {}^{\exists}(0 \subset E_1 \subset E) : \text{Harder-Narasimhan filtration} \\ \text{such that } v(E_1) = v_1, v(E/E_1) = v_2 \end{array} \right\}.$$

We call $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ the moduli stack of Harder-Narasimhan filtrations with type (v_1, v_2). (in detail, see Definition 2.3.3)

Remark 2.1.2. Note that $\mathscr{M}^{\mathrm{ss}}(v) \neq \emptyset$ if and only if $\langle v_0, v_0\rangle \geq -2$ ([63, Corollary 0.3]). And, we can compute the dimensions of $\mathscr{M}^{\mathrm{tf}}(v)$ at each point by using Theorem 2.1.1 and Lemma 2.3.11.

The second purpose of this paper is classifying the irreducible components of Brill-Noether loci of Hilbert schemes of points on K3 surfaces by using the first result. Originally, in [56], components of Brill-Noether loci of Hilbert schemes of points on ruled surfaces were classified. In [56], Castelnuovo-Mumford regularity and the Bertini's theorem were mainly used. However, we need more detailed analysis to achieve the application for K3 surfaces. Namely, we focus on the method of the proof of the Bertini theorem ([4]) and more recent results about K3 surfaces ([24], [63]). Our second result is the following.

Theorem 2.1.3. *Let X be a K3 surface of $\rho(X) = 1$ over $\mathbb{C}$, let $v := (2, nH, \frac{n^2}{2}H^2 - N + 2) = mv_0$ (v_0 : primitive Mukai vector, $m \in \mathbb{Z}$) and let*

nH be an effective divisor on X ($n \in \mathbb{Z}_{\geq 0}, H$: the generator of Pic(X)). *We assume $N \leq h^0(\mathscr{O}(nH))$. Then, we classify the irreducible components of*

$$W_N^0(nH) = \{[Z] \in \mathrm{Hilb}^N(X) \mid h^1(\mathscr{I}_Z(nH)) \geq 1\}$$

into one of the following.

(α) : *for all (v_1, v_2) satisfying $v = v_1 + v_2$ and $v_1 > v_2$ (about this notation, see Definiton 2.3.2), if $\langle v_1, v_2 \rangle \leq 1, -1 < [v_2]_2$, and $[v_1]_1, [v_2]_1$ are non zero effective divisors, there exists a unique irreducible component of $W_N^0(nH)$ such that, for a general member Z, the torsion-free sheaf E fitting into the extension*

$$0 \to \mathscr{O}_X \to E \to \mathscr{I}_Z(nH) \to 0$$

is contained in $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$.

(β) : *if $\langle v_0, v_0 \rangle \geq -2$ except for the case " $H^2 = 2$ and $v = (2, 3H, 5)$ ", there exists a unique irreducible component of $W_N^0(nH)$ such that for a general member Z, the torsion-free sheaf E fitting into the extension*

$$0 \to \mathscr{O}_X \to E \to \mathscr{I}_Z(nH) \to 0$$

is contained in $\mathscr{M}^{ss}(v)$.

Remark 2.1.4. If $N > h^0(\mathscr{O}(nH))$, then $W_N^0(nH) = \mathrm{Hilb}^N(X)$. And by using Theorem 2.1.3, we see not only whether $W_N^0(nH)$ is empty or not but also the dimensions and the number of the irreducible components of $W_N^0(nH)$.

Remark 2.1.5. About what happens in the exceptional case " $H^2 = 2$ and $v = (2, 3H, 5)$ " in Theorem 2.1.3, see Claim 2.4.7 and a few paragraphs after that.

2.2 Preliminaries

In this paper, the word *a surface* means a two-dimensional algebraic variety over $\mathbb{C}$. The word *an algebraic stack* means an Artin stack over $\mathbb{C}$. In addition, the word *open (resp. closed, resp. locally closed) substack* means a strictly substack whose inclusion map is an open (resp. closed, resp. locally closed) immersion (in detail, see [26] or [50]).

2.2.1 Mukai vectors

Definition 2.2.1 (Mukai vectors [17])**.** Let X be a K3 surface and let E be a coherent sheaf on X.Then the Mukai vector $v(E)$ of E is (rank(E), $c_1(E)$, $\frac{c_1(E)^2}{2} - c_2(E)$+rank($E$)) $\in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$.

Definition 2.2.2 (Mukai paring [17])**.** Let X be a K3 surface and let $v := ([v]_0, [v]_1, [v]_2)$, $v' := ([v']_0, [v']_1, [v']_2) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$. Then, we define the Mukai pairing of v and v' to be $\langle v, v' \rangle := -[v]_0[v']_2 + [v]_1[v']_1 - [v]_2[v']_0 \in \mathbb{Z}$.

Definition 2.2.3 (([17])). For any $v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, v is primitive if "$v' \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, $m \in \mathbb{Z}$, $v = mv' \Rightarrow m = 1$ or -1"

2.2.2 Moduli stacks

Definition 2.2.4 (Moduli stacks of torsion-free sheaves). Let X be a K3 surface over $\mathbb{C}$, and let $v \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$. we define the moduli stack $\mathscr{M}^{\mathrm{tf}}(v)$ of torsion-free sheaves with Mukai vector v on X to be the following category

1. Objects: $(S,\, E)$, where S : scheme over $\mathbb{C}$, E : quasi-coherent locally of finite presentation sheaves over $X \times_{\mathbb{C}} S (=: Z)$ and flat over S, and E_t : torsion-free sheaf over $Z_t = X_{k(t)}$ such that $v(E) = v$, $(\forall t \in S)$;
2. Morphisms : morphisms from $(S,\, E)$ to $(S',\, E')$ are the pairings $(\varphi : S \to S', \alpha : \varphi^* E' \to E)$ such that α is an isomorphism.

Remark 2.2.5. $\mathscr{M}^{tf}(v)$ is an algebraic stack. And, we can define moduli stacks $\mathscr{M}(v)$ of coherent sheaves with Mukai vector v on X in the same way.

Definition 2.2.6 (Points of algebraic stacks [26], [50]). Let $\mathscr{X}$ be an algebraic stack. Then,

$$|\mathscr{X}| := \coprod_{K/\mathbb{C}:\text{extension of fields}} \mathscr{X}(\mathrm{Spec}(K))/\sim,$$

where if let $E \in \mathscr{X}(\mathrm{Spec}(K))$, let $E' \in \mathscr{X}(\mathrm{Spec}(K'))$ and let K, K' be extensions of $\mathbb{C}$, we write $E \sim E'$ if there exists a extension K'' of K, K' such that $E\,|_{X_{\mathrm{Spec}(K'')}} \simeq E'\,|_{X_{\mathrm{Spec}(K'')}}$.

Definition 2.2.7 (Topological spaces of algebraic stacks [26], [50]). Let $\mathscr{X}$ be an algebraic stack. Then the set $\{U \subseteq |\mathscr{X}| \mid \exists \mathscr{U}\colon$ open substack of $\mathscr{X}$ such that $|\mathscr{U}| = U\}$ satisfies the axiom of open sets of $\mathscr{X}$. We think of $|\mathscr{X}|$ as a topological space by applying the definition.

Definition 2.2.8 (Relative dimensions [26],[50]). Let $P : U \to \mathscr{X}$ be a morphism from a scheme, and we assume $u \in U$ maps to $x \in |\mathscr{X}|$. Then, we define $\dim_u(P)$ as follows. In the commutative diagram

$$\begin{array}{ccc} U \times_{\mathscr{X}} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & \square & \downarrow{\scriptstyle x} \\ U & \xrightarrow{\quad P \quad} & \mathscr{X}, \end{array}$$

$$\dim_u(P) := \dim_x(U \times_{\mathscr{X}} \mathrm{Spec}(k)).$$

Definition 2.2.9 (Dimensions of algebraic stacks at points [26], [50]). Let $\mathscr{X}$ be an algebraic stack, let $x \in \mathscr{X}(\mathrm{Spec}(K))$ where $K/\mathbb{C}$ is an extension and let $P : U \to \mathscr{X}$ be a smooth morphism from a scheme. We assume $u \in U$ maps to $x \in |\mathscr{X}|$. Then

$$\dim_x(\mathscr{X}) := \dim_u(U) - \dim_u(P).$$

Remark 2.2.10. If there is no confusion, we do not distinguish $\mathscr{X}$ with $|\mathscr{X}|$. And, Irreducible decomposition of $\mathscr{X}$ means irreducible decomposition of $|\mathscr{X}|$.

2.2.3 Harder-Narasimhan filtrations and polygons

Theorem 2.2.11 (Harder-Narasimhan(HN) filtration [17])**.** *Let X be a projective surface over $\mathbb{C}$, let H be an ample divisor on X and let E be a torsion-free sheaf on X. Then, for E and H, there exists a unique filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$$

such that E_i/E_{i-1} is semistable (in the sense of Gieseker) with respect to H $(i = 1, \cdots s)$ and

$$p(E_1/E_0) > p(E_2/E_1) > \cdots > p(E_{s-1}/E_{s-2}) > p(E_s/E_{s-1})$$

, where $p(E_i/E_{i-1})$ are the reduced Hilbert polynomials (cf.[17, Def 1.2.3]) of E_i/E_{i-1} and each $p(E_i/E_{i-1}) > p(E_{i+1}/E_i)$ means $p(E_i/E_{i-1})(m) > p(E_{i+1}/E_i)(m)$ for $m \gg 0$. It is called Harder-Narasimhan(HN) filtration of E for stability (in the sense of Gieseker) with respect to H.

In the same way, we have HN filtration of E for μ-stability w.r.t. H.

Definition 2.2.12 (Harder-Narasimhan polygon[39],[45])**.** Let $\mathbb{Q}[\lambda]$ be the polynomial ring in one variable over $\mathbb{Q}$. Let X be a projective surface over $\mathbb{C}$, H be an ample divisor on X and E be a torsion-free sheaf on X. We assume that E has the HN filtration for stability w.r.t. H

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E.$$

Then, we define the HN polygon $\mathrm{HNP}(E)$ of E for stability w.r.t. H to be the subset of $\mathbb{Z} \times \mathbb{Q}[\lambda]$ which is the union of the segments $\overline{x_i x_{i+1}}$ for $0 \leq i \leq s$, where $x_i := (\mathrm{rank}(E_i), p(E_i))$ and each $\overline{x_i x_{i+1}}$ consists of all $(a, f) \in \mathbb{Z} \times \mathbb{Q}[\lambda]$ such that $(a, f) = tx_i + (1-t)x_{i+1}$ for some $t \in \mathbb{Q}$ with $0 \leq t \leq 1$.

We can also define the HN polygon $\mathrm{HNP}^{\mu}(E)$ of E for μ-stability w.r.t. H.

Remark 2.2.13. In this paper, *a HN-filtration* (resp. *polygon*) means a HN-filtration (resp. polygon) for stability (not for μ-stability).

2.3 Irreducible decomposition of $\mathscr{M}^{\mathrm{tf}}(v)$

Notation 2.3.1. In this and next section, X always means a K3 surface of $\rho(X) = 1$ and H means the ample generator of $\mathrm{Pic}(X)$. We denote the open substack of semi stable sheaves and of μ-semi stable sheaves of $\mathscr{M}^{\mathrm{tf}}(v)$ by $\mathscr{M}^{\mathrm{ss}}(v)$ and $\mathscr{M}^{\mu\mathrm{ss}}(v)$. If $\overline{\{p\}} \ni p'$, then we write $p \rightsquigarrow p'$, where p, p' denote points of a topological space and say that p specializes p'. We always assume that $[v]_0 = 2$.

Definition 2.3.2. Let $v_i := (1, d_iH, a_i) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}, (i = 1, 2)$. Then, we say $v_1 > v_2$ if (i) $d_1 > d_2$, or (ii) $d_1 = d_2$ and $a_1 > a_2$.

Definition 2.3.3. Let $v_i := (1, d_iH, a_i) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}, (i = 1, 2)$ such that $v = v_1 + v_2$ and $v_1 > v_2$. We define $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ to be a substack of $\mathscr{M}^{\mathrm{tf}}(v)$ whose objects and morphisms are defined as follows.

Objects: $E \in \mathscr{M}^{\mathrm{tf}}(v)$ such that E's HN-filtration is $0 \subset E_1 \subset E$ with $v(E_1) = (1, d_1H, a_1)$, $v(E/E_1) = (1, d_2H, a_2)$;

Morphisms: $\alpha : E \to E'$: an isomorphism preserving their HN-filtrations.

Remark 2.3.4. In Definition 2.3.3, if the condition "$v_1 > v_2$" does not hold, then $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) = \emptyset$.

Notation 2.3.5. Let v be an element of $\mathbb{Z} \bigoplus \mathrm{Pic}(Z) \bigoplus \mathbb{Z}$. We define

$$Q_X(F, v) := \{F \twoheadrightarrow E \mid E : \text{coherent on } X, v(E) = v\},$$

$$R^{N,m}(v) := \left\{ [\varphi : \mathscr{O}_X(-m)^{\oplus N} \twoheadrightarrow E] \in Q_X(\mathscr{O}_X(-m)^{\oplus N}, v) \;\middle|\; \begin{matrix} H^0(\varphi(m)) : \text{isomorphism} \\ H^i(E(m)) = 0 (i > 0) \end{matrix} \right\},$$

$$R^{N,m}_{\mathrm{tf}} := R^{N,m} \times_{\mathscr{M}(v)} \mathscr{M}^{\mathrm{tf}}(v),$$

$$R^{N,m}_{\mathrm{ss}} := R^{N,m} \times_{\mathscr{M}(v)} \mathscr{M}^{\mathrm{ss}}(v) \simeq R^{N,m}_{\mathrm{tf}} \times^{\mathrm{tf}}_{\mathscr{M}} (v) \mathscr{M}^{\mathrm{ss}}(v),$$

$$R^{N,m}_{(v_1,v_2)} := R^{N,m} \times_{\mathscr{M}(v)} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \simeq R^{N,m} \times_{\mathscr{M}^{\mathrm{tf}}(v)} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v).$$

Remark 2.3.6. $[R^{N,m}_{\mathrm{ss}}/\mathrm{GL}(N)] \to \mathscr{M}^{\mathrm{ss}}(v)$ and $[R^{N,m}_{(v_1,v_2)}/\mathrm{GL}(N)] \to \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ are open immersions because $[R^{N,m}/\mathrm{GL}(N)] \to \mathscr{M}(v)$ is an open immersion ([20, Proposition 9.6]). In addition, we have $\dim[R^{N,m}_{\mathrm{ss}}/\mathrm{GL}(N)] = \dim R^{N,m}_{\mathrm{ss}} - \dim \mathrm{GL}(N)$ and $\dim[R^{N,m}_{(v_1,v_2)}/\mathrm{GL}(N)] = \dim R^{N,m}_{(v_1,v_2)} - \dim \mathrm{GL}(N)$.

2.3.1 Irreducibility of moduli stacks of sheaves and known results

In this subsection, we refer to irreducibility of moduli stacks of HN-filtrations and known results needed to prove the our results.

Lemma 2.3.7 ([23, Theorem 1.2])**.** *Let X be a K3 surface of Picard number 1. If $\langle v, v \rangle > 0$, then $\mathscr{M}^{\mathrm{ss}}(v)$ is an irreducible algebraic stack.* □

Remark 2.3.8. When $\langle v, v \rangle \leq 0$ and $\mathscr{M}^{\mathrm{ss}}(v) \neq \emptyset$, the topological spaces of moduli stacks and moduli schemes are homeomorphic because the stacks are quotient stacks and all semistable sheaves are polystable. Therefore, the moduli stacks are irreducible.

Lemma 2.3.9 ([60, Lemma 2.5])**.** *Let $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$ be the moduli stack of torsion-free sheaves with Mukai vector v whose Harder-Narasimhan type is (v_1, v_2). Then*

1. *the morphism $\mathscr{M}^{HN}_{(v_1,v_2)}(v) \to \mathscr{M}(v)$ is an immersion;*

2. *Let $E \in \mathscr{M}^{HN}_{(v_1,v_2)}(v)$, whose HN-filtration corresponds to $0 \to F_1 \to E \to F_2 \to 0$ and let $\mathscr{M}^{HN}_{(v_1,v_2)}(v) \to \mathscr{M}^{ss}(v_1) \times \mathscr{M}^{ss}(v_2)$ be a morphism which sends $[E] \mapsto ([F_1],[F_2])$. Then, all irreducible components of $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$ are obtained as the pullback of an irreducible component of $\mathscr{M}^{ss}(v_1) \times \mathscr{M}^{ss}(v_2)$.*

Corollary 2.3.10. *$\mathscr{M}^{HN}_{(v_1,v_2)}(v)$ is an irreducible algebraic stack.*

We explain facts about $\dim \mathscr{M}^{\mathrm{ss}}(v)$ and $\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$, which are necessary to prove a proposition later.

Lemma 2.3.11 ([23, Theorem 1.2] [24, Lemma 5.3], [31, Lemma 5.3.2])**.** *Let X be K3 surface of Picard number 1. $v = lv_0$ with v_0 : primitive and $l \in \mathbb{Z}$. Then,*

$$\dim \mathscr{M}^{ss}(v) = \begin{cases} \langle v,v\rangle + 1 & \langle v,v\rangle > 0 \\ \langle v,v\rangle + l & \langle v,v\rangle = 0 \\ \langle v,v\rangle + l^2 & \langle v_0,v_0\rangle = -2 \end{cases}$$

$$\dim \mathscr{M}^{HN}_{(v_1,v_2)}(v) = \langle v_1,v_1\rangle + \langle v_2,v_2\rangle + \langle v_1,v_2\rangle + 2.$$

We also explain facts which are necessary to prove Theorem 2.1.1.

Lemma 2.3.12 ([23, Proposition 1.1])**.** *The dimensions of all irreducible components of $\mathscr{M}(v)$ is more than(or equal to) $\langle v,v\rangle + 1$.*

Lemma 2.3.13 ([11, Lemma 2.21])**.** *Let $\mathscr{X}$ be a pseudo-catenary, jacobson, and locally noetherian algebraic stack. If $|\mathscr{X}|$ is irreducible, then $\dim_x \mathscr{X}$ is constant for all $x \in |\mathscr{X}|$.*

Remark 2.3.14. (i) algebraic stacks which are locally of finite type satisfy the assumption of Lemma 2.3.13.

(ii) by Lemma 2.3.12 and Lemma 2.3.13, we get $\dim_x \mathscr{M}^{tf}(v) \geq \langle v,v\rangle + 1$ ($\forall x \in |\mathscr{M}^{tf}(v)|$).

2.3.2 A criterion of the irreducible components of $\mathscr{M}^{\mathrm{tf}}(v)$

In this section, we classify the irreducible components of $\mathscr{M}^{\mathrm{tf}}(v)$.

Let $v_1, v_2, v_1', v_2' \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$ such that $v = v_1 + v_2 = v_1' + v_2'$, $v_1 > v_2$ and $v_1' > v_2'$. We also suppose $v_1 \neq v_1'$ and $v_2 \neq v_2'$.

