Applications of Serre correspondence to classification problems of ACM bundles and globally generated vector bundles

セール対応のACM束および大域生成ベクトル束の分類問題への応用

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Chapter 1

Introduction

In this thesis, we investigate the classification of ACM bundles and globally generated vector bundles on nonsingular projective algebraic varieties. We use the Serre correspondence for rank two vector bundles on surfaces and the generalized Hartshorne-Serre correspondence for vector bundles of higher rank on threefolds. The classification problem of vector bundles is reduced to the classification problem of subvarieties of codimension 2 in the base space by these correspondences. This thesis consists of two main parts.

In Chapter 2, we consider rank 2 Arithmetically Cohen–Macaulay (ACM) bundles. On a nonsingular projective variety X of dimension n with a polarization $\mathcal{O}_X(h)$, ACM bundles are defined as locally free sheaves $\mathcal{E}$ such that $h^i(X, \mathcal{E}(th)) = 0$ for $t \in \mathbb{Z}, 1 \leq i \leq n-1$. In a more algebraic context, an ACM bundle corresponds to a maximal Cohen-Macaulay module over the homogeneous coordinate ring of X. From this perspective as well, ACM bundles have been a subject of research for a long time.

The property of being ACM is not affected by tensoring by $\mathcal{O}_X(th)$, thus we can restrict our attention to ACM bundles which satisfy the conditions $h^0(X, \mathcal{E}) \neq 0$ and $h^0(X, \mathcal{E}(-h)) = 0$ (we briefly say that $\mathcal{E}$ is initialized) and which are indecomposable, i.e. bundles which do not split as a direct sum of bundles of lower ranks. On a fixed pair $(X, \mathcal{O}_X(h))$, there exist initialized ACM bundles with the maximal number of global sections, known as Ulrich bundles. Ulrich bundles have various intriguing properties, particularly being globally generated. This motivates us to embark on a study in Chapter 3.

The classification of ACM bundles on algebraic varieties is a fundamental and classical topic in algebraic geometry. In 1964, G. Horrocks [27] proved that on $\mathbb{P}^n$, a vector bundle $\mathcal{E}$ splits as a direct sum of line bundles if and only if $\mathcal{E}$ is an ACM bundle. This marked the starting point for the study of ACM bundles.

In 1988, D. Eisenbud and J. Herzog [22] provided the complete list of varieties that have only finitely many initialized and indecomposable ACM bundles. For instance, on $\mathbb{P}^n$, the only initialized and indecomposable ACM bundle is $\mathcal{O}_{\mathbb{P}^n}$ by the theorem of Horrocks. Among these varieties is the smooth quadric surface $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$ in $\mathbb{P}^3$, where we write $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ for $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$.

Recently, there has been a focus on the classification of ACM bundles of low rank on other surfaces and threefolds, such as hypersurfaces of low degrees and del Pezzo threefolds. For instance, G, Casnati, D. Faenzi, M. Filip and F. Malaspina classified rank 2 ACM bundles on the del Pezzo threefolds of degree 6 and 7 [15, 16, 17].

The classification of ACM bundles on a specific variety heavily depends on the chosen embedding. For instance, while the quadric surface $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$ in $\mathbb{P}^3$ is a well-known example of a surface with only finitely many initialized indecomposable ACM bundles, there are currently no known results regarding the classification of ACM bundles of low rank on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to the general polarization $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$. Hence we pose the following problem.

Problem 1.0.1. *Classify ACM bundles of rank* 2 *on* $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b))$.

Faenzi and Malaspina treated the case of $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2))$ in Theorem 3.1 of [24]. Therefore, we consider rank 2 ACM bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to the polarization $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)$, which is a del Pezzo surface of degree 8. On the classification of ACM bundles on del Pezzo surfaces, Faenzi [23] classified ACM bundles of rank 2 on cubic surfaces.

Our result is the following.

Theorem 1.0.2. *Let $\mathcal{E}$ be an initialized indecomposable ACM bundle of rank* 2 *on* $(X := \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_X(2,2))$, *with Chern classes* $c_1 = \mathcal{O}_X(a,b)$ *(we write* $c_1 = (a,b)$ *for short), where we may assume* $a \le b$ *by symmetry, and* $c_2 \in \mathbb{Z}$.

Then one of the following assertions holds.

1. $c_1 = (0,0), c_2 = 1$. *The zero–locus of its general section is a point.*
2. $c_1 = (0,1), c_2 = 1$. *The zero–locus of its general section is a point.*
3. $c_1 = (3,4), c_2 = 7$. *The zero–locus of its general section is a* 0*-dimensional degree* 7 *subscheme contained in a pencil of hyperplanes and which is not the complete intersection of curves* $C_1 \in |\mathcal{O}_X(2,1)|$ *and* $C_2 \in |\mathcal{O}_X(1,3)|$.
4. $c_1 = (3,5), c_2 = 9$. *In this case $\mathcal{E}$ is an Ulrich bundle. The zero–locus of its general section is a* 0*-dimensional non–degenerate degree* 9 *subscheme.*
5. $c_1 = (4,4), c_2 = 10$. *In this case $\mathcal{E}$ is an Ulrich bundle. The zero–locus of its general section is a* 0