Lemma 2.3.15. *if $\dim \mathscr{M}^{HN}_{(v_1,v_2)}(v) \geq \dim \mathscr{M}^{HN}_{(v_1',v_2')}(v)$, We have $\overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} \not\subseteq \overline{\mathscr{M}^{HN}_{(v_1',v_2')}(v)}$.*

Proof. We assume that $\overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)} \subseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v'_1,v'_2)}(v)}$ holds. If let p and p' be the generic points of $\overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ and $\overline{\mathscr{M}^{\mathrm{HN}}_{(v'_1,v'_2)}(v)}$ respectively, there exist $N, m \in \mathbb{Z}_{\geq 0}$ such that the morphism $R^{N,m}_{(v_1,v_2)} \to \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ is dominant, i.e., $R^{N,m}_{(v_1,v_2)} \ni \exists q \mapsto p \in \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$. Note that $R^{N,m}_{(v_1,v_2)}$ is irreducible.

By the fact that $p' \rightsquigarrow p$ and [26] , there exists $q' \in R^{N,m}_{(v'_1,v'_2)}$ such that $R^{N,m}_{(v'_1,v'_2)} \ni q' \mapsto p' \in \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ and $q' \rightsquigarrow q$ in $R^{N,m}_{\mathrm{tf}}$ and we think of q' as the generic point of $R^{N,m}_{(v'_1,v'_2)}$. So, we get $\dim\overline{R^{N,m}_{(v_1,v_2)}} = \dim R^{N,m}_{(v_1,v_2)}$ and $\dim\overline{R^{N,m}_{(v'_1,v'_2)}} = \dim R^{N,m}_{(v'_1,v'_2)}$. In addition, we have $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \neq \mathscr{M}^{\mathrm{HN}}_{(v'_1,v'_2)}(v)$. Therefore, it holds that $\dim R^{N,m}_{(v'_1,v'_2)} > \dim R^{N,m}_{(v_1,v_2)}$. (if $\dim R^{N,m}_{(v'_1,v'_2)} = \dim R^{N,m}_{(v_1,v_2)}$, we have $\overline{R^{N,m}_{(v'_1,v'_2)}} = \overline{R^{N,m}_{(v_1,v_2)}}$, this contradicts to uniqueness of generic points.)

On the other hand, the condition $\dim\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \geq \dim\mathscr{M}^{\mathrm{HN}}_{(v'_1,v'_2)}(v)$ is equivalent to $\dim R^{N,m}_{(v_1,v_2)} \geq \dim R^{N,m}_{(v'_1,v'_2)}$ because of the irreducibility of $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ (cf.Corollary 2.3.10) and Remark 2.3.6. This contradicts to the above inequality $\dim R^{N,m}_{(v'_1,v'_2)} > \dim R^{N,m}_{(v_1,v_2)}$.

□

Lemma 2.3.16. *We have* $\overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} \not\subseteq \overline{\mathscr{M}^{HN}_{(v'_1,v'_2)}(v)}$.

Proof. We assume $\overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)} \subseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v'_1,v'_2)}(v)}$. Let p and p' be the generic points of $\overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ and $\overline{\mathscr{M}^{\mathrm{HN}}_{(v'_1,v'_2)}(v)}$ respectively. Then, we have $p' \rightsquigarrow p$. On the other hand, the map $|\mathscr{M}^{\mathrm{tf}}(v)| \ni p'' \mapsto \mathrm{HNP}(p'')$ is upper semicontinuous by [45] or [39], where $\mathrm{HNP}(p'') := \mathrm{HNP}(E'')$($E''$ is a corresponding object in $\mathscr{M}^{\mathrm{tf}}(v)$ to p''). So, we have $\mathrm{HNP}(p) \geq \mathrm{HNP}(p')$. Let $v_1 := (1, mH, \frac{m^2H^2}{2} - \ell_1 + 1)$ and $v'_1 := \langle 1, m'H, \frac{m'^2H^2}{2} - \ell'_1 + 1\rangle$. Then, this means $m \geq m'$.

If $v_2 := (1, (n-m)H, \frac{(n-m)^2H^2}{2} - \ell_2 + 1)$, then

$$\begin{aligned}
\dim\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) &= \langle v_1, v_1\rangle + \langle v_2, v_2\rangle + \langle v_1, v_2\rangle + 2 \\
&= (2\ell_1 - 2) + (2\ell_2 - 2) \\
&\quad + \left\{ m(n-m)H^2 - \left(\frac{m^2H^2}{2} - \ell_1 + 1\right) - \left(\frac{(n-m)^2H^2}{2} - \ell_2 + 1\right)\right\} \\
&\quad + 2 \\
&= 3(\ell_1 + \ell_2) - 4 + m(n-m)H^2 - \frac{m^2H^2}{2} - \frac{(n-m)^2H^2}{2} \\
&= -2m(n-m)H^2 - \frac{m^2H^2}{2} - \frac{(n-m)^2H^2}{2} + (3c_2 - 4) \\
&= H^2\left(m - \frac{n}{2}\right)^2 + 3c_2 - 4 - \frac{3n^2H^2}{4}.
\end{aligned}$$

Note that $\ell_1 + \ell_2 + m(n-m)H^2 = c_2$ in the above calculation. From this calculation, we have $\dim\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \geq \dim\mathscr{M}^{\mathrm{HN}}_{(v'_1,v'_2)}(v)$ by the above calculation.

This contradicts to Lemma 2.3.15. Therefore, we get $\overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)} \nsubseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1',v_2')}(v)}$. $\square$

Remark 2.3.17.

1. It is shown that $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \supseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ implies $\dim \mathscr{M}^{\mathrm{ss}}(v) > \dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ by the method of the proof of Lemma 2.3.15, Remark 2.3.6 and the irreducibility of $\mathscr{M}^{\mathrm{ss}}(v)$ and $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ (cf. Lemma 2.3.7, Corollary 2.3.10).

2. If $\mathscr{M}^{\mathrm{ss}}(v) \neq \emptyset$, we have $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \nsubseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$. Actually, if $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \subseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$, then we have $\mathrm{HNP}(p_1) \geq \mathrm{HNP}(p_2)$ where p_1 and p_2 are the generic points of $\mathscr{M}^{\mathrm{ss}}(v)$ and $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ respectively. However, this does not occur.

2.3.3 The proof of Theorem 2.1.1

We prove Theorem 2.1.1 by using the lemmas before.

Proof. Any Mukai vector v satisfies one of the following disjoint conditions ;

$$\begin{aligned}
&(a) : \langle v, v\rangle > 0,\\
&(b) : \langle v, v\rangle = 0, -2 \text{ and } v \text{ is primitive },\\
&(c) : \langle v_0, v_0\rangle = 0, -2 \text{ and } v \text{ is non-primitive },\\
&(d) : \langle v, v\rangle < -2 \text{ and } \langle v_0, v_0\rangle \neq -2.
\end{aligned}$$

And, we will prove Theorem 2.1.1 in each case. We first have a stratification of $\mathscr{M}^{\mathrm{tf}}(v)$ by $\mathscr{M}^{\mathrm{ss}}(v)$ and $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$. (About HN stratification, for example, see [39] or [16], Section 5). In the case of (a) and (b), if $\langle v_1, v_2\rangle \leq 1$,

$$\begin{aligned}
\dim \mathscr{M}^{\mathrm{ss}}(v) &= \langle v, v\rangle + 1 = \langle v_1, v_1\rangle + \langle v_2, v_2\rangle + 2\langle v_1, v_2\rangle + 1\\
&= \dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) + \langle v_1, v_2\rangle - 1.
\end{aligned}$$

By Remark 2.3.17, we get $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \nsupseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ and $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \nsubseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$. On the other hand, we consider the case $\langle v_1, v_2\rangle > 1$. We assume $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \nsupseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$. Then, for general $x \in \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$, we have $\dim_x \mathscr{M}^{\mathrm{tf}}(v) < \langle v, v\rangle + 1$ and this contradicts Remark 2.3.14. So we have $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \supseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$. By Lemma 2.3.7, Remark 2.3.8, and Corollary 2.3.10, the stacks $\mathscr{M}^{\mathrm{ss}}(v)$ and $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ are irreducible. Thus by Lemma 2.3.16, we get the irreducible decomposition of $\mathscr{M}^{\mathrm{tf}}(v)$ as the statements of the theorem.

In the case of (c), we can show that for any stack of HN-filtrations, $\langle v_1, v_2\rangle \leq 0$ and $\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \geq \dim \mathscr{M}^{\mathrm{ss}}(v)$. Let $v := (2, nH, \frac{n^2H^2}{2} - c_2 + 2)$. Then, we have

$$\frac{1}{4}\langle v, v\rangle = \langle v_0, v_0\rangle = \frac{-n^2H^2}{4} + c_2 - 2 = 0 \text{ or } -2.$$

And let $v_1 := (1, kH, \frac{k^2H^2}{2} - \ell_1 + 1), v_2 := (1, lH, \frac{l^2H^2}{2} - \ell_2 + 1)$. Then,

$$\begin{aligned}\langle v_1, v_2 \rangle &= klH^2 - \frac{n^2H^2}{2} + c_2 - 2 = klH^2 - \frac{n^2H^2}{4} + (\frac{n^2H^2}{4} + c_2 - 2) \\ &= -\frac{H^2}{4}(n^2 - 4kl) + (\frac{n^2H^2}{4} + c_2 - 2) \\ &= -\frac{H^2}{4}(k-l)^2 + (\frac{n^2H^2}{4} + c_2 - 2) \\ &= -\frac{H^2}{4}(k-l)^2 + \begin{cases} 0 & \langle v_0, v_0 \rangle = 0 \\ -2 & \langle v_0, v_0 \rangle = -2 \end{cases} \leq 0.\end{aligned}$$

So, we have $\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) = \langle v, v \rangle - \langle v_1, v_2 \rangle + 2 \geq \dim \mathscr{M}^{\mathrm{ss}}(v)$. we get $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \not\supseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ and $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \not\subseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ for any pair (v_1, v_2) such that $v = v_1 + v_2$ by Remark 2.3.17. This induces the statement of the theorem.

In the case (d), we have $\mathscr{M}^{\mathrm{ss}}(v) = \emptyset$ by [63, Corollary 0.3]. So, we can classify the irreducible components. □

2.4 An application to Brill-Noether theory of Hilbert schemes of points

In [56], an application of the irreducible components of moduli stacks of torsion-free sheaves on ruled surfaces are performed. In this section, we replace ruled surfaces by K3 surfaces. For a K3 surface X, let N be a non-negative integer and let D be an effective divisor on X such that $h^0(X, \mathscr{O}(D)) \geq N$. And let $\mathrm{Hilb}^N(X)$ be the Hilbert scheme of finite schemes of length N on X. For the Hilbert schemes $\mathrm{Hilb}^N(X)$ of finite schemes of length N on X, We define $W^i_N(D)$ as follows.

$$W^i_N(D) := \{[Z] \in \mathrm{Hilb}^N(X) \mid h^1(\mathscr{I}_Z(D)) \geq i + 1\}.$$

Then, it is known that $W^i_N(D) \subseteq \mathrm{Hilb}^N(X)$ is a closed subscheme from upper semicontinuity of cohomology of flat families of sheaves and $h^1(\mathscr{I}_Z(D)) = i+1$ for general members of each irreducible component of $W^i_N(D)$. In particular, if $i = 0$, we have a bijection between the irreducible components of $W^i_N(D)$ and the irreducible components of $\mathscr{M}^{\mathrm{tf}}(v)$ whose general member E satisfies the conditions $(1) : H^1(X, E) = H^2(X, E) = 0$ and $(2) : \exists s \in H^0(X, E)$ such that $E/s\mathscr{O}_X$ is torsion-free. where, $v := (2, D, \frac{D^2}{2} - N + 2)$. Note that the conditions (1) and (2) are open conditions. Moreover, if E is a general member of an irreducible component $\mathscr{M}'$ of $\mathscr{M}^{\mathrm{tf}}(v)$ which satisfies (1), (2) and let the corresponding irreducible component of $W^i_N(D)$ be V, then

$$\dim V = \dim \mathscr{M}' + h^0(E). \qquad (\spadesuit)$$

Proof of Theorem 2.1.3. We get the claim of Theorem 2.1.3 by the above comment, Lemma 2.4.1, Lemma 2.4.6 and calculating and rearranging $\chi(v) > 0$ and $h^0(\mathscr{O}_X(n-m)) > \ell_2$.

For example, a not semistable component $\overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)} \subset \mathscr{M}^{\mathrm{tf}}(v)$ corresponds to a component of $W^0_N(nH)$ if and only of the following conditions hold :

- $\langle v_1, v_2\rangle \le 1$ (because $\overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ is an irreducible component of $\mathscr{M}^{\mathrm{tf}}(v)$),
- $2m \ge n > m > 0 \Leftrightarrow [v_1]_1, [v_2]_1$: non zero effective divisors and $v_1 > v_2$,
- $\chi(v) > 0$ (this always holds by the assumption $N \le h^0(\mathscr{O}(n))$ and the Riemann-Roch formula),
- $h^0(\mathscr{O}_X(n-m)) > \ell_2 \Leftrightarrow -1 < [v_2]_2$,

where $v_1 := (1, mH, \frac{m^2H^2}{2} - \ell_1 + 1)$ and $v_2 := (1, (n-m)H, \frac{(n-m)^2H^2}{2} - \ell_2 + 1)$. Thus, we have ($\alpha$) of Theorem 2.1.3. In the same way, we have (β) of Theorem 2.1.3. □

2.4.1 About not semistable components

Lemma 2.4.1. *Let $v := (2, nH, \frac{n^2H^2}{2} - N + 2)$, let $v_1 := (1, mH, \frac{m^2H^2}{2} - \ell_1 + 1)$, and let $v_2 := (1, (n-m)H, \frac{(n-m)^2H^2}{2} - \ell_2 + 1)$ such that $v = v_1 + v_2$ and $v_1 > v_2$. We assume that E is a general member of $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$. Then, E satisfies the conditions (1), (2) if and only if the following conditions hold. (a) : $2m \ge n > m > 0$, (b) : $\chi(E) > 0$, (c) : $h^0(\mathscr{O}_X(n-m)) > \ell_2$.*

Proof.

(1), (2) ⇒ (a), (b), (c) If the conditions (1), (2) are satisfied, it is clear that a general E satisfies (b). Let

$$0 \to \mathscr{I}_{Z_1}(m) \to E \to \mathscr{I}_{Z_2}(n-m) \to 0$$

be the exact sequence corresponding to the HN-filtration of E, where $v(\mathscr{I}_{Z_1}(m)) = v_1, v(\mathscr{I}_{Z_2}(n-m)) = v_2$. Then, we have (a) because $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ and $h^2(\mathscr{O}(m)) = h^2(\mathscr{O}(n-m)) = 0$. We also have $h^0(\mathscr{O}(n-m)) > \ell_2$ because $h^1(\mathscr{I}_{Z_2}(n-m)) = 0$. Note that the condition $h^0(\mathscr{O}(n-m)) = \ell_2$ does not occur. If $h^0(\mathscr{O}(n-m)) = \ell_2$, then any global section s of a general sheaf $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ is included in $H^0(\mathscr{I}_{Z_1}(m))$. Because all non-zero sections in $H^0(\mathscr{I}_{Z_1}(m))$ never induce torsion-free quotients, so $E/s\mathscr{O}_X$ includes a torsion sheaf $\mathscr{I}_{Z_1}(m)/s\mathscr{O}_X$. This contradicts to the condition (2). So, we have the condition (c).

(a), (b), (c) ⇒ (1),(2) Conversely, we assume that the conditions (a), (b) and (c) are satisfied. First, we prove the condition (1) by induction for ℓ_1 (cf. [56, Lem 3.3 and Lem 4.5]). Note that $H^2(E) = 0$ for a general $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ because $H^2(\mathscr{I}_{Z_1}(m)) = H^2(\mathscr{I}_{Z_2}(n-m)) = 0$. If $\ell_1 = 0$, then we have $H^1(E) = 0$ in the same way. For general $\ell_1 > 0$, we prove $H^1(E) = 0$ for a general $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$. We assume E' fits in the exact sequence

$$0 \to \mathscr{I}_{Z_1'}(m) \to E' \to \mathscr{I}_{Z_2'}(n-m) \to 0$$

which is the HN filtration of E' with $\ell(Z_1') = \ell_1 - 1$ and $\ell(Z_2') = \ell_2$. If $H^1(E') = 0$ and E' satisfies the conditions (a), (b) and (c), then, E' have a nonzero global section s. And, for a general point $x \in X$ and a general one dimensional quotient $E' \twoheadrightarrow E' \otimes k(x) \twoheadrightarrow k(x)$ of the fiber of E' at x denoted by φ, we have $\varphi(s) \neq 0$. Note that we can assume $x \notin Z_1'$. Let E be the kernel of φ. Then, we have $h^0(E) = h^0(E') - 1$ and $H^1(E) = 0$. And, we get the HN filtration of E

$$0 \to \mathscr{I}_{Z_1' \cup \{x\}}(m) \to E \to \mathscr{I}_{Z_2'}(n-m) \to 0$$

because of the HN-filtration of E' and the assumption $x \notin Z_1'$. So, we get condition (1) for general $\ell_1 > 0$.

Next, we prove the condition (2) under the condition (1). We consider the conditions $(\alpha) : 2m = n, (\beta) : \ell_2 = 1$. And, we divide our proof into two cases: (i). (α) or (β) is not true, (ii). both (α) and (β) are true.

Case (i) It is enough to prove the following claim.

Claim 2.4.2. *Let k be a positive integer. We assume that $\ell_1 = 0$. Then, we have*

$$h^0(E(-k)) + \dim|kH| < h^0(E).$$

We show Claim 2.4.2 induces the condition (2) before proving it. If the claim is true, then a general $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ with $\ell_1 = 0$ is a vector bundle because the Cayley-Bacharach property (cf. [17, Thm 5.1.1]) holds for a pair $(Z_2, \mathscr{O}_X(n-2m))$ by the choice of m and ℓ_2, where Z_2 is a general set of ℓ_2 points. And, the set $H^0(E) \setminus \bigcup_{C \in |kH|, k \in \mathbb{N}} H^0(E(-C))$ is a non-empty open set from Claim 2.4.2. So, a general section s of a general $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ with $\ell_1 = 0$ defines a torsion-free quotient $E/s\mathscr{O}_X$ because the zero set $Z(s)$ of s is a finite set (cf. [40, Ch. 1, §5]).

In the case $\ell_1 > 0$, we have a vector bundle E' fitting into the sequence

$$0 \to \mathscr{O}_X(m) \to E' \to \mathscr{I}_{Z_2}(n-m) \to 0 \quad (\ell(Z_2) = \ell_2)$$

whose general section s determines a torsion-free quotient because of the case $l_1 = 0$.

In addition, E' is generically globally generated. Note that we say that E' is generically generated if the evaluation map ev : $H^0(E') \otimes \mathscr{O}_X \to E'$

is surjective on an open set of X. Actually, from the condition (a) and (c), $\mathscr{O}_X(m)$ and $\mathscr{I}_{Z_2}(n-m)$ is generically globally generated. So, a simple diagram chase shows that E' is generically global generated.

Let U be the subset of $H^0(E')$ of the sections defining torsion-free quotients. Then, a natural $\mathbb{C}$-linear homomorphism $\widetilde{\mathrm{ev}} : H^0(E') \to H^0(E' \otimes k(x))$ obtained from ev above is surjective and $\widetilde{\mathrm{ev}}|_U$ is dominant for general $x \in X$ because E' is generically globally generated. So, we can take general ℓ_1 points $x_1, \cdots, x_{\ell_1}$ on X and a general section s such that $s \notin \mathscr{O}_X(m) \otimes k(x_i)$ for all i and s defines a torsion-free quotient. Then, we can take one-dimensional quotients $\varphi_i : E' \twoheadrightarrow E' \otimes k(x_i) \twoheadrightarrow k(x_i)(1 \le i \le \ell_1)$ such that $\varphi_i|_{\mathscr{O}_X(m)} \neq 0$ for all i and $\varphi_i(s) = 0$ for all i. We consider the quotient $\varphi : E' \twoheadrightarrow \bigoplus_{i=1}^{\ell_1} k(x_i)$ obtained from φ_i. If let E be the kernel of φ, then $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ and a general section s of E defines a torsion-free quotient.

Proof of Claim 2.4.2. We prove the claim by cases. In this paper, we only consider the case

$$2m > n > m > 0, \qquad \ell(Z_2) > h^0(\mathscr{O}(n-m-1) = \frac{(n-m-1)^2}{2}H^2 + 2,$$
$$m \ge 3, \qquad n - m \ge 3.$$

The other cases can be proved in the same way or more easily.

Because $h^0(\mathscr{O}(n-m-1) < \ell(Z_2) < h^0(\mathscr{O}(n-m))$, we have $H^0(\mathscr{I}_{Z_2}(n-m-k)) = 0$ for all positive integer k and general Z_2. So, we have $H^0(E(-k)) = H^0(\mathscr{O}(m-k))$.

In this condition, we have $H^0(E(-k)) = \chi(\mathscr{O}(m-k)) = \frac{(m-k)^2}{2}H^2 + 2$ and $\dim|kH| = h^0(\mathscr{O}(kH)) - 1 = \frac{k^2}{2}H^2 + 1$. Note that $h^0(E) = \chi(E) = \frac{m^2}{2}H^2 + \frac{(n-m)^2}{2}H^2 + 4 - \ell(Z_2)$. Then, we can calculate as follows.

$$\begin{aligned} h^0(E) - \{h^0(E(-k)) + \dim|kH|\} &= \frac{m^2}{2}H^2 + \frac{(n-m)^2}{2}H^2 \\ &\quad - \frac{(m-k)^2}{2}H^2 - \frac{k^2}{2}H^2 + 1 - \ell(Z_2). \end{aligned}$$

In addition, we have $\ell(Z_2) < h^0(\mathscr{O}(n-m)) = \frac{(n-m)^2}{2}H^2 + 2$. So,

$$h^0(E) - \{h^0(E(-k)) + \dim|kH|\} > H^2k(m-k) - 1 > 0(\because k, m-k > 0).$$

□

Case (ii) Next, we suppose (α) and (β) are true. In this case, note that $\ell_1 = 0$ because $\chi(v_1) > \chi(v_2)$ and every sheaf $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ is isomorphic to $\mathscr{O}_X(m) \oplus \mathscr{I}_x(m)$ for some $x \in X$. We take sections $s_1, s_2 \in H^0(\mathscr{O}(m))$ such that $Z(s_1) \cap Z(s_2)$ is a finite set, where $Z(s_i)$ is the zero set of $s_i (i = 1, 2)$. If $x \in Z(s_2)$, $s_1 \oplus s_2 \in H^0(\mathscr{O}_X(m) \oplus \mathscr{I}_x(m))$. Since $s_1 \oplus s_2 \in \mathscr{O}_X(m) \oplus \mathscr{O}_X(m)$ defines a torsion-free quotient $\mathscr{O}_X(m)\oplus\mathscr{O}_X(m)/(s_1\oplus s_2)\mathscr{O}_X$, $\mathscr{O}_X(m)\oplus\mathscr{I}_x(m)/(s_1\oplus s_2)\mathscr{O}_X$ is also torsion-free. □

2.4.2 About semistable components

We will use the following lemmas to prove the Lemma 2.4.6.

Lemma 2.4.3 ([63] Lemma 1.4 or [61] Lemma 2.1)**.** *Let n be an odd integer. If the exact sequence*

$$0 \to \mathscr{O}\left(\frac{n-1}{2}\right) \to E \to \mathscr{I}_Z\left(\frac{n+1}{2}\right) \to 0$$

does not split, then E is a μ-stable sheaf, where $\mathscr{I}_Z$ is the ideal sheaf of a finite subscheme Z.

Lemma 2.4.4 ([63] Proposition 0.5 and Section 3.3)**.** *Let $v := (2, nH, \frac{n^2H^2}{2} - N + 2)$. We assume that v is primitive and " $v \neq (2, nH, \frac{n^2H^2}{4} - 1)$ and n is even". Then, there exists a stable vector bundle with Mukai vector v.*

Remark 2.4.5. In the Lemma 2.4.4, if n is odd, then any stable sheaf is μ-stable . However, if n is even, a stable sheaf is not necessarily μ-stable.

Lemma 2.4.6. *Let E be a general member of the stack $\mathscr{M}^{ss}(v)$. Then, the conditions (1) and (2) are equal to the conditions $\chi(E) > 0$ and "$H^2 \neq 2$ or $v \neq (2, 3H, 5)$".*

Proof. If (1) and (2) satisfy, we have $H^1(E) = 0$ and $H^0(E) \neq 0$. Therefore, we have $\chi(E) > 0$.

We will prove Lemma 2.4.6 only when n is an odd integer. We can also prove this lemma in the same way when n is even. Note, for a general E, we have $H^2(E) = 0$ by semistability.

We assume $\chi(v) > 0$. Note we do not assume the latter of the conditions (The case " $H^2 = 2$ and $v = (2, 3H, 5)$" is excluded in Claim 2.4.7 below.).

When $N > \frac{n^2+1}{4}H^2 + 3$ with odd n First, we assume that $N > \frac{n^2+1}{4}H^2 + 3$. This is equivalent to the condition that the closure of the stacks of Harder-Narasimhan filtrations whose general sheaf is an extension

$$0 \to \mathscr{I}_{Z_1}\left(\frac{n+1}{2}\right) \to E \to \mathscr{I}_{Z_2}\left(\frac{n-1}{2}\right) \to 0$$

is contained in the closure of $\mathscr{M}^{ss}(v)$. Then, we can show that some E in the closure of $\mathscr{M}^{ss}(v)$ have no higher cohomology in the same way as in Lemma 2.4.1. Moreover, we can prove a general E have a global section which give a torsion-free quotient.

When $\frac{n^2+1}{4}H^2+3 \geq N$ with odd n Next we assume that $\frac{n^2+1}{4}H^2+3 \geq N$. From Lemma 2.4.4 and Remark 2.4.5, there exists a μ-stable vector bundle E with Mukai vector v. We next consider $E(-\frac{n-1}{2})$. Let $E' := E(-\frac{n-1}{2})$ and $v' := v(E')$. Then, E' fits into the following exact sequence

$$0 \to \mathscr{O}_X \to E' \to \mathscr{I}_Z(1) \to 0 \qquad (\clubsuit)$$

, where Z is a finite subscheme of X. Indeed, we have $\hom(E'^{\vee}, \mathscr{O}_X) = \operatorname{ext}^2(\mathscr{O}_X, E'^{\vee}) = h^2(E'^{\vee}) = h^0(E') \neq 0$ and $\hom(E', \mathscr{O}_X) = \operatorname{ext}^2(\mathscr{O}_X, E') = h^2(E') = 0$ because $\chi(E') > 0$ and E' is also μ-stable. So, we have the above exact sequence by using these. Let $v' := v(E') = (2, H, \frac{H^2}{2} - \ell(Z) + 2)$. Because any non-split extension of $\mathscr{I}_Z(1)$ by $\mathscr{O}_X$ is a μ-stable sheaf from Lemma 2.4.3, the unique irreducible component of $W^0_{\ell(Z)}(H)$ corresponds to $\mathscr{M}^{\mathrm{ss}}(v')$. This means that a general sheaf in $\mathscr{M}^{\mathrm{ss}}(v')$ satisfies the conditions (1), (2) and any sheaf in irreducible components of $\mathscr{M}^{\mathrm{tf}}(v')$ whose general member is not semistable does not satisfy them.

Then, we see the following claim holds.

Claim 2.4.7. *$h^1(\mathscr{I}_{Z'}(2)) = 0$ for a general $Z' \in W^0_{\ell(Z)}(H)$ except for the case "$H^2 = \ell(Z) = 2$ ". In the case " $H^2 = \ell(Z) = 2$ ", $W^0_{\ell(Z)}(H) = W^0_{\ell(Z)}(2H)$ and $h^1(\mathscr{I}_{Z'}(3)) = 0$ for a general $Z' \in W^0_{\ell(Z)}(H)$.*

If the claim holds, this induces condition (1) except for the case "$H^2 = 2, v = (2, 3H, 5)$ " and condition (1) never hold in this exceptional case. Before proving the claim, we show this.

First, note that "$n = 3, H^2 = \ell(Z) = 2$" $\Rightarrow v = (2, 3H, 5)$ and a general $F \in \mathscr{M}^{\mathrm{ss}}(v)$ fits into an exact sequence

$$0 \to \mathscr{O}_X(1) \to F \to \mathscr{I}_{Z'}(2) \to 0$$

, where $Z' \in W^0_2(H)$. Here, $W^0_2(H) = W^0_2(2H)$ from the claim and we have $h^1(\mathscr{I}_{Z'}(2)) \neq 0$ for any $Z' \in W^0_2(H)$. Thus, $h^1(F) \neq 0$ from the long exact sequence of cohomology obtained from the above. This shows the condition (1) never hold in the case " $H^2 = 2, v = (2, 3H, 5)$. On the other hand, except for this case, a general $F \in \mathscr{M}^{\mathrm{ss}}(v)$ fits into the following exact sequence

$$0 \to \mathscr{O}_X\left(\frac{n-1}{2}\right) \to F \to \mathscr{I}_{Z'}\left(\frac{n+1}{2}\right) \to 0$$

, where $Z' \in W^0_{\ell(Z)}(H)$. Here, $h^1(\mathscr{I}_{Z'}(2)) = 0$ for a general $Z' \in W^0_{\ell(Z)}(H)$ and $h^1(\mathscr{I}_{Z'}(k)) \geq h^1(\mathscr{I}_{Z'}(k+1))$ for all $k > 0$. This induces the condition (1) holds in a general $F \in \mathscr{M}^{\mathrm{ss}}(v)$.

Proof of Claim 2.4.7. First, note that we have $\frac{H^2}{2} + 3 \geq \ell(Z)$ because $\chi(E') > 0$. If $\frac{H^2}{2} + 3 = \ell(Z)$, then $\mathrm{Hilb}^{\ell(Z)}(X) = W^0_{\ell(Z)}(H)$. So we have $h^1(\mathscr{I}_{Z'}(2)) = 0$ for a general $Z' \in \mathrm{Hilb}^{\ell(Z)}(X)$ because $h^0(\mathscr{O}_X(2)) \geq \ell(Z)$.

If $\frac{H^2}{2}+2 \geq \ell(Z)$, then $\mathrm{Hilb}^{\ell(Z)}(X) \neq W^0_{\ell(Z)}(H)$. We only consider the case $\frac{H^2}{2}+2 \geq \ell(Z)$ in the following. Let $v'' := (2, 2H, 2H^2 - \ell(Z) + 2)$. We divide the rest of the proof into 4 steps.

1. We have $\mathscr{M}^{\mathrm{ss}}(v'') = \emptyset$ unless "$H^2 = \ell(Z) = 4$" or "$H^2 = \ell(Z) = 2$" by [63, Cor 0.3].

   Moreover, there is not an irreducible component of $\mathscr{M}^{\mathrm{tf}}(v'')$ whose general member is a HN-filtration satisfying the conditions (a), (b) and (c) of Lemma 2.4.1 unless "$H^2 = 2$ and $\ell(Z) = 3$". This is because a general sheaf F in such a component fits into

   $$0 \to \mathscr{I}_{Y_1}(H) \to F \to \mathscr{I}_{Y_2}(H) \to 0 \quad (Y_1, Y_2: \text{ finite schemes})$$

   and $v'' = v(F) = v(\mathscr{I}_{Y_1}(H)) + v(\mathscr{I}_{Y_2}(H))$.

   So, we have $W^0_{\ell(Z)}(2H) = \emptyset$ except for the three cases. Thus, $h^1(\mathscr{I}_{Z'}(2)) = 0$ for a general $Z' \in W^0_{\ell(Z)}(H)$ except for the three cases.

2. When "$H^2 = \ell(Z) = 4$", $W^0_{\ell(Z)}(2H)$ may not be empty. If $W^0_{\ell(Z)}(2H)$ is not empty, the unique irreducible component corresponds to $\mathscr{M}^{\mathrm{ss}}(v'')$. Moreover, we can calculate the dimensions of $W^0_{\ell(Z)}(H)$ and $W^0_{\ell(Z)}(2H)$ by using Lemma 2.3.11 and the formula ♠ and get $\dim W^0_{\ell(Z)}(H) = 7$ and $\dim W^0_{\ell(Z)}(2H) = 4$. This means that $W^0_{\ell(Z)}(H) \supsetneq W^0_{\ell(Z)}(2H)$ and we have the claim.

3. When "$H^2 = \ell(Z) = 2$", $W^0_{\ell(Z)}(2H) \neq \emptyset$ because the unique point of $\mathscr{M}^{\mathrm{ss}}(v'')$ is $\mathscr{O}_X(H)^{\oplus 2}$ (cf. Remark 2.4.8). We also have $\dim W^0_{\ell(Z)}(H) = \dim W^0_{\ell(Z)}(2H) = 2$. So, we have $W^0_{\ell(Z)}(H) = W^0_{\ell(Z)}(2H)$. However, $W^0_{\ell(Z)}(3H) = \emptyset$ as above.

4. In the same way as in Step 2, we get the claim when "$H^2 = 2$ and $\ell(Z) = 3$".

Therefore, we get $h^1(\mathscr{I}_{Z'}(2)) = 0$ for a general $Z' \in W^0_{\ell(Z)}(H)$ when $H^2 \neq 2$ or $\ell(Z) \neq 2$, $h^1(\mathscr{I}_{Z'}(2)) \neq 0$ for any $Z' \in W^0_{\ell(Z)}(H)$ when $H^2 = \ell(Z) = 2$ and $h^1(\mathscr{I}_{Z'}(3)) = 0$ for a general $Z' \in W^0_{\ell(Z)}(H)$ when $H^2 = \ell(Z) = 2$.

□

Remark 2.4.8. We will explain how we calculate $\dim W^0_{\ell(Z)}(H)$ and $\dim W^0_{\ell(Z)}(2H)$ when " $H^2 = \ell(Z) = 4$" here. Similarly, we can also do when "$H^2 = \ell(Z) = 2$" and "$H^2 = 2, \ell(Z) = 3$".

It is sufficient to calculate $\dim \mathscr{M}^{\mathrm{ss}}(v')$, $\dim \mathscr{M}^{\mathrm{ss}}(v'')$, $h^0(E')$ and $h^0(E'')$ from the formula ♠, where E'' is a general member of $\mathscr{M}^{\mathrm{ss}}(v'')$. We can calculate $\dim \mathscr{M}^{\mathrm{ss}}(v')$ and $\dim \mathscr{M}^{\mathrm{ss}}(v'')$ by using Lemma 2.3.11. We also do $h^0(E')$ by using the exact sequence ♣. Moreover, we can obtain $h^0(E'')$ by the fact that the unique member of $\mathscr{M}^{\mathrm{ss}}(v'')$ is $\mathscr{O}_X(H)^{\oplus 2}$ (in detail, see [37], [21]).

Next,we prove the condition (2). It is enough to prove $h^0(E(-k)) + \dim|kH| < h^0(E)$ as in the same way of the proof of Lemma 2.4.1 because a general sheaf in $\mathscr{M}^{\mathrm{ss}}(v)$ is a vector bundle by Lemma 2.4.4. Note that we have the following exact sequence for a general E,

$$0 \to \mathscr{O}\left(\frac{n-1}{2}\right) \to E \to \mathscr{I}_Z\left(\frac{n+1}{2}\right) \to 0$$

, where Z is a finite subscheme of X and $h^1(\mathscr{I}_Z\left(\frac{n+1}{2}\right)) = 0$. So, for $\frac{n-1}{2} \geq k > 0$,

$$\begin{aligned}
&h^0(E) - \{h^0(E(-k)) + \dim|kH|\} \\
&\geq (h^0(\mathscr{I}_Z(\frac{n+1}{2})) + \chi(\mathscr{O}(\frac{n-1}{2})) \\
&- (h^0(\mathscr{I}_Z(\frac{n+1}{2} - k)) + \chi(\mathscr{O}(\frac{n-1}{2} - k)) + \dim|kH|) \\
&= kH^2(\frac{n-1}{2} - k) - 1 + h^0(\mathscr{I}_Z(\frac{n+1}{2})) - h^0(\mathscr{I}_Z(\frac{n+1}{2} - k)) > 0.
\end{aligned}$$

(In the case of $k = \frac{n-1}{2}$, we use $h^0(\mathscr{I}_Z(\frac{n+1}{2})) = \frac{(n+1)^2}{8}H^2 - \ell(Z) + 2$ and $h^0(\mathscr{I}_Z(1)) = \frac{1}{2}H^2 - \ell(Z) + 3$)

Remark 2.4.9. (In the case n is even) When n is even, we can prove that a general sheaf $E \in \mathscr{M}^{\mathrm{ss}}(v)$ have a section defining a torsion-free quotient as in the same way as in the proof above except $v = (2, nH, \frac{n^2H^2}{2})$ or $(2, nH, \frac{n^2H^2}{4} - 1)$. In these case, any sheaf of $\mathscr{M}^{\mathrm{ss}}(v)$ is not vector bundle and the closure of $\mathscr{M}^{\mathrm{ss}}(v)$ dose not contain any stack of HN-filtrations. However, we can prove the condition (1), (2) in the same way of the proof of Lemma 2.4.1. In the former case, note that any semistable sheaf is isomorphic to a sheaf of the form $\mathscr{I}_x\left(\frac{n}{2}\right) \oplus \mathscr{I}_y\left(\frac{n}{2}\right) (x, y \in X)$. In the latter case, note that a general quotient $\mathscr{O}\left(\frac{n}{2}\right) \to \oplus_{i=1}^{3} k(x_i)(x_i \in X)$ and any non split extension $0 \to \mathscr{I}_{\{y_1,y_2\}}\left(\frac{n}{2}\right) \to E \to \mathscr{I}_{y_3}\left(\frac{n}{2}\right) \to 0 (y_j \in X, j = 1, 2, 3)$ is a semistable sheaf with the Mukai vector $v = (2, nH, \frac{n^2H^2}{4} - 1)$ when n is even (cf. [63, Prop 3.4]).

□

Chapter 3

Some examples of noncommutative projective Calabi-Yau schemes

3.1 Introduction

Calabi-Yau varieties are rich objects and play an important role in mathematics and physics. In noncommutative algebraic geometry, (skew) Calabi-Yau algebras are often treated as noncommutative analogues of Calabi-Yau varieties. Calabi-Yau algebras have a deep relationship with quiver algebras ([13], [51]). For example, many known Calabi-Yau algebras are constructed by using quiver algebras. They are also used to characterize Artin-Schelter regular algebras ([43], [42]). In particular, a connected graded algebra A over a field k is Artin-Schelter regular if and only if A is skew Calabi-Yau.

On the other hand, a triangulated subcategory of the derived category of a cubic fourfold in $\mathbb{P}^5$, which is obtained by some semiorthogonal decompositions, has the 2-shift functor [2] as the Serre functor. Moreover, the structure of Hochschild (co)homology is the same as that of a projective K3 surface ([25]). However, some such categories are not obtained as the derived categories of coherent sheaves of projective K3 surfaces and called noncommutative K3 surfaces.

Artin and Zhang constructed a framework of noncommutative projective schemes in [3], which are defined from noncommutative graded algebras. In this framework, we can think of Artin-Schelter regular algebras as noncommutative analogues of projective spaces, which are called quantum projective spaces. Our objective is to produce examples of noncommutative projective Calabi-Yau schemes that are not obtained from commutative Calabi-Yau varieties. In the future, it would be an interesting question to compare the derived category of a noncommutative projective Calabi-Yau scheme created in the framework of Artin-Zhang's noncommutative projective schemes with a noncommutative K3 surface obtained as a triangulated subcategory of the derived category of a cubic fourfold.

As the definition of noncommutative projective Calabi-Yau schemes, we adopt the definition introduced by Kanazawa ([22]). His definition is a direct generalization of the definition of commutative Calabi-Yau varieties to noncommutative projective schemes. He also constructed the first examples of noncommutative projective Calabi-Yau schemes that are not isomorphic to commutative Calabi-Yau varieties as hypersurfaces of quantum projective spaces. Recently, some examples constructed by Kanazawa play an important role in noncommutative Donaldson-Thomas theory ([27], [28]).

In this chapter, we construct new examples of noncommutative projective Calabi-Yau schemes by using noncommutative Segre products and weighted hypersurfaces. There are many known examples of Calabi-Yau varieties in algebraic geometry. Some of them are complete intersections in products of projective spaces. Moreover, Reid gave a list of Calabi-Yau surfaces, which are hypersurfaces in weighted projective spaces ([19, Table 1 in Section 13.3], [41, Theorem 4.5]). Motivated by these two facts, we construct noncommutative analogues of the two types of examples of Calabi-Yau varieties (Theorem 3.3.3, Theorem 3.3.15) in Section 3.3.

In order to prove that a noncommutative projective scheme is Calabi-Yau, we use the methods of Kanazawa. However, they are not sufficient because the algebras we treat are more complicated than the ones he considered. In order to construct noncommutative projective Calabi-Yau schemes as noncommutative analogues of complete intersections in Segre products, we perform a more detailed analysis of noncommutative projective schemes defined by $\mathbb{Z}^2$-graded algebras, which were studied by Van Rompay ([53]). A different approach to noncommutative Segre products is also studied in [15]. In order to construct noncommutative projective Calabi-Yau schemes as noncommutative analogues of weighted hypersurfaces, we consider quotients of weighted quantum polynomial rings. In commutative algebraic geometry, the category $\mathrm{Coh}(\mathrm{Proj}(k[x_0, \cdots, x_n]))$ of coherent sheaves on the projective spectrum $\mathrm{Proj}(k[x_0, \cdots, x_n])$ of a weighted polynomial ring is not necessarily equivalent to $\mathrm{qgr}(k[x_0, \cdots, x_n])$, where $\mathrm{qgr}(k[x_0, \cdots, x_n])$ is the quotient category associated to $k[x_0, \cdots, x_n]$ constructed in [3]. In fact, $\mathrm{qgr}(k[x_0, \cdots, x_n])$ is equivalent to the category of coherent sheaves on a weighted projective space constructed as a Deligne-Mumford stack. Moreover, $\mathrm{qgr}(k[x_0, \cdots, x_n])$ is thought of as a nonsingular model of $\mathrm{Proj}(k[x_0, \cdots, x_n])$ (see [46, Example 4.9]). We use this idea to construct new noncommutative projective Calabi-Yau schemes. In addition, it should be noted that local structures of noncommutative projective schemes of quotients of weighted quantum polynomial rings are somewhat complicated. An analysis of the local structures was performed by Smith ([46]). We show that the local structure obtained in [46] is described by the notion of quasi-Veronese algebras introduced by Mori ([32]).

In Section 3.4, we compare our constructions from weighted hypersurfaces in Section 3.3 with commutative Calabi-Yau varieties and the first examples constructed in [22], focusing on noncommutative projective Calabi-Yau schemes of dimensions 2. We show that some of our constructions in Section 3.3 are not isomorphic to any of the commutative Calabi-Yau varieties and the first examples constructed in [22] (Proposition 3.4.9). When we consider moduli spaces of point modules of noncommutative projective schemes obtained from weighted hypersurfaces in Section 3.3, there is a problem, which is that in general weighted quantum polynomial rings are not generated in degree 1. So, the notion of point modules is not necessarily useful in this case. In this paper, we use theories of closed points studied in [33], [47] and [48], etc. A different approach to closed points of weighted quantum polynomial rings is studied in [49]. The notion of point modules defined in [49] corresponds to those of ordinary and thin points in [33]. To show that some of our constructions are not isomorphic to the examples obtained in [22], we use Morita theory of noncommutative schemes, which is established in [8] (see also [3, Section 6]). In the theory, we need to calculate the centers of noncommutative rings. By using these calculations, we can do a detailed analysis and some classifications of noncommutative projective Calabi-Yau surfaces.

3.2 Preliminaries

Notation and Terminology 3.2.1. In this chapter, k denotes an algebraically closed field of characteristic 0. We suppose $\mathbb{N}$ contains 0. Let A be a k-algebra, M be an A-bimodule and ψ, ϕ be algebra automorphisms of A. Then, we denote the associated A-bimodule by ${}^{\psi}M^{\phi}$, i.e. ${}^{\psi}M^{\phi} = M$ as k-modules and the new bimodule structure is given by $a * m * b := \psi(a)m\phi(b)$ for all $a, b \in A$ and all $m \in M$. Let $\mathcal{C}$ be a k-linear abelian category. We denote the global dimension of $\mathcal{C}$ by gl.dim($\mathcal{C}$). An $\mathbb{N}$-graded k-algebra A is connected if $A_0 = k$.

For any $\mathbb{N}$-graded k-algebra $A = \bigoplus_{i=0}^{\infty} A_i$, we denote the category of graded right A-modules (resp. finitely generated graded right A-modules) by $\mathrm{Gr}(A)$ (resp. $\mathrm{gr}(A)$). Let $M \in \mathrm{Gr}(A)$ and A° be the opposite algebra of A. We define the Matlis dual $M^* \in \mathrm{Gr}(A^{\circ})$ by $M_i^* := \mathrm{Hom}_k(M_{-i}, k)$ and the shift $M(n) \in \mathrm{Gr}(A)$ by $M(n)_i := M_{i+n}$ $(i, n \in \mathbb{Z})$. For $M, N \in \mathrm{Gr}(A)$, we write $\mathrm{Hom}_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr(A)}}(M, N(n)) \in \mathrm{Gr}(A)$. For $M \in \mathrm{Gr}(A)$ and a homogeneous element $m \in M$, we denote the degree of m by $\deg(m)$. We define the truncation $M_{\geq n} := \bigoplus_{i \geq n} M_i \in \mathrm{Gr}(A)$ $(n \in \mathbb{Z})$. An element $m \in M$ is called torsion if $mA_{\geq n} = 0$ for $n \gg 0$. We say M is a torsion module if any element of M is torsion. We denote the subcategory of torsion modules in $\mathrm{Gr}(A)$ (resp. $\mathrm{gr}(A)$) by $\mathrm{Tor}(A)$ (resp. $\mathrm{tor}(A)$).

3.2.1 Noncommutaive schemes

Definition 3.2.2 ([3, Section 2])**.** Let A be a right noetherian $\mathbb{N}$-graded k-algebra. We define the quotient categories $\mathrm{QGr(A)} := \mathrm{Gr}(A)/\mathrm{Tor}(A)$ and qgr(A) := gr(A)/tor(A). We denote the projection functor by π and its right adjoint functor by ω. The general (resp. noetherian) projective scheme of A is defined as $\mathrm{Proj}(A) := (\mathrm{QGr}(A), \pi(A))$ (resp. $\mathrm{proj}(A) := (\mathrm{qgr}(A), \pi(A))$).

Definition 3.2.3 ([3, Section 2], [47, Chapter 3]). A quasi-scheme over k is a pair $(\mathcal{C}, \mathcal{O})$ where $\mathcal{C}$ is a k-linear abelian category and $\mathcal{O}$ is an object in $\mathcal{C}$. A morphism from a quasi-scheme $(\mathcal{C}, \mathcal{O})$ to another quasi-scheme $(\mathcal{C}', \mathcal{O}')$ is a pair (F, φ) consisting of a k-linear right exact functor $F : \mathcal{C} \to \mathcal{C}'$ and an isomorphism $\varphi : F(\mathcal{O}) \xrightarrow{\simeq} \mathcal{O}'$. We call (F, φ) is an isomorphism if F is an equivalence.

When A is as in Definition 3.2.2, we think of proj(A) = $(\mathrm{qgr}(A), \pi(A))$ as a quasi-scheme. For any (commutative) noetherian scheme X, $(\mathrm{Coh}(X), \mathcal{O}_X)$ is also a quasi-scheme. From this observation, we regard X as a quasi-scheme.

3.2.2 Dualizing complexes

Definition 3.2.4 ([52, Section 4], [58, Section4]). Let A, B be $\mathbb{N}$-graded k-algebras and m_A be $A_{\geq 1}$. We define the torsion functor $\Gamma_{m_A} : \mathrm{Gr}(A \otimes_k B^\circ) \to \mathrm{Gr}(A \otimes_k B^\circ)$ by $\Gamma_{m_A}(M) := \{m \in M \mid mA_{\geq n} = 0 \text{ for some } n \in \mathbb{N}\}$. We write $H^i_{m_A} := \mathrm{R}^i\Gamma_{m_A}$.

Definition 3.2.5 ([52, Definition 6.1, 6.2], [58, Definition 3.3, 4.1]). Let A be a right and left noetherian connected $\mathbb{N}$-graded k-algebra and A^e be the enveloping algebra of A. Let R be an object of $\mathrm{D^b}(\mathrm{Gr}(A^e))$. Then, R is called a dualizing complex of A if (1) R has finite injective dimension over A and A°, (2) The cohomologies of R are finitely generated as both A and A°-modules, (3) The natural morphisms $A \to \mathrm{RHom}_A(R, R)$ and $A \to \mathrm{RHom}_{A^\circ}(R, R)$ are isomorphisms in $\mathrm{D^b}(\mathrm{Gr}(A^e))$. Moreover, R is called balanced if $\mathrm{R}\Gamma_{m_A}(R) \simeq A^*$ and $\mathrm{R}\Gamma_{m_{A^\circ}}(R) \simeq A^*$ in $\mathrm{D^b}(\mathrm{Gr}(A^e))$.

3.3 Calabi-Yau conditions

Definition 3.3.1 ([22, Section 2.2]). Let A be a connected right noetherian $\mathbb{N}$-graded k-algebra. Then, proj(A) is a projective Calabi-Yau n scheme if the global dimension of qgr(A) is n and the Serre functor of the derived category $\mathrm{D^b}(\mathrm{qgr}(A))$ is the n-shift functor $[n]$.

Remark 3.3.2. Actually, we do not need the condition that the global dimension of qgr(A) is n. If the Serre functor of the derived category $\mathrm{D^b}(\mathrm{qgr}(A))$ is the n-shift functor $[n]$, then we can easily show that this condition holds. However, when we prove the existence of the Serre functor of $\mathrm{D^b}(\mathrm{qgr}(A))$, we essentially need the condition that the global dimension of qgr(A) is n (cf. [9, Theorem A.4, Corollary A.5], Lemma 3.3.10).

3.3.1 $\mathbb{Z}^2$-graded algebras and Segre products

In commutative algebraic geometry, a smooth complete intersection $X \subset \mathbb{P}^n \times \mathbb{P}^m$ of bidegrees $(n+1, 0)$ and $(0, m+1)$ provides a Calabi-Yau variety. We also have a little more complicated example that gives a Calabi-Yau variety. That is a smooth complete intersection of bidegrees $(n, 0)$ (resp. $(n+1, 0)$) and $(1, n+1)$ in $\mathbb{P}^n \times \mathbb{P}^n$ (resp. $\mathbb{P}^{n+1} \times \mathbb{P}^n$). We construct noncommutative analogues of these examples.

Let C be an $\mathbb{N}^2$-graded k-algebra. We denote the category of $\mathbb{Z}^2$-graded right C-modules (resp. finitely generated $\mathbb{Z}^2$-graded right C-modules) by $\mathrm{BiGr}(C)$ (resp. $\mathrm{bigr}(C)$). Let $M \in \mathrm{BiGr}(C)$. We denote by C° (resp. C^e) the opposite (resp. enveloping) algebra of C. We define the Matlis dual $M^* \in \mathrm{BiGr}(C^\circ)$ by $M^*_{i,j} := \mathrm{Hom}_k(M_{-i,-j}, k)$ and the shift $M(n,m) \in \mathrm{BiGr}(C)$ by $M(m,n)_{i,j} := M_{i+m,j+n}$ $(m,n,i,j \in \mathbb{Z})$. For $M, N \in \mathrm{BiGr}(C)$, we write $\mathrm{Hom}_C(M,N) := \bigoplus_{m,n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{BiGr(C)}}(M, N(m,n))$. For a bihomogeneous element $m \in M$, we denote the bidegree of m by $\mathrm{bideg}(m)$.

Let $M \in \mathrm{BiGr}(C)$. We define the truncation $M_{\geq n,\geq n} := \bigoplus_{i\geq n, j\geq n} M_{i,j} \in \mathrm{BiGr}(C)$ $(n \in \mathbb{Z})$. We say $m \in M$ is torsion if $mC_{\geq n,\geq n} = 0$ for $n \gg 0$. If all $m \in M$ are torsion, then M is called a torsion C-module. We denote the category of $\mathbb{Z}^2$-graded torsion C-modules by $\mathrm{Tor}(C)$. We also define $\mathrm{tor}(C)$ to be the intersection of $\mathrm{bigr}(C)$ and $\mathrm{Tor}(C)$. When we assume that C is right noetherian, we have the quotient categories $\mathrm{QBiGr}(C) := \mathrm{BiGr}(C)/\mathrm{Tor}(C)$ and $\mathrm{qbigr}(C) := \mathrm{bigr}(C)/\mathrm{tor}(C)$ (cf. [53, Section 2]). We denote the projection functor by π and its right adjoint functor by ω. We can define the general (resp. noetherian) projective scheme $\mathrm{Proj}(C)$ (resp. $\mathrm{proj}(C)$) associated to C and the notion of noncommutative projective Calabi-Yau schemes as in the case of $\mathbb{N}$-graded algebras.

Let D be an $\mathbb{N}^2$-graded algebra. We take the tensor product $C \otimes_k D^\circ$ of C and D° over k. We think of $C \otimes_k D^\circ$ as an $\mathbb{N}^2$-graded algebra by $(C \otimes_k D^\circ)_{i,j} := \bigoplus_{i_1+i_2=i, j_1+j_2=j} C_{i_1,j_1} \otimes_k D^\circ_{i_2,j_2}$. We define $m_{C_{++}} := C_{\geq 1,\geq 1}$ and the torsion functor $\Gamma_{m_{C_{++}}} : \mathrm{BiGr}(C \otimes_k D^\circ) \to \mathrm{BiGr}(C \otimes_k D^\circ)$ by $\Gamma_{m_{C_{++}}}(M) := \{m \in M \mid mC_{\geq n,\geq n} = 0 \text{ for some } n \in \mathbb{N}\}$. We write $m_C := \bigoplus_{i+j\geq 1} C_{i,j}$ and define another torsion functor $\Gamma_{m_C} : \mathrm{BiGr}(C \otimes_k D^\circ) \to \mathrm{BiGr}(C \otimes_k D^\circ)$ by $\Gamma_{m_C}(M) := \{m \in M \mid mC_{\geq n} = 0 \text{ for some } n \in \mathbb{N}\}$, where $C_{\geq n} := \bigoplus_{i+j\geq n} C_{i,j} \in \mathrm{BiGr}(C)$. See [42, Section 3] for details of Γ_{m_C}. We write $H^i_{m_{C_{++}}} := \mathrm{R}^i\Gamma_{m_{C_{++}}}$ and $H^i_{m_C} := \mathrm{R}^i\Gamma_{m_C}$. The reason we define the functor $\Gamma_{m_{C_{++}}}$ is that we can describe the Serre duality in $\mathrm{D^b}(\mathrm{qbigr}(C))$ by using $\mathrm{R}\Gamma_{m_{C_{++}}}$ (cf. Lemma 3.3.10). However, it is not easy to calculate the functor $\mathrm{R}\Gamma_{m_{C_{++}}}$ directly. The reason we define the functor Γ_{m_C} is that we can use the theory of $\mathbb{Z}$-graded modules to calculate $\mathrm{R}\Gamma_{m_C}$ and we can reduce the calculation of $\mathrm{R}\Gamma_{m_{C_{++}}}$ to that of $\mathrm{R}\Gamma_{m_C}$ (cf. Lemma 3.3.6, the proof of Theorem 3.3.3).

Theorem 3.3.3. *Let* $A := k\langle x_0, \cdots, x_n\rangle/(x_jx_i - q_{ji}x_ix_j)_{i,j}$, $B := k\langle y_0, \cdots, y_m\rangle/(y_jy_i - q'_{ji}y_iy_j)_{i,j}$ *and* $C := A \otimes_k B$, *where* $q_{ji}, q'_{ji} \in k^\times$ *for all* i, j. *We regard* C *as an* $\mathbb{N}^2$*-graded algebra with* $\mathrm{bideg}(x_i) = (1, 0)$ *and* bideg$(y_i) = (0,1)$ *for all* i.

1. *Let* $f := \sum_{i=0}^{n} x_i^{n+1}$ *and* $g := \sum_{i=0}^{m} y_i^{m+1}$. *We assume that (i)* $q_{ii} = q_{ij}q_{ji} = q_{ij}^{n+1} = 1$ *for all* i, j, *(ii)* $q'_{ii} = q'_{ij}q'_{ji} = q'^{m+1}_{ij} = 1$ *for all* i, j.

   Then, $\mathrm{proj}(C/(f,g))$ *is a noncommutatative projective Calabi-Yau scheme of dimension* $(n+m-2)$ *if and only if* $\prod_{i=0}^{n} q_{ij}$ *and* $\prod_{i=0}^{m} q'_{ij}$ *are independent of* j, *respectively.*

2. *Suppose that* $m = n+1$ *(resp.* $m = n$*) and* $q'_{ij} = 1$ *for all* i, j. *Let* $f := \sum_{i=0}^{n} x_i^{n+1} y_i$ *and* $g := \sum_{i=0}^{n+1} y_i^{n+1}$ *(resp.* $\sum_{i=0}^{n} y_i^{n}$*). We assume that* $q_{ii} = q_{ij}q_{ji} = q_{ij}^{n+1} = 1$ *for all* i, j.

   Then, $\mathrm{proj}(C/(f,g))$ *is a noncommutatative projective Calabi-Yau scheme of dimension* $(2n-1)$*(resp.* $(2n-2)$*) if and only if* $\prod_{i=0}^{n} q_{ij}$ *is independent of* j.

Notation 3.3.4. For simplicity, we denote the bidegrees of f, g in the theorem by $(d_0, d_1), (e_0, e_1)$, respectively.

Remark 3.3.5.

- f, g are central elements in C because of the choices of $\{q_{ij}\}, \{q'_{ij}\}$.
- We have $n+m-2 = d_0+d_1+e_0+e_1-4$ in (1). We have $2n-1$(resp. $2n-2) = d_0+d_1+e_0+e_1-4$ also in (2).
- In (2) of the theorem, even if we do not assume $q'_{ij} = 1$, the condition for f, g to be central in C implies $q'_{ij} = 1$ for all i, j after all.
- In the theorem, all equations appearing except for g of (2) are Fermat-type equations.

To prove the theorem, we need to show some lemmas. Perhaps some experts may know the following lemmas. However, to the best of the author's knowledge, there are no references written on those lemmas, so the proofs are given below. In addition, the following proofs do not depend on whether (1) or (2) in the theorem is considered (except for Lemma 3.3.8).

Lemma 3.3.6. *Let* $\mathcal{R} := \pi(\mathrm{R}\Gamma_{m_{C/(f,g)_{++}}}(C/(f,g))^*)$ *and* $\mathcal{R}' := \pi(\mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g))^*)$. *Then, the functors* $-\otimes^{\mathbb{L}}\mathcal{R}$ *and* $-\otimes^{\mathbb{L}}\mathcal{R}'[-1]$ *between* $\mathrm{D}(\mathrm{QBiGr}(C/(f,g)))$ *and itself are naturally isomorphic.*

Proof. Let I_1, I_2 be the ideals generated by $\{x_0, \cdots, x_n\}, \{y_0 \cdots, y_m\}$, respectively. Then, we have $m_{C/(f,g)_{++}} = I_1 \cap I_2$, $m_{C/(f,g)} = I_1 + I_2$ and have the following long exact sequence in $\mathrm{BiGr}(C/(f,g)^e)$

$$\cdots \to H^i_{m_{C/(f,g)}}(C/(f,g)) \to H^i_{I_1}(C/(f,g)) \oplus H^i_{I_2}(C/(f,g)) \to H^i_{m_{C/(f,g)_{++}}}(C/(f,g)) \to \cdots$$

by using the Mayer-Vietoris sequence in the sense of [7, Chapter 3] , where $\Gamma_{I_j} (j = 1,2)$ is defined not by using the degrees of I_j but by using powers of

I_j (i.e., $\Gamma_{I_j}(M) := \{m \in M \mid mI_j^n = 0 \text{ for some } n\}$). Note that we can use the Mayer-Vietoris sequence in our case because I_1, I_2 are generated by normal elements and this implies that I_1, I_2 satisfy Artin-Rees property. We also have the exact triangle in $\mathrm{D}(\mathrm{BiGr}(C/(f,g)^e))$

$$\mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g)) \to \mathrm{R}\Gamma_{I_1}(C/(f,g)) \oplus \mathrm{R}\Gamma_{I_2}(C/(f,g)) \to \mathrm{R}\Gamma_{m_{C/(f,g)_{++}}}(C/(f,g)).$$

Moreover, $H^i_{I_1}(C/(f,g))^*$ and $H^i_{I_2}(C/(f,g))^*$ are torsion modules for $m_{C/(f,g)_{++}}$ from Sub-Lemma 3.3.7. So, the cohomologies of $\mathrm{R}\Gamma_{I_1}(C/(f,g))^* \oplus \mathrm{R}\Gamma_{I_2}(C/(f,g))^*$ are torsion. From this result, the above triangle gives an isomorphism between $\mathcal{R}$ and $\mathcal{R}'[-1]$ after taking the dual and applying π. Hence, we get the claim.

□

Sub-Lemma 3.3.7. *Let I_1, I_2 be as in the proof of Lemma 3.3.6. $H^i_{I_1}(C/(f,g))^*$ and $H^i_{I_2}(C/(f,g))^*$ are torsion modules for $m_{C/(f,g)_{++}}$ for any i.*

Proof. We only show that $H^i_{I_1}(C/(f,g))^*$ are torsion modules for $m_{C/(f,g)_{++}}$. We can show that $H^i_{I_2}(C/(f,g))^*$ are torsion in the same way.

First, we prove that $H^i_{I_1}(C)^*$ is torsion. We have $\Gamma_{I_1} = \Gamma_{I_1^{n+1}}$. Moreover, if J_1 is the ideal generated by $x_0^{n+1}, \cdots, x_n^{n+1}$, then we have $\Gamma_{I_1^{n+1}} = \Gamma_{J_1}$. Note that $x_0^{n+1}, \cdots, x_n^{n+1}$ are central elements in C from the choice of $\{q_{ij}\}$.

Let $M \in \mathrm{Gr}(C)$ be injective. Then, we have a surjective localization map $M \to M[x_i^{-(n+1)}]$ for any i and $\Gamma_{J_1}(M)$ is injective in $\mathrm{Gr}(C)$ because J_1 satisfies Artin-Rees property (cf. [55, Example 3.13], [10, Lemma A1.4]). When M' is injective in $\mathrm{Gr}(C^e)$, then M' is injective in $\mathrm{Gr}(C)$, where $\mathrm{Res}_C : \mathrm{Gr}(C^e) \to \mathrm{Gr}(C)$ is the restriction functor ([58, Lemma 2.1]). Thus, we can calculate $\mathrm{Res}_C(H^i_{J_1}(C))$ by using a Čech complex $\mathscr{C}(x_0^{n+1}, \cdots, x_n^{n+1}; C)$ (cf. [30, Chapter 2, 3], [10, Theorem A1.3]). Then, we have $\mathscr{C}(x_0^{n+1}, \cdots, x_n^{n+1}; C) = \mathscr{C}(x_0^{n+1}, \cdots, x_n^{n+1}; A) \otimes_k B$. This induces that $\mathrm{Res}_C(H^i_{J_1}(C)) \simeq H^i_{m_A}(A) \otimes_k B$. Because $H^i_{m_A}(A)_{>0} = 0$ ([22, Proposition 2.4]), $H^i_{J_1}(C)^* \simeq H^i_{I_1}(C)^*$ is torsion.

Finally, we consider the exact sequences of C-bimodules

$$0 \to C(-d_0, -d_1) \overset{\times f}{\to} C \to C/(f) \to 0, \tag{3.3.1.1}$$

$$0 \to C/(f)(-e_0, -e_1) \overset{\times g}{\to} C/(f) \to C/(f,g) \to 0. \tag{3.3.1.2}$$

Then, we take the long exact sequence for Γ_{I_1} and we get the claim since $H^i_{I_1}(C)^*$ is torsion. □

Lemma 3.3.8. $\mathrm{gl.dim}(\mathrm{qbigr}(C/(f,g))) = d_0 + d_1 + e_0 + e_1 - 4$.

Proof. We show the proposition only in (1) of the theorem. In (2) of the theorem, the proposition can be shown in the same way (cf. Remark 3.3.9). We consider a bigraded (commutative) algebra $D := k[s_0, \cdots, s_n, t_0, \cdots, t_m]/(\sum_{i=0}^n s_i, \sum_{i=0}^m t_i)$ with $s_i = x_i^{n+1}, t_i = y_i^{m+1}$ and the projective spectrum biProj(D) in the sense of [18, Section 1]. Then, $C/(f,g)$ is a finite D-module. So, qbigr$(C/(f,g))$ can be thought of as the category of modules over a sheaf $\mathcal{A}$ of $\mathcal{O}_{\mathrm{biProj}(D)}$-algebras, where $\mathcal{A}$ is the sheaf on biProj(D) which is locally defined by the algebra $(k[x_0,\cdots,x_n,y_0,\cdots,y_m]/(f,g)_{x_iy_j})_{(0,0)}$ on each open affine scheme $\mathrm{D}_+(s_it_j) \simeq \mathrm{Spec}((D_{s_it_j})_{(0,0)})$. Hence, it is enough to prove that

$$\begin{aligned}\mathrm{gl.dim}((k[x_0,\cdots,x_n,y_0,\cdots,y_m]/(f,g)_{x_iy_j})_{(0,0)}) &= d_0+d_1+e_0+e_1-4\\ &= n+m-2.\end{aligned}$$

We can complete the rest of the proof in the same way as in [22, Section 2.3]. We give its sketch. For simplicity, we prove the claim when $i=j=0$. We define a k-algebra E by

$$E := k[S_1,\cdots,S_n,T_1,\cdots,T_m] \Bigg/ \left(1+\sum_{i=1}^{n} S_i, 1+\sum_{i=0}^{m} T_i\right)$$

with $S_i = s_i/s_0, T_i = t_i/t_0$. We also define an E-algebra F by

$$F := k\langle X_1,\cdots,X_n,Y_1,\cdots,Y_m\rangle \Bigg/ \begin{pmatrix} X_iX_j-(q_{0i}q_{ij}q_{j0})X_jX_i,\\ Y_iY_j-(q'_{0i}q'_{ij}q'_{j0})Y_jY_i,\\ 1+\sum_{l=1}^{n} X_l^{n+1}, 1+\sum_{l=1}^{m} Y_l^{m+1}\end{pmatrix}_{i,j}$$

with $X_i = x_i/x_0, Y_i = y_i/y_0$. The module structure of F is given by the identifications $S_i = X_i^{n+1}, T_i = Y_i^{m+1}$. Let $F_{\widetilde{m}}$ be the localization of F at a maximal ideal

$$\widetilde{m} := (S_1-a_1,\cdots,S_n-a_n,T_1-b_1,\cdots,T_m-b_m)$$

of E with $1+\sum_{i=1}^{n} a_i = 1+\sum_{i=1}^{m} b_i = 0$ $(a_i,b_i \in k)$. Then, it is enough to prove that the global dimension of $F_{\widetilde{m}}$ is $n+m-2$ ([22, Lemma 2.6, 2.7]).

If all a_i,b_i are not 0, then $F/\widetilde{m}F$ is a twisted group ring and hence semisimple. Moreover, $S_1-a_1,\cdots,S_n-a_n,T_1-b_1,\cdots,T_m-b_m$ is a regular sequence in $F_{\widetilde{m}}$. This induces the claim ([29, Theorem 7.3.7]).

On the other hand, assume that one of $\{a_1,\cdots,a_n,b_1,\cdots,b_m\}$ is 0. For example, assume $a_1=0$. We consider $F/(X_1)$. Then, we can show that the global dimension of $(F/(X_1))_{\widetilde{m}} = n+m-3$ because $\mathrm{pd}_F(S) = \mathrm{pd}_{F/(X_1)}(S)+1$ for any simple F-module S with $\mathrm{Ann}(S) = \widetilde{m}$ ([29, Theorem 7.3.5]). If some other a_i,b_j are 0, we repeat taking quotients and can reduce to considering the global dimension of the algebra $k[X,Y]/(X^{n+1}+1,Y^{m+1}+1)$, which is 0. □

Remark 3.3.9. To prove Lemma 3.3.8 in (2) of the theorem, consider the projective spectrum $X :=$ biProj$(k[s_0,\cdots,s_n,t_0,\cdots,t_{n+1}]/\ (\sum_{i=0}^{n} s_it_i, \sum_{i=0}^{n+1} t_i^{n+1}))$

(resp. $\mathrm{biProj}(k[s_0, \cdots, s_n, t_0, \cdots, t_n]/\ (\sum_{i=0}^n s_i t_i, \sum_{i=0}^n t_i^n)))$ and the sheaf $\mathcal{A}$ of algebras on X associated to $C/(f,g)$.

Proof of Theorem 3.3.3. First, we calculate $\mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g))^*$. From [22, Proposition 2.4] (or [42, Example 5.5]) and the proof of [42, Lemma 6.1], we have

$$\begin{aligned}\mathrm{R}\Gamma_{m_C}(C)^* &\simeq \mathrm{R}\Gamma_{m_A}(A)^* \otimes \mathrm{R}\Gamma_{m_B}(B)^* \\ &\simeq {}^{\phi}A^1(-d_0-e_0) \otimes_k {}^{\psi}B^1(-d_1-e_1)[d_0+d_1+e_0+e_1],\end{aligned} \tag{3.3.1.3}$$

where ϕ (resp. ψ) is the graded automorphism of A (resp. B) which maps $x_j \mapsto \prod_{i=0}^n q_{ji}x_j$ (resp. $y_j \mapsto \prod_{i=0}^m q'_{ji}y_j$). Then, we consider the distinguished triangles

$$\mathrm{R}\Gamma_{m_C}(C(-d_0,-d_1)) \xrightarrow{\times f} \mathrm{R}\Gamma_{m_C}(C) \longrightarrow \mathrm{R}\Gamma_{m_{C/(f)}}(C/(f)), \tag{3.3.1.4}$$

$$\mathrm{R}\Gamma_{m_{C/(f)}}((C/(f))(-e_0,-e_1)) \xrightarrow{\times g} \mathrm{R}\Gamma_{m_{C/(f)}}(C/(f)) \longrightarrow \mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g)) \tag{3.3.1.5}$$

obtained from the exact sequences (3.3.1.1) and (3.3.1.2) of C-bimodules. Combining the formula (3.3.1.3) and the triangle (3.3.1.4), we have

$$\mathrm{R}\Gamma_{m_{C/(f)}}(C/(f))^* \simeq {}^{\phi\otimes\psi}(A \otimes_k B/(f))^1(-e_0,-e_1)[d_0+d_1+e_0+e_1-1]. \tag{3.3.1.6}$$

In addition, combining the triangle (3.3.1.5) and the formula (3.3.1.6), we have

$$\mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g))^* \simeq {}^{\phi\otimes\psi}(A \otimes_k B/(f,g))^1[d_0+d_1+e_0+e_1-2]. \tag{3.3.1.7}$$

On the other hand, we have the Serre duality in $\mathrm{D^b}(\mathrm{qbigr}(C/(f,g)))$ from Lemma 3.3.10. Thus, $-\otimes^{\mathbb{L}} \pi(\mathrm{R}\Gamma_{m_{C/(f,g)_{++}}}(C/(f,g))^*)[-1]$ is the Serre functor of $\mathrm{D^b}(\mathrm{qbigr}(C/(f,g)))$ because this functor induces an equivalence from Lemma 3.3.6 and the formula (3.3.1.7). Finally, the Serre functor $-\otimes^{\mathbb{L}} \pi(\mathrm{R}\Gamma_{m_{C/(f,g)_{++}}}(C/(f,g))^*)[-1]$ induces the $[d_0+d_1+e_0+e_1-4]$-shift functor if and only if $\prod_{i=0}^n q_{ij}$ and $\prod_{i=0}^m q'_{ij}$ are independent of j (cf. [22, Remark 2.5]). This completes the proof. □

The following lemma is well-known in the case of $\mathbb{N}$-graded algebras (for example, see [9], [59]).

Lemma 3.3.10 (Local Duality and Serre Duality for $\mathbb{N}^2$-graded algebras)**.** *Let D be a connected right noetherian $\mathbb{N}^2$-graded k-algebra (connected means $D_{0,0} = k$). Let E be a connected $\mathbb{N}^2$-graded k-algebra. We assume that $\Gamma_{m_{D_{++}}}$ has finite cohomological dimension.*

1. *Let* $Q := \omega \circ \pi : \mathrm{BiGr}(D) \to \mathrm{BiGr}(D)$. *Let* $M \in \mathrm{D}(\mathrm{BiGr}(D \otimes_k E^\circ))$. *Then,*

$$\mathrm{R}\Gamma_{m_{D_{++}}}(M)^* \simeq \mathrm{RHom}_D(M, \mathrm{R}\Gamma_{m_{D_{++}}}(D)^*), \tag{a}$$
$$\mathrm{R}Q(M)^* \simeq \mathrm{RHom}_D(M, \mathrm{R}Q(D)^*) \tag{b}$$

in $\mathrm{D}(\mathrm{BiGr}(D^\circ \otimes_k E))$, *where we denote the natural extension of* Q *to a functor between* $\mathrm{BiGr}(D \otimes_k E^\circ)$ *and itself by the same notation.*

2. *We assume that* $\mathrm{qbigr}(D)$ *has finite global dimension. Let* $\mathcal{M} := \pi(M)$, $\mathcal{N} := \pi(N)$ $(M, N \in \mathrm{D^b}(\mathrm{bigr}(D)))$. *Let* $\mathcal{R}_D := \pi(\mathrm{R}\Gamma_{m_{D_{++}}}(D)^*) \in \mathrm{D^b}(\mathrm{qbigr}(D^e))$. *Then,* $\mathcal{N} \otimes^{\mathbb{L}} \mathcal{R}_D \in \mathrm{D^b}(\mathrm{qbigr}(D))$ *and*

$$\mathrm{Hom}_{\mathrm{D^b}(\mathrm{qbigr}(D))}(\mathcal{N}, \mathcal{M}) \simeq \mathrm{Hom}_{\mathrm{D^b}(\mathrm{qbigr}(D))}(\mathcal{M}, (\mathcal{N} \otimes^{\mathbb{L}} \mathcal{R}_D)[-1])',$$

which is functorial in $\mathcal{M}$ *and* $\mathcal{N}$. *Here,* $(-)'$ *denotes the* k*-dual.*

Proof. Since $\mathrm{R}^i\Gamma_{m_{D_{++}}}(-) \simeq \lim_{n\to\infty} \mathrm{Ext}^i(D/D_{\geq n, \geq n}, -)$ and D is right noetherian, one can check that $\mathrm{R}^i\Gamma_{m_{D_{++}}}(-)$ commutes with direct limits as in [57, Proposition 16.3.19]. In addition, if K is a complex of graded free right D-modules and L is a complex of graded right D^e-modules, then $\Gamma_{m_{D_{++}}}(K \otimes_D L) \simeq K \otimes_D \Gamma_{m_{D_{++}}}(L)$ (cf. [35, Lemma 6.10]). So, we can apply the argument of [52, Theorem 5.1] (or [34, Theorem 2.1]) to prove (a) of (1).

In order to prove (b) of (1), note that we have the canonical exact sequence and the isomorphism (see also [6, Lemma 4.1.4, 4.1.5])

$$0 \to \Gamma_{m_{D_{++}}}(M) \to M \to Q(M) \to \lim_{n\to\infty} \mathrm{Ext}^1(D/D_{\geq n, \geq n}, M) \to 0,$$
$$\mathrm{R}^i Q(M) \simeq \mathrm{R}^{i+1}\Gamma_{m_{D_{++}}}(M), \quad (1 \leq i, M \in \mathrm{BiGr}(D)).$$

So, from the previous paragraph, Q has finite cohomological dimension, R^iQ commutes with direct limits. We also have $Q(K \otimes_D L) \simeq K \otimes_D Q(L)$, where K, L are as above (cf. [36, Lemma 3.28]). Hence, we can also apply the argument of [52, Theorem 5.1] (or [36, Theorem 3.29]) to prove (b) of (1).

We can prove (2) in the same way as in [9, Lemma A.1, Theorem A.4] by using (b) of (1). Note that we have a natural equivalence $\mathrm{D^b}(\mathrm{qbigr}(D)) \simeq \mathrm{D^b_f}(\mathrm{QBiGr}(D))$, where $\mathrm{D^b_f}(\mathrm{QBiGr}(D))$ is the full subcategory of $\mathrm{D^b}(\mathrm{QBiGr}(D))$ consisting of complexes with cohomology in $\mathrm{qbigr}(D)$ ([9, Lemma 2.2]). □

As a corollary of Theorem 3.3.3, we construct examples of noncommutative projective Calabi-Yau schemes by using Segre products. Let A, B, f and g be as in Theorem 3.3.3.

Definition 3.3.11. 1. The Segre product $A \circ B$ of A and B is the $\mathbb{N}$-graded k-algebra with $(A \circ B)_i = A_i \otimes_k B_i$.

2. Let $M \in \mathrm{bigr}(C)$. We define a right graded $A \circ B$-module M_Δ as the graded $A \circ B$-module with $(M_\Delta)_i = M_{i,i}$.

Lemma 3.3.12 ([53, Theorem 2.4]). *We have the following natural isomorphism*

$$\mathrm{qbigr}(C) \longrightarrow \mathrm{qgr}(A \circ B), \qquad \pi(M) \longmapsto \pi(M_\Delta).$$

In addition, the functor defined by $- \otimes_{A\circ B} C$ is the inverse of this equivalence.

Remark 3.3.13. Let $J := (f, g) \in \mathrm{bigr}(C)$. We similarly obtain an equivalence

$$\mathrm{qbigr}(C/J) \simeq \mathrm{qgr}(A \circ B/J_\Delta).$$

Combining Theorem 3.3.3 with Remark 3.3.13, we get the following.

Corollary 3.3.14. *Let $J := (f, g) \in \mathrm{bigr}(C)$. Then,* $\mathrm{proj}(A \circ B/J_\Delta)$ *is a noncommutative projective Calabi-Yau scheme.*

3.3.2 Weighted hypersurfaces

Reid produced the list of all commutative weighted Calabi-Yau hypersurfaces of dimensions 2 (for example, see [19], [41]). In this section, we construct noncommutative projective Calabi-Yau schemes from noncommutative weighted projective hypersurfaces. Let A be a right noetherian $\mathbb{N}$-graded k-algebra. Then, the r-th Veronese algebra $A^{(r)}$ is the $\mathbb{N}$-graded k-algebra with $A^{(r)}_i = A_{ri}$. We consider the (commutative) weighted polynomial ring $A = k[x_0, \cdots, x_n]$ with $\deg(x_i) = d_i$. Then, $\mathrm{Coh}(\mathrm{Proj}(A))$ is in general not equivalent to $\mathrm{qgr}(A)$, but to $\mathrm{qgr}(A^{(n+1)\mathrm{lcm}(d_0,\cdots,d_n)})$. However, we can think of $\mathrm{qgr}(A)$ as a resolution of singularities of $\mathrm{Coh}(\mathrm{Proj}(A))$ (cf. [46, Example 4.9]). Moreover, we have $\mathrm{qgr}(A) \simeq \mathrm{Coh}([(\mathrm{Spec}(A)\backslash\{0\})/\mathbb{G}_m])$ and $[(\mathrm{Spec}(A)\backslash\{0\})/\mathbb{G}_m]$ is a smooth Deligne-Mumford stack whose coarse moduli space is $\mathrm{Proj}(A)$.

Theorem 3.3.15. *Let $(d_0, \cdots, d_n) \in \mathbb{Z}^{n+1}_{>0}$ and $d := \sum_{i=0}^n d_i$ such that d is divisible by d_i for all i. Let $C := k\langle x_0, \cdots, x_n\rangle/(x_j x_i - q_{ji} x_i x_j)_{i,j}$, where $q_{ji} \in k^\times, \deg(x_i) = d_i$ for all i, j. Let $f := \sum_{i=0}^n x_i^{h_i}$, where $h_i := d/d_i$.*

We assume that $q_{ii} = q_{ij} q_{ji} = q_{ij}^{h_i} = q_{ij}^{h_j} = 1$ for all i, j. Then, $\mathrm{proj}(C/(f))$ *is a noncommutative projective Calabi-Yau scheme of dimension $(n-1)$ if and only if there exists $c \in k$ such that $c^{d_j} = \prod_{i=0}^n q_{ij}$ for all j.*

Remark 3.3.16.

- f is a central element in C from the choice of $\{q_{ij}\}$.
- Theorem 3.3.15 is a generalization of [22, Theorem 1.1].

Lemma 3.3.17. *The balanced dualizing complex of $C/(f)$ is isomorphic to ${}^\phi(C/(f))^1[n]$, where ϕ is a graded automorphism of C which maps $x_j \mapsto \prod_{i=0}^n q_{ji} x_j$.*

Proof. Since C is Artin-Schelter regular, C is skew Calabi-Yau ([42, Lemma 1.2]). This implies that the balanced dualizing complex of C is isomorphic to ${}^{\phi}C^1(-d)[n+1]$, where ϕ is the Nakayama automorphism of C. From [42, Example 5.5], the automorphism ϕ is the map which maps $x_j \mapsto \prod_{i=0}^{n} q_{ji}x_j$.

By using this result, we can obtain the claim in the same way as in the proof of Theorem 3.3.3 after Remark 3.3.9. □

In general, $C/(f)$ is not generated in degree 0 and 1. This fact prevents us from using the idea of the proof of Lemma 3.3.8 to calculate the global dimension of $\mathrm{qgr}(C/(f))$. So, we need to find a right noetherian $\mathbb{N}$-graded k-algebra R which is generated in degree 0 and 1 and satisfies $\mathrm{qgr}(R) \simeq \mathrm{qgr}(C/(f))$. Quasi-Veronese algebras are effective in achieving this objective. We recall the notion of quasi-Veronese algebras below. In detail, see [32, Section 3].

Definition 3.3.18 ([32, Section 3])**.** Let A be an $\mathbb{N}$-graded k-algebra. The l-th quasi-Veronese algebra $A^{[l]}$ of A is a graded k-algebra defined by

$$A^{[l]} := \bigoplus_{i\in\mathbb{N}} A_i^{[l]} := \bigoplus_{i\in\mathbb{N}} \begin{pmatrix} A_{li} & A_{li+1} & \cdots & A_{li+l-1} \\ A_{li-1} & A_{li} & \cdots & A_{li+l-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{li-l+1} & A_{li-l+2} & \cdots & A_{li}. \end{pmatrix}.$$

Remark 3.3.19.

1. We have $\mathrm{Gr}(A) \simeq \mathrm{Gr}(A^{[l]})$ ([32, Lemma 3.9]). The equivalence is obtained by the functor $\Psi : \mathrm{Gr}(A) \to \mathrm{Gr}(A^{[l]})$, which is defined by $\Psi(M) := \bigoplus_{i\in\mathbb{Z}} \left(\bigoplus_{j=0}^{l-1} M_{li-j}\right)$
2. When A is right noetherian, $A^{[l]} \simeq \bigoplus_{0\le i,j\le n-1} A(j-i)^{(l)} \in \mathrm{gr}(A^{(l)})$, where $A^{(l)}$ is the l-th Veronese algebra of A and the $A^{(l)}$-module structure of $A^{[l]}$ is given by the natural inclusion $A^{(l)} \subset A^{[l]}$ (cf. the proof of [33, Proposition 4.11]). Then, $A^{[l]}$ is also right noetherian since $A^{(l)}$ is right noetherian. In this case, Ψ induces an equivalence between $\mathrm{qgr}(A)$ and $\mathrm{qgr}(A^{[l]})$.

Lemma 3.3.20. *Let A be an $\mathbb{N}$-graded k-algebra which is generated by homogeneous elements $y_0, \cdots, y_h$ with $\deg(y_i) > 0$ as an A_0-algebra. Let $l \ge \max\{\deg(y_0), \cdots, \deg(y_h)\}$. Then, $A^{[l]}$ is generated in degree 0 and 1.*

Proof. For any $i \in \mathbb{N}$ and any $a, b \in \{0, 1, \cdots, l-1\}$, it is enough to show that every homogeneous element m of the form

$$m = \begin{pmatrix} m_{0,0} & \cdots & m_{0,\beta} & \cdots & m_{0,l-1} \\ \vdots & & \vdots & & \vdots \\ m_{\alpha,0} & \cdots & m_{\alpha,\beta} & \cdots & m_{\alpha,l-1} \\ \vdots & & \vdots & & \vdots \\ m_{l-1,0} & \cdots & m_{l-1,\beta} & \cdots & m_{l-1,l-1} \end{pmatrix} \in A_i^{[l]}, \quad \begin{pmatrix} m_{\alpha,\beta} \in \left(A_i^{[l]}\right)_{\alpha,\beta} := A_{li+\beta-\alpha}, \\ m_{\alpha,\beta} = 0 \text{ when } (\alpha,\beta) \neq (a,b) \\ 0 \le \alpha,\beta \le l-1 \end{pmatrix}$$

is generated in degree 0 and 1. Moreover, we can assume that $m_{a,b} = \prod_{j=0}^{n_1} y_{i_j}$ $(i_j \in \{0, \cdots, h\}, n_1 \in \mathbb{N})$.

If $m_{a,b}$ is decomposed into $\prod_{j=0}^{n_1} y_{i_j} = \prod_{j=0}^{n_2} y_{i_j} \prod_{j=n_2+1}^{n_1} y_{i_j}$ $(n_2 \in \mathbb{N})$ such that $l - a \leq \deg(\prod_{j=1}^{n_2} y_{i_j}) \leq 2l - a - 1$, then we have $\prod_{j=0}^{n_2} y_{i_j} \in (A_1^{[l]})_{a,c} = A_{l+c-a}$ and $\prod_{j=n_2+1}^{n_1} y_{i_j} \in (A_{i-1}^{[l]})_{c,b} = A_{l(i-1)+b-c}$ $(0 \leq {}^{\exists}c \leq l-1)$. In this case, we can show the claim by using induction on the degree of m. So, it is sufficient to show that we have such a decomposition for all m. Indeed, we can find at least one such decomposition from $(2l-a-1)-(l-a)+1 = l$ and the choice of l. In detail, we have $l - a \leq \deg(y_{i_0}) \leq 2l-a-1$ or there exists $n_3 \in \mathbb{N}$ such that $\deg(y_{i_0} y_{i_1} \cdots y_{i_{n_3}}) < l - a$ and $l - a \leq \deg(y_{i_0} y_{i_1} \cdots y_{i_{n_3}} y_{i_{n_3}+1}) \leq 2l-a-1$ since $0 < \deg(y_i) \leq l$. □

Lemma 3.3.21. $\mathrm{gl.dim}(\mathrm{qgr}(C/(f))) = n - 1$.

Proof. We use the idea of the proof of Lemma 3.3.8. We consider an $\mathbb{N}$-graded k-algebra $B := k[s_0, \cdots, s_n]/(\sum_{i=0}^{n} s_i)$ with $s_i = x_i^{h_i}$. Then, $A^{[d]}$ is right noetherian and $\mathrm{qgr}(C/(f)) \simeq \mathrm{qgr}((C/(f))^{[d]})$ from Remark 3.3.19. So, it is enough to prove that $\mathrm{gl.dim}(\mathrm{qgr}((C/(f))^{[d]})) = n - 1$. Because $C/(f)$ is finite over B, a B-submodule $Z(C/(f))^{(d)}$ of $C/(f)$ is finite over B. From [33, Proposition 4.10, 4.11], $(C/(f))^{[d]}$ is finite over $Z(C/(f))^{(d)}$. So, $(C/(f))^{[d]}$ is finite over B. In addition, $(C/(f))^{[d]}$ is generated in degrees 0 and 1 from Lemma 3.3.20. So, $\mathrm{qgr}((C/(f))^{[d]})$ is equivalent to the category of coherent modules over a sheaf $\mathcal{A}$ of $\mathcal{O}_{\mathrm{Proj}(B)}$-algebra, where $\mathcal{A}$ is the sheaf on the projective spectrum $\mathrm{Proj}(B)$ which is locally defined by a tiled matrix algebra

$$N_i = \begin{pmatrix} E_{i,0} & E_{i,1} & \cdots & E_{i,d-1} \\ E_{i,-1} & E_{i,0} & \cdots & E_{i,d-2} \\ \vdots & \vdots & \cdots & \vdots \\ E_{i,-d+1} & E_{i,-d+2} & \cdots & E_{i,0} \end{pmatrix}$$

on each $D_+(s_i)$. Here, $E_i := (C/(f))[x_i^{-1}]$ and $E_{i,j}$ is the degree j part of E_i. As in the proof of Lemma 3.3.8, it is enough to show that the global dimension of N_i is $n - 1$ for all i.

On the other hand, two graded algebras

$$R_1 := E_i \oplus E_i(1) \oplus \cdots \oplus E_i(d-2) \oplus E_i(d-1),$$
$$R_2 := E_i \oplus E_i(1) \oplus \cdots \oplus E_i(d_i-2) \oplus E_i(d_i-1)$$

are progenerators in $\mathrm{Gr}(E_i)$. So, the category of right $\mathrm{End}_{\mathrm{gr}}(R_1)$-modules and the category of right $\mathrm{End}_{\mathrm{gr}}(R_2)$-modules are equivalent because they are equivalent to the category of graded right E_i-modules (cf. [47, Lemma 4.8], [46, Remarks after Proposition 4.5]). We also have $\mathrm{End}_{\mathrm{gr}}(R_1) \simeq N_i$ and

$$\mathrm{End}_{\mathrm{gr}}(R_2) \simeq M_i := \begin{pmatrix} E_{i,0} & E_{i,1} & \cdots & E_{i,d_i-1} \\ E_{i,-1} & E_{i,0} & \cdots & E_{i,d_i-2} \\ \vdots & \vdots & \cdots & \vdots \\ E_{i,-d_i+1} & E_{i,-d_i+2} & \cdots & E_{i,0} \end{pmatrix}.$$

So, it is sufficient to prove the global dimension of M_i is $n-1$ for each i.

For simplicity, we assume $i=0$. When $i \neq 0$, we can show the claim in the same way. Let $D = k[S_1, \cdots, S_n]/(1+\sum_{j=0}^{n} S_j)$ with $S_j = s_j/s_0$. We show that the global dimension of the D-algebra M_0 is $n-1$. The module structure of M_0 is given by the identification $S_j = (x_j^{h_j}/x_0^{h_0})I_{d_0} \in M_0$, where I_{d_0} is the $(d_0 \times d_0)$-identity matrix. Let

$$\widetilde{m} = (S_1 - a_1, \cdots, S_n - a_n) \ (a_j \in k)$$

be a maximal ideal of D with $1 + \sum_{j=1}^{n} a_j = 0$. It is sufficient to show that $\mathrm{gl.dim}((M_0)_{\widetilde{m}}) = n-1$, where $(M_0)_{\widetilde{m}}$ is the localization of M_0 at $\widetilde{m}$ (cf. the second paragraph of the proof of Lemma 3.3.8). We divide the proof of this claim into two cases.

Case (a) : all a_j are not 0. Because $S_1 - a_1, \cdots, S_n - a_n$ is a regular sequence in $(M_0)_{\widetilde{m}}$, we show that the global dimension of $(M_0)_{\widetilde{m}}/\widetilde{m}(M_0)_{\widetilde{m}} \simeq M_0/\widetilde{m}M_0$ is 0 (cf. the third paragraph of the proof of Lemma 3.3.8).

First, the category of $M_0/\widetilde{m}M_0$-modules is equivalent to the category of graded E_0'-modules, where

$$E_0' := E_0/(x_1^{h_1}/x_0^{h_0} - a_1, \cdots, x_n^{h_n}/x_0^{h_0} - a_n)E_0.$$

This is a Morita equivalence obtained from the isomorphism $\mathrm{End}_{\mathrm{gr}}(E_0') \simeq M_0/\widetilde{m}M_0$ (cf. the three previous paragraph).

Next, we see that E_0' is strongly graded. Since

$$E_0 \simeq (C[x_0^{-1}])/(1 + (x_1^{h_1}/x_0^{h_0}) + \cdots + (x_n^{h_n}/x_0^{h_0})),$$

we have

$$E_0' \simeq (C[x_0^{-1}])/(x_1^{h_1}/x_0^{h_0} - a_1, \cdots, x_n^{h_n}/x_0^{h_0} - a_n).$$

For any $l \in \mathbb{Z}$, if $\widetilde{x} := x_0^{l_0} x_1^{l_1} \cdots x_n^{l_n} \in (E_0')_l$ ($l_0 \in \mathbb{Z}, l_1, \cdots l_n \in \mathbb{N}$), then there exist $k_1, \cdots, k_n \in \mathbb{N}$ such that $\widetilde{x}' := x_0^{(-\sum k_i)h_0 - l_0} x_1^{k_1 h_1 - l_1} \cdots x_n^{k_n h_n - l_n} \in (E_0')_{-l}$. Because $\widetilde{x}\,\widetilde{x}' \in k^*$, we get $1 \in (E_0')_l (E_0')_{-l}$ and E_0' is strongly graded.

Since E_0' is strongly graded, we have $\mathrm{Gr}(E_0') \simeq \mathrm{Mod}((E_0')_0)$. Then, $(E_0')_0$ is a twisted group algebra, where a k-basis of $(E_0')_0$ is

$$\left\{ x_0^{e_0} x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n} \in (E_0')_0 \;\middle|\; \sum_{j=0}^{n} e_j d_j = 0 \text{ and } 0 \le e_j < h_j \ ({}^\forall j = 1, 2, \cdots, n) \right\}.$$

In particular, $(E_0')_0$ is semisimple. Hence, the graded global dimension of E_0' is 0 and $\mathrm{gl.dim}(M_0/\widetilde{m}M_0) = 0$.

Case (b) : some of a_j are 0. For example, we assume $a_1 = 0$. Then, $(x_1^{h_1}/x_0^{h_0})I_{d_0}$ is an annihilator of any simple M_0-module N. On the other hand, we have a unique integer r_1 such that $0 \le \deg(x_1/x_0^{r_1}) \le d_0 - 1$. If

$\deg(x_1/x_0^{r_1}) = 0$, then $J = x_1/x_0^{r_1} I_{d_0}$ annihilates N. Otherwise, the matrix

$$J = \left(\begin{array}{c|c} O & \begin{matrix} x_1/x_0^{r_1} & & \\ & \ddots & \\ & & x_1/x_0^{r_1} \end{matrix} \\ \hline \begin{matrix} x_1/x_0^{r_1+1} & & \\ & \ddots & \\ & & x_1/x_0^{r_1+1} \end{matrix} & O \end{array}\right) \in M_0$$

annihilates N because $\exists n_J \in \mathbb{N}$ such that $J^{n_J} = (x_1^{h_1}/x_0^{h_0}) I_{d_0}$ (the reduction of N_i to M_i is used here). Thus, it is enough to prove that the global dimension of $(M_0/JM_0)_{\widetilde{m}} = n-2$ (cf. the fourth paragraph of the proof of Lemma 3.3.8). Note that we have

$$M_0/JM_0 \simeq \begin{pmatrix} F_{0,0} & F_{0,1} & \cdots & F_{0,d_0-1} \\ F_{0,-1} & F_{0,0} & \cdots & F_{0,d_0-2} \\ \vdots & \vdots & \cdots & \vdots \\ F_{0,-d_0+1} & F_{0,-d_0+2} & \cdots & F_{0,0} \end{pmatrix}, \qquad (3.3.2.1)$$

where

$$F_0 := E_0/x_1E_0 \simeq k\langle x_0, x_2, \cdots, x_n\rangle/(x_jx_i - q_{ji}x_ix_j, x_0^{h_0} + x_2^{h_2} + \cdots + x_n^{h_n})_{i,j}[x_0^{-1}]$$

and $F_{0,j}$ is the degree j part of F_0.

If any of $a_2, \cdots, a_n$ is not 0, we can reduce to the case (a) from (3.3.2.1). If some of $a_2, \cdots, a_n$ are 0, repeat the above process until we can reduce to the case (a). □

Proof of Theorem 3.3.15. gl.dim(qgr($C/(f)$)) is finite. So, the balanced dualizing complex ${}^{\phi}(C/(f))^1[n]$ of $C/(f)$ induces the Serre functor of qgr($C/(f)$) from [9, Theorem A.4]. We complete the proof as in the proof of Theorem 3.3.3. □

3.4 Comparison and closed points

In this section, we calculate closed points of noncommutative projective Calabi-Yau schemes of dimensions 2 obtained in Section 3.3.2 and compare our examples with commutative Calabi-Yau varieties and the first examples constructed in [22]. In particular, we show that a noncommutative projective Calabi-Yau scheme in Section 3.3.2 gives essentially a new example of noncommutative projective Calabi-Yau schemes.

3.4.1 Closed points of noncommutative weighted hypersurfaces

Example 3.4.1. Any weight (d_0, d_1, d_2, d_3) of noncommutative projective Calabi-Yau schemes of dimensions 2 in Theorem 3.3.15 such that gcd$(d_0, d_1, d_2, d_3) = 1$ is one of the following (obtained by using a computer):

$$\begin{aligned}(d_0, d_1, d_2, d_3) =&(1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 2, 2), (1, 1, 2, 4), (1, 1, 4, 6),\\ &(1, 2, 2, 5), (1, 2, 3, 6), (1, 2, 6, 9), (1, 3, 4, 4), (1, 3, 8, 12),\\ &(1, 4, 5, 10), (1, 6, 14, 21), (2, 3, 3, 4), (2, 3, 10, 15).\end{aligned}$$

From now, we focus on the closed points of noncommutatative projective Calabi-Yau schemes of dimensions 2 in Theorem 3.3.15 whose weights are of type $(1, 1, a, b)$. We recall the notion of closed points of noncommutative projective schemes.

For simplicity, we often call an $\mathbb{N}$-graded k-algebra of the form $k\langle z_0, \cdots, z_m\rangle/(z_j z_i - p_{ji} z_i z_j)_{i,j}$ $(p_{ji} \in k^\times, m \in \mathbb{N})$ with $\deg(z_i) > 0$ and $p_{ji}p_{ij} = 1$ a weighted quantum polynomial ring. (p_{ji}) is called the quantum parameter.

Definition 3.4.2 ([33, Section 3.1])**.** Let A be a finitely generated right noetherian connected $\mathbb{N}$-graded k-algebra. A closed point of proj(A) is an object of qgr(A) represented by a 1-critical module of A. We denote by $|\mathrm{proj}(A)|$ the set of closed points of proj(A). For the definition of 1-critical modules, see [33, Definition 3.1].

Remark 3.4.3 ([33, Section 3.1]). If A is a quotient of a weighted quantum polynomial ring, then every closed point of proj(A) is one of the following:

1. An ordinary point, which is represented by a finitely generated 1-critical module of multiplicity 1.
2. A fat point, which is represented by a finitely generated 1-critical module of multiplicity > 1.
3. A thin point, which is represented by a finitely generated 1-critical module of multiplicity < 1.

For the definition of multiplicities, see [33, Definition 3.10]. In addition, if A is generated in degree 1, the notion of ordinary points and that of point modules are the same, and there are no thin points.

Let $C := k\langle x_0, x_1, x_2, x_3\rangle/(x_j x_i - q_{ji} x_i x_j)_{i,j}$ whose weight is of type $(d_0, d_1, d_2, d_3) = (1, 1, a, b)$ $(0 < a \leq b)$. We assume that $q_{ij}q_{ji} = q_{ii} = 1$ for all i, j. Since $d_0 = 1$, $C[x_0^{-1}]$ is strongly graded. So, from [33, Theorem 4.20], we have

$$|\mathrm{proj}(C)| = |\mathrm{spec}(C[x_0^{-1}]_0)| \bigsqcup |\mathrm{proj}(C/(x_0))|,$$

where we denote by $|\mathrm{spec}(C[x_0^{-1}]_0)|$ the set of simple modules of $C[x_0^{-1}]_0$. In this equality, the 1 (resp. $n > 1$)-dimensional simple modules of spec$(C[x_0^{-1}]_0)$ correspond to ordinary (resp. fat) points in proj(C). Similarly, we have

$$|\mathrm{proj}(C)| = |\mathrm{spec}(C[x_0^{-1}]_0)| \bigsqcup |\mathrm{spec}(C/(x_0)[x_1^{-1}]_0)| \bigsqcup |\mathrm{proj}(C/(x_0, x_1))|.$$

We have an isomorphism $C[x_0^{-1}]_0 \xrightarrow{\simeq} k\langle X_1, X_2, X_3\rangle/(X_jX_i - q'_{ji}X_iX_j)_{i,j}$ which sends $x_1x_0^{-1}, x_2x_0^{-a}$ and $x_3x_0^{-b}$ to X_1, X_2 and X_3, respectively. Here, $q'_{ji} := q_{0j}^{d_i}q_{ji}q_{i0}^{d_j}$ $(i, j \neq 0)$. In the same way, $C/(x_0)[x_1^{-1}]_0$ is also isomorphic to $k\langle Y_2, Y_3\rangle/(Y_3Y_2 - p_{32}Y_2Y_3)$, where $p_{32} := q_{13}^a q_{32} q_{21}^b$.

Let $C_1 := k\langle x'_0, x'_1, x'_2, x'_3\rangle/(x'_jx'_i - q'_{ji}x'_ix'_j)_{i,j}$, where $\deg(x'_i) = 1$, $q'_{0i} = q'_{j0} = 1$ for all i, j. Let $C_2 := k\langle y_1, y_2, y_3\rangle/(y_jy_i - p_{ji}y_iy_j)_{i,j}$, where $\deg(y'_i) = 1$, $p_{1i} = p_{j1} = 1$ for all i, j. Then, we can consider the point scheme of $\mathrm{proj}(C_1)$ (resp. $\mathrm{proj}(C_2)$), which is isomorphic to the set of ordinary points $|\mathrm{proj}(C_1)|_{\mathrm{ord}}$ (resp. $|\mathrm{proj}(C_2)|_{\mathrm{ord}}$) as sets. Thus, we regard $|\mathrm{proj}(C_1)|_{\mathrm{ord}}$ (resp. $|\mathrm{proj}(C_2)|_{\mathrm{ord}}$) as the point scheme of $\mathrm{proj}(C_1)$ (resp. $\mathrm{proj}(C_2)$).

Let $|\mathrm{spec}(C[x_0^{-1}]_0)|_1$ (resp. $|\mathrm{spec}(C/(x_0)[x_1^{-1}]_0)|_1$) be the set of 1-dimensional simple modules of $C[x_0^{-1}]_0$ (resp. $C/(x_0)[x_1^{-1}]_0$). Because $C_1[x_0'^{-1}]_0 \simeq C[x_0^{-1}]_0$ and $C_2[y_1^{-1}]_0 \simeq C/(x_0)[x_1^{-1}]_0$, we can think of $|\mathrm{spec}(C[x_0^{-1}]_0)|_1$ (resp. $|\mathrm{spec}(C/(x_0)[x_1^{-1}]_0)|_1$) as a locally closed subscheme of $|\mathrm{proj}(C_1)|_{\mathrm{ord}}$ (resp. $|\mathrm{proj}(C_2)|_{\mathrm{ord}}$) from [33, Theorem 4.20].

Lemma 3.4.4. *1. If $q'_{ji} \neq 1$ for all $i, j \neq 0$, $|\mathrm{spec}(C[x_0^{-1}]_0)|_1$ is a union of three affine lines.*

2. If $p_{32} \neq 1$, $|\mathrm{spec}(C/(x_0)[x_1^{-1}]_0)|_1$ is a union of two affine lines. Otherwise, $|\mathrm{spec}(C/(x_0)[x_1^{-1}]_0)|_1 \simeq \mathbb{A}^2$.

Proof. (2) is well-known (for example, see [47, Section 4.3]). Regarding (1), under the assumption of the lemma, $\mathrm{proj}(C_1)$ belongs to case (3) or case (4) in [54, Corollary 5.1]. This shows that $|\mathrm{spec}(C_1[x_0'^{-1}]_0)|_1$ is isomorphic to $\bigcup_{i\neq j} Z(X'_i, X'_j) \subset \mathbb{A}^3 = \mathrm{Spec}(k[X'_1, X'_2, X'_3])$ (cf. [54, Proposition 4.2] or [5, Theorem 1]). □

Remark 3.4.5. We consider the weights $(1, 1, a, b)$ and the quantum parameters which give noncommutative projective Calabi-Yau schemes of dimensions 2 in Theorem 3.3.15. Then, we can check that if $p_{32} \neq 1$, then $q'_{ji} \neq 1$ for all $i, j \neq 0$ by using a computer. Moreover, if $p_{32} = 1$, then $q'_{ji} = 1$ for all $i, j \neq 0$. In this case, $|\mathrm{spec}(C[x_0^{-1}]_0)|_1 \simeq \mathbb{A}^3$.

We consider $C/(x_0, x_1) = k\langle x_2, x_3\rangle/(x_3x_2 - q_{32}x_2x_3)$. Then, it is known that a weighted quantum polynomial ring of 2 variables is a twisted algebra of a commutative weighted polynomial ring $k[x, y]$ with $\deg(x) = a > 0, \deg(y) = b > 0$ (for example, see [49, Example 4.1] or [64, Example 3.6]). So, it is enough to consider the closed points of $\mathrm{proj}(k[x, y])$. We want to study the closed points of $\mathrm{proj}(k[x, y])$ in the case of $(a, b) = (2, 2), (2, 4)$ or $(4, 6)$. Note that when $(a, b) = (1, 1)$ or $(1, 3)$, they are classified in [33, Theorem 3.16]. We treat a more general setting below.

Lemma 3.4.6. *Let $R = k[x, y]$ be a commutative weighted polynomial ring with $\deg(x) = a > 0, \deg(y) = b > 0$. Let $g := \gcd(a, b), a' := a/g$ and $b' := b/g$. Then, every closed point of $\mathrm{proj}(R)$ is one of the following:*

1. $\pi R/(x)(-i),\ i = 0, \cdots, b-1$.

2. $\pi R/(y)(-j),\ j = 0, \cdots, a-1$.

3. $\pi R/(\beta x^{b'} - \alpha y^{a'})(-k)$, *where* $(\alpha, \beta) \in \mathbb{P}^1 \backslash \{(0,1),(1,0)\}$ *and* $k = 0, \cdots, g-1$.

Moreover, all of them are not isomorphic in $\mathrm{proj}(R)$.

Proof. The proof is almost the same as the proof of [33, Lemma 3.15, Theorem 3.16]. We give the sketch of the proof.

Firstly, every closed point of $\mathrm{proj}(R)$ is represented by a cyclic critical Cohen-Macaulay module of depth 1. Then, $M \in \mathrm{gr}(R)$ satisfies these conditions and is generated in degree 0 if and only if M is isomorphic to one of $R/(x), R/(y)$ or $R/(\beta x^{b'} - \alpha y^{a'})$ $(\alpha, \beta \in k^{\times})$. Since being cyclic critical Cohen-Macaulay of depth 1 is invariant under shifting, any closed point is represented by some shifts of one of the above modules (that is, $R/(x)(-l), R/(y)(-l), R/(\beta x^{b'} - \alpha y^{a'})(-l),\ l \in \mathbb{Z}$).

Finally, we classify the isomorphic classes of these modules in $\mathrm{proj}(R)$. We have no isomorphisms between the three types of closed points by considering their Hilbert polynomials and multiplicities. Then, we have $\pi R/(\beta x^{b'} - \alpha y^{a'}) \simeq \pi R/(\beta x^{b'} - \alpha y^{a'})(-gl)$, $({}^{\forall} l \in \mathbb{Z}, {}^{\forall}(\alpha, \beta) \in \mathbb{P}^1 \backslash \{(1,0),(0,1)\})$. We also have $\pi R/(\beta x^{b'} - \alpha y^{a'}) \simeq \pi R/(\beta' x^{b'} - \alpha' y^{a'})$ if and only if $(\alpha, \beta) = (\alpha', \beta')$ in $\mathbb{P}^1 \backslash \{(1,0),(0,1)\}$. In addition, we can show that $\pi R/(x) \simeq \pi R/(x)(-i)$ (resp. $\pi R/(y) \simeq \pi R/(y)(-j)$) if and only if $i \equiv 0 \pmod b$ (resp. $j \equiv 0 \pmod a$). From these discussions, we get the claim. □

3.4.2 Closed points of noncommutatative projective Calabi-Yau schemes and a result about comparison

We can study ordinary and thin points of noncommutative projective Calabi-Yau schemes of dimensions 2 in Theorem 3.3.15 by using the above investigations. We give examples of noncommutative projective Calabi-Yau schemes whose moduli of ordinary closed points are different from those in [22, Proposition 3.4] and commutative Calabi-Yau varieties.

Example 3.4.7. We consider the weight $(1,1,2,2)$ and the quantum parameter

$$\mathbf{q} = (q_{ij}) = \begin{pmatrix} 1 & 1 & 1 & \omega^2 \\ 1 & 1 & \omega^2 & 1 \\ 1 & \omega & 1 & 1 \\ \omega & 1 & 1 & 1 \end{pmatrix}, \quad \omega := \frac{-1 + i\sqrt{3}}{2}.$$

Then, we have

$$\mathbf{q}' = (q'_{ij}) = \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad q_{13}^2 q_{32} q_{21}^2 = \omega^2.$$

From Lemma 3.4.4 and Lemma 3.4.6, the set of ordinary and thin points

$$|\mathrm{proj}(C/(f))|_{\text{ord \& thin}} = |\mathrm{spec}(C/(f)[x_0^{-1}]_0)|_1 \bigsqcup |\mathrm{spec}(C/(f,x_0)[x_1^{-1}]_0)|_1$$
$$\bigsqcup |\mathrm{proj}(C/(f,x_0,x_1))|$$

is 24 points. To be more precise, we have $|\mathrm{spec}(C/(f)[x_0^{-1}]_0)|_1 = \bigsqcup_{i\neq j} Z(X_i, X_j, 1 + X_1^6 + X_2^3 + X_3^3) \subset \mathbb{A}^3$, $|\mathrm{spec}(C/(f,x_0)[x_1^{-1}]_0)|_1 = \bigsqcup_{i=1,2} Z(Y_i, 1+Y_2^3+Y_3^3)$ and $|\mathrm{proj}(C/(f,x_0,x_1))| = \{3\mathrm{pts}\} \sqcup \{3\mathrm{pts}\}$.

This calculation shows that for a fixed weight, if the set of ordinary and thin points of $\mathrm{proj}(C/(f))$ is finite, then the cardinality is independent of the quantum parameters.

From the method in Example 3.4.7, Remark 3.4.5 and a direct computation, we have the following.

Proposition 3.4.8. *For a weight* $(1,1,a,b)$ *in Example 3.4.1 and a quantum parameter* $\mathbf{q}$ *which gives a noncommutative projective Calabi-Yau scheme, if the set of ordinary and thin points of* $\mathrm{proj}(C/(f))$ *is finite, then the cardinality is always* 24.

The following proposition shows that some of noncommutative projective Calabi-Yau schemes of dimensions 2 in Theorem 3.3.15 are essentially new examples.

Proposition 3.4.9. *There exists a noncommutative projective Calabi-Yau scheme of dimension 2 which is obtained in Theorem 3.3.15 and not isomorphic to either commutative Calabi-Yau surfaces or noncommutative projective Calabi-Yau schemes of dimensions 2 obtained in [22].*

Proof. We divide the proof into four steps.

Step 1. We choose the weight $(1,1,a,b)$ and the quantum parameter $\mathbf{q}$ as in Example 3.4.7. Then, the number of ordinary and thin points of $\mathrm{proj}(C/(f))$ is finite. So, $\mathrm{proj}(C/f)$ is not isomorphic to any commutative Calabi-Yau surfaces.

Step 2. We prove that $\mathrm{proj}(C/(f))$ is not isomorphic to any noncommutative projective Calabi-Yau schemes of dimensions 2 in [22]. To prove this, we use the theory established in [8]. First, note that we can think of $\mathrm{qgr}(C/(f))$ as the category of coherent modules of a sheaf $\mathcal{A}$ of algebras on the projective spectrum $\mathrm{Proj}(k[s_0,s_1,s_2,s_3]/(s_0+s_1+s_2+s_3))$ (cf. the proof of Lemma 3.3.21). We define a sheaf $\mathcal{Z}_{\mathcal{A}}$ to be the sheaf whose sections are

$$\Gamma(U,\mathcal{Z}_{\mathcal{A}}) = \{s \in \Gamma(U,\mathcal{A}) \mid s|_V \in Z(\Gamma(V,\mathcal{A})), {}^{\forall}V \subset U : \text{open}\}$$

for all open subsets U (cf. [8, Proposition 2.11]). In particular, if U is affine, $\Gamma(U,\mathcal{Z}_{\mathcal{A}}) = Z(\Gamma(U,\mathcal{A}))$. Then, we show that $\mathrm{Spec}(Z(\Gamma(D_+(s_i),\mathcal{A})))$ has 4 singular points when $i = 0,1$ and a 1-dimensional singular locus when $i = 2,3$. In the following, we verify this claim for $i = 0,2$. Similarly, the claim is proved for $i = 1, 3$. In the following, we write Z_i as $Z(\Gamma(D_+(s_i), \mathcal{A}))$ for any i. We also use the notations in the proof of Lemma 3.3.21.

When $i = 0$, any $m \in Z_0$ is of the form $m = \left(\begin{smallmatrix} \mu_1 e & 0 \\ 0 & \mu_2 e \end{smallmatrix}\right) \in N_0$, $(e \in E_{0,0}, \mu_1, \mu_2 \in k^\times)$ from the definition of $\mathcal{A}$. We have

$$E_{0,0} \simeq k\langle X_1, X_2, X_3\rangle (X_j X_i - q'_{ji} X_i X_j, 1 + X_1^6 + X_2^3 + X_3^3)_{i,j},$$

which is obtained from the identifications $X_1 = x_1 x_0^{-1}, X_2 = x_2 x_0^{-2}$ and $X_3 = x_3 x_0^{-2}$. Here, the q'_{ji} are as in Example 3.4.7. So, we have

$$Z(E_{0,0}) \simeq k[Y, Z, W, U]/(1 + Y^2 + Z + W, YZW - \lambda_1 U^3) \ (\lambda_1 \in k^\times),$$

which is obtained from the identifications $Y = (x_1 x_0^{-1})^3, Z = (x_2 x_0^{-2})^3, W = (x_3 x_0^{-2})^3$ and $U = (x_1 x_0^{-1})(x_2 x_0^{-2})(x_3 x_0^{-2})$. On the other hand, we define the inclusion $\phi : Z(E_{0,0}) \to N_0$ in which Y, Z, W are mapped naturally and U to $\left(\begin{smallmatrix} U & 0 \\ 0 & \omega U \end{smallmatrix}\right)$. It is easy to see that $\phi(Z(E_{0,0})) \subset Z_0$. Because the choice of μ_1 determines μ_2 in the above form of m, the map ϕ induces $Z_0 \simeq Z(E_{0,0})$. Thus, one can show that $\mathrm{Spec}(Z_0)$ has 4 singular points by using the Jacobi criterion.

When $i = 2$, any $m \in Z_2$ is of the form $m = \left(\begin{smallmatrix} \mu_1 e & 0 \\ 0 & \mu_2 e \end{smallmatrix}\right) \in N_2$, $(e \in E_{2,0}, \mu_1, \mu_2 \in k^\times)$ from the definition of $\mathcal{A}$. We also have

$$E_{2,0} \simeq k\langle X_0, X_1, X_2, X_3\rangle \Big/ \begin{pmatrix} X_j X_i - q''_{ji} X_i X_j, \\ 1 + X_0^6 + X_1^6 + X_3^3, X_0 X_1 - \lambda_2 X_2^2 \end{pmatrix}_{i,j} \ (\lambda_2 \in k^\times),$$

which is obtained from the identifications $X_0 = x_0^2 x_2^{-1}, X_1 = x_1^2 x_2^{-1}, X_2 = x_0 x_1 x_2^{-1}$ and $X_3 = x_3 x_2^{-1}$. Here, the q''_{ij} are defined by the matrix

$$(q''_{ij}) = \begin{pmatrix} 1 & \omega & \omega^2 & \omega \\ \omega^2 & 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 & 1 \\ \omega^2 & \omega & 1 & 1 \end{pmatrix}.$$

So, we have

$$Z(E_{2,0}) \simeq k[X, Y, W, U, V] \Big/ \begin{pmatrix} X + Y + 1 + W, \\ XY - \lambda_3 U^2, XYW - \lambda_4 V^2 \end{pmatrix} \ (\lambda_3, \lambda_4 \in k^\times),$$

which is obtained from the identifications $X = (x_0^2 x_2^{-1})^3, Y = (x_1^2 x_2^{-1})^3, W = (x_3 x_2^{-1})^3, U = (x_0 x_1 x_2^{-1})^3$ and $V = (x_0 x_1 x_2^{-1})(x_3 x_2^{-1})$. On the other hand, we define the inclusion $\phi : Z(E_{2,0}) \to N_2$ in which X, Y, W, U are mapped naturally and V to $\left(\begin{smallmatrix} V & 0 \\ 0 & \omega V \end{smallmatrix}\right)$. It is easy to see that $\phi(Z(E_{2,0})) \subset Z_2$. Because the choice of μ_1 determines μ_2 in the above form of m, the map ϕ induces $Z_2 \simeq Z(E_{2,0})$. Thus, one can show that $\mathrm{Spec}(Z_2)$ has a 1-dimensional singular locus by using the Jacobi criterion.

Step 3. We consider the weight $(1, 1, 1, 1)$ and take a quantum parameter which gives a noncommutative projective Calabi-Yau scheme $\mathrm{proj}(C'/(f'))$ whose point scheme is finite. $\mathrm{qgr}(C'/(f'))$ is thought of as the category of coherent modules of a sheaf $\mathcal{B}$ of algebras on the projective spectrum $\mathrm{Proj}(k[t_0, t_1, t_2, t_3]/(t_0 + t_1 + t_2 + t_3))$.

The number of the choices of quantum parameters (q_{ij}) which satisfy the conditions of Theorem 3.3.15 and give a noncommutative projective Calabi-Yau scheme whose moduli space of point modules is finite is 20 up to permutating variables (we get the list below by using a computer and hand calculations):

$$1.\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix},2.\begin{pmatrix}1&1&1&1\\1&1&-i&i\\1&i&1&-i\\1&-i&i&1\end{pmatrix},3.\begin{pmatrix}1&1&1&-1\\1&1&-1&1\\1&-1&1&1\\-1&1&1&1\end{pmatrix},4.\begin{pmatrix}1&1&1&-1\\1&1&-i&-i\\1&i&1&i\\-1&i&-i&1\end{pmatrix},$$

$$5.\begin{pmatrix}1&1&1&i\\1&1&-1&-i\\1&-1&1&-i\\-i&i&i&1\end{pmatrix},6.\begin{pmatrix}1&1&1&i\\1&1&-i&-1\\1&i&1&1\\-i&-1&1&1\end{pmatrix},7.\begin{pmatrix}1&1&1&-i\\1&1&-1&i\\1&-1&1&i\\i&-i&-i&1\end{pmatrix},8.\begin{pmatrix}1&1&1&-i\\1&1&-i&1\\1&i&1&-1\\i&1&-1&1\end{pmatrix},$$

$$9.\begin{pmatrix}1&1&-1&-1\\1&1&-i&i\\-1&i&1&i\\-1&-i&-i&1\end{pmatrix},10.\begin{pmatrix}1&1&-1&i\\1&1&i&-1\\-1&-i&1&-1\\-i&-1&-1&1\end{pmatrix},11.\begin{pmatrix}1&1&-1&-i\\1&1&-i&-1\\-1&i&1&-1\\i&-1&-1&1\end{pmatrix},12.\begin{pmatrix}1&1&i&i\\1&1&-i&-i\\-i&i&1&-1\\-i&i&-1&1\end{pmatrix},$$

$$13.\begin{pmatrix}1&1&i&-i\\1&1&-i&i\\-i&i&1&1\\i&-i&1&1\end{pmatrix},14.\begin{pmatrix}1&-1&-1&-1\\-1&1&-1&-1\\-1&-1&1&-1\\-1&-1&-1&1\end{pmatrix},15.\begin{pmatrix}1&-1&-1&-1\\-1&1&-i&i\\-1&i&1&-i\\-1&-i&i&1\end{pmatrix},16.\begin{pmatrix}1&-1&-1&i\\-1&1&-1&i\\-1&-1&1&i\\-i&-i&-i&1\end{pmatrix},$$

$$17.\begin{pmatrix}1&-1&-1&-i\\-1&1&-1&-i\\-1&-1&1&-i\\i&i&i&1\end{pmatrix},18.\begin{pmatrix}1&-1&i&i\\-1&1&i&i\\-i&-i&1&-1\\-i&-i&-1&1\end{pmatrix},19.\begin{pmatrix}1&i&i&i\\-i&1&-i&i\\-i&i&1&-i\\-i&-i&i&1\end{pmatrix},20.\begin{pmatrix}1&i&i&-i\\-i&1&-i&-i\\-i&i&1&i\\i&i&-i&1\end{pmatrix}.$$

When we choose one (q_{ij}) of the above 20 quantum parameters, then for any l, we have

$$\Gamma(D_+(t_l),\mathcal{B}) \simeq k\langle Y_1,Y_2,Y_3\rangle/(Y_iY_j - q'_{ij}Y_jY_i, Y_1^4+Y_2^4+Y_3^4+1)_{1\le i,j\le 3},$$

where (q'_{ij}) is represented by one of the following matrices (we can verify this with direct calculations):

$$(a).\begin{pmatrix}1&-1&-1\\-1&1&-1\\-1&-1&1\end{pmatrix},\qquad (b).\begin{pmatrix}1&-i&i\\i&1&-i\\-i&i&1\end{pmatrix}.$$

We write $Z'_l := Z(\Gamma(D_+(t_l),\mathcal{B}))$. When (q'_{ij}) is type (a), $\mathrm{Spec}(Z'_l)$ has 6 singular points because Z'_l is generated by Y_1^2, Y_2^2, Y_3^2 and $Y_1Y_2Y_3$ as a k-algebra. When (q'_{ij}) is type (b), $\mathrm{Spec}(Z'_l)$ has 3 singular points because Z'_l is generated by Y_1^4, Y_2^4, Y_3^4 and $Y_1Y_2Y_3$ as a k-algebra. Moreover, for any (q_{ij}) in the above table, if $\mathcal{B}$ is type (a) (resp. (b)) on $D_+(t_l)$ for some l, it is also type (a) (resp. (b)) on $D_+(t_l)$ for any other l.

Step 4. If $\mathrm{qgr}(C/(f))$ is equivalent to $\mathrm{qgr}(C'/(f'))$ then, we must have an isomorphism of schemes between $\mathrm{Spec}(\mathcal{Z}_\mathcal{A})$ and $\mathrm{Spec}(\mathcal{Z}_\mathcal{B})$ by [8, Theorem 4.4] (cf. [3, Section 6]). Since $\mathrm{Spec}(\mathcal{Z}_\mathcal{A})$ has infinitely many singular points, but, $\mathrm{Spec}(\mathcal{Z}_\mathcal{B})$ has finitely many singular points, such a situation does not happen. Hence, we complete the proof. □

Bibliography

[1] E. Arbarello et al. *Geometry of algebraic curves. Vol. I.* Vol. 267. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985.

[2] E. Arbarello, M. Cornalba, and P. A. Griffiths. *Geometry of algebraic curves. Volume II.* Vol. 268. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With a contribution by Joseph Daniel Harris. Springer, Heidelberg, 2011.

[3] M. Artin and J. J. Zhang. "Noncommutative Projective Schemes". *Adv. Math.* **109.2** (1994), pp. 228–287.

[4] L. Badescu. *Projective geometry and formal geometry.* Vol. 65. Birkhauser, 2012.

[5] P. Belmans, K. De Laet, and L. Le Bruyn. "The point variety of quantum polynomial rings". *Journal of Algebra* **463** (2016), pp. 10–22.

[6] A. Bondal and M. Van den Bergh. "Generators and representability of functors in commutative and noncommutative geometry". *Mosc. Math. J.* **3.1** (2003), pp. 1–36, 258.

[7] M. P. Brodmann and R. Y. Sharp. *Local cohomology: an algebraic introduction with geometric applications.* Vol. 136. Cambridge university press, 2012.

[8] I. Burban and Y. Drozd. "Morita theory for non–commutative Noetherian schemes". *Adv. Math.* **399** (2022), pp. 1–42.

[9] K. De Naeghel and M. Van den Bergh. "Ideal classes of three-dimensional Sklyanin algebras". *J. Algebra* **276.2** (2004), pp. 515–551.

[10] D. Eisenbud. *The geometry of syzygies: A second course in commutative algebra and algebraic geometry.* Vol. 229. Springer, 2005.

[11] M. Emerton and T. Gee. *Dimension theory and components of algebraic stacks.* 2017. arXiv: `1704.07654 [math.AG]`.

[12] P. Gabriel. "Des catégories abéliennes". *Bull. Soc. Math. France* **90** (1962), pp. 323–448.

[13] V. Ginzburg. *Calabi-Yau algebras.* 2006. arXiv: `math / 0612139 [math.AG]`.

[14] T. L. Gómez, I. Sols, and A. Zamora. "A GIT interpretation of the Harder–Narasimhan filtration". *Rev. Mat. Complut.* **28**.**1** (2015), pp. 169–190.

[15] J.-W. He and K. Ueyama. "Twisted Segre products". *J. Algebra* **611** (2022), pp. 528–560.

[16] V. Hoskins. "Stratifications for moduli of sheaves and moduli of quiver representations". *Algebr. Geom.* **5**.**6** (2018), pp. 650–685.

[17] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves.* Cambridge University Press, 2010.

[18] E. Hyry. "The diagonal subring and the Cohen-Macaulay property of a multigraded ring". *Trans. Amer. Math. Soc.* **351**.**6** (1999), pp. 2213–2232.

[19] A. R. Iano-Fletcher. "Working with weighted complete intersections". *Explicit Birational Geometry of 3-folds.* Vol. 281. London Mathematical Society Lecture Note Series. Cambridge University Press, 2000, 101–174.

[20] D. Joyce and Y. Song. *A theory of generalized Donaldson-Thomas invariants.* American Mathematical Soc., 2012.

[21] D. Kaledin, M. Lehn, and C. Sorger. "Singular symplectic moduli spaces". *Inventiones mathematicae* **164**.**3** (2006), pp. 591–614.

[22] A. Kanazawa. "Non-commutative projective Calabi–Yau schemes". *J. Pure Appl. Algebra* **219**.**7** (2015), pp. 2771–2780.

[23] M. Kimura and K. Yoshioka. "Birational Maps of Moduli Spaces of Vector Bundles on $K3$ Surfaces". *Tokyo J. Math.* **34**.**2** (2011), pp. 473–491.

[24] K. Kurihara and K. Yoshioka. "Holomorphic vector bundles on non-algebraic tori of dimension 2". *Manuscripta Math.* **126**.**2** (2008), pp. 143–166.

[25] A. Kuznetsov. "Calabi–Yau and fractional Calabi–Yau categories". *J. Reine Angew. Math.* **2019**.**753** (2019), pp. 239–267.

[26] G. Laumon and L. Moret-Bailly. *Champs algébriques.* Vol. 39 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Berlin: Springer, 2000.

[27] Y.-H. Liu. "Donaldson–Thomas theory of quantum Fermat quintic threefolds I". *arXiv.1911.07949* (2019).

[28] Y.-H. Liu. "Donaldson–Thomas theory of quantum Fermat quintic threefolds II". *arXiv:2004.10346* (2020).

[29] J. C. McConnell, J. C. Robson, and L. W. Small. *Noncommutative Noetherian rings.* Vol. 30. American Mathematical Soc., 2001.

[30] R. Mckemey. *Relative local cohomology.* Thesis, University of Manchester. 2012.

[31] H. Minamide, S. Yanagida, and K. Yoshioka. "The wall-crossing behavior for Bridgeland's stability conditions on abelian and K3 surfaces". *J. Reine Angew. Math.* **2018.735** (2018), pp. 1–107.

[32] I. Mori. "B-construction and C-construction". *Comm. Algebra* **41.6** (2013), pp. 2071–2091.

[33] I. Mori. "Regular modules over 2-dimensional quantum Beilinson algebras of Type S". *Math. Z.* **279.3** (2015), pp. 1143–1174.

[34] I Mori and A Nyman. "Corrigendum to "Local duality for connected $\mathbb{Z}$-algebras" [J. Pure Appl. Algebra 225 (9)(2019) 106676]". *J. Pure Appl. Algebra* **228.3** (2024), p. 107493.

[35] I. Mori and A. Nyman. "Local duality for connected $\mathbb{Z}$-algebras". *J. Pure Appl. Algebra* **225.9** (2021), Paper No. 106676, 22.

[36] I. Mori and A. Nyman. *A categorical characterization of quantum projective $\mathbb{Z}$-spaces*. 2023. arXiv: 2307.15253 [math.RA].

[37] S. Mukai. "On the moduli space of bundles on K3 surfaces. I". *Tata Inst. Fundam. Res. Stud. Math* **11** (1984), pp. 341–413.

[38] S. Mukai. "Symplectic structure of the moduli space of sheaves on an abelian or K3 surface". *Invent. Math.* **77.1** (1984), pp. 101–116.

[39] N. Nitsure. "Schematic harder–narasimhan stratification". *Internat. J. Math.* **22.10** (2011), pp. 1365–1373.

[40] C. Okonek, M. Schneider, and H. Spindler. *Vector Bundles on Complex Projective Spaces: With an Appendix by SI Gelfand*. Springer Science & Business Media, 2011.

[41] M. Reid. *Canonical 3-folds*. Journees de geometrie algebrique, Angers/France 1979, 273-310. 1980.

[42] M. Reyes, D. Rogalski, and J. J. Zhang. "Skew Calabi–Yau algebras and homological identities". *Adv. Math.* **264** (2014), pp. 308–354.

[43] M. L. Reyes and D. Rogalski. "Graded twisted Calabi–Yau algebras are generalized Artin–Schelter regular". *Nagoya Math. J.* **245** (2022), pp. 100–153.

[44] J. P. Serre. "Faisceaux algébriques cohérents". *Ann. of Math. (2)* **61** (1955), pp. 197–278.

[45] S. S. Shatz. "The decomposition and specialization of algebraic families of vector bundles". *Compos. Math.* **35.2** (1977), pp. 163–187.

[46] S. P. Smith. "Maps between non-commutative spaces". *Trans. Amer. Math. Soc.* **356.7** (2004), pp. 2927–2944.

[47] S. P. Smith. *Noncommutaive algebraic geometry*. Lecture Notes, University of Washington. 2000. URL: https://sites.math.washington.edu/~smith/Research/nag.pdf.

[48] S. P. Smith. "Subspaces of non-commutative spaces". *Trans. Amer. Math. Soc.* **354.6** (2002), pp. 2131–2171.

[49] D. R. Stephenson. "Quantum planes of weight $(1, 1, n)$". *J. Algebra* **225.1** (2000), pp. 70–92.

[50] The Stacks Project Authors. *Stacks Project.* URL: `https://stacks.math.columbia.edu`.

[51] M. Van den Bergh. "Calabi-Yau algebras and superpotentials". *Selecta Math. (N.S.)* **21.2** (2015), pp. 555–603.

[52] M. Van den Bergh. "Existence Theorems for Dualizing Complexes over Non-commutative Graded and Filtered Rings". *J. Algebra* **195.2** (1997), pp. 662–679.

[53] K. Van Rompay. "Segre Product of Artin–Schelter Regular Algebras of Dimension 2 and Embeddings in Quantum $\mathbb{P}^3$'s". *J. Algebra* **180.2** (1996), pp. 483–512.

[54] J. Vitoria. *Equivalences for noncommutative projective spaces.* 2011. arXiv: `1001.4400 [math.RA]`.

[55] R. Vyas and A. Yekutieli. "Weak proregularity, weak stability, and the noncommutative MGM equivalence". *J. Algebra* **513** (2018), pp. 265–325.

[56] C. H. Walter. "Components of the stack of torsion-free sheaves of rank 2 on ruled surfaces". *Math. Ann.* **301** (1995), pp. 699–715.

[57] A. Yekutieli. *Derived Categories.* Vol. 183. Cambridge University Press, 2019.

[58] A. Yekutieli. "Dualizing complexes over noncommutative graded algebras". *J. Algebra* **153.1** (1992), pp. 41–84.

[59] A. Yekutieli and J. J. Zhang. "Serre duality for noncommutative projective schemes". *Proc. Amer. Math. Soc.* **125.3** (1997), pp. 697–707.

[60] K. Yoshioka. "Fourier–Mukai transform on abelian surfaces". *Math. Ann* **345.3** (2009), pp. 493–524.

[61] K. Yoshioka. "Some examples of Mukai's reflections on K3 surfaces". *J. Reine Angew. Math.* **515** (1999), pp. 97–123.

[62] K. Yoshioka. "Twisted stability and Fourier–Mukai transform I". *Compos. Math.* **138.3** (2003), pp. 261–288.

[63] K. Yoshioka. *Irreducibility of moduli spaces of vector bundles on K3 surfaces.* 1999. arXiv: `math/9907001 [math.AG]`.

[64] J. J. Zhang. "Twisted graded algebras and equivalences of graded categories". *Proc. Lond. Math. Soc.* **3.2** (1996), pp. 281–311.

List of papers by Yuki MIZUNO

(1) Y. Mizuno, *Classifying the irreducible components of moduli stacks of torsion-free sheaves on K3 surfaces and an application to Brill-Noether theory*, J. Geom. Phys. 179.

(2) Y. Mizuno, *Some examples of noncommutative projective Calabi-Yau schemes*, to appear in Canad. Math. Bull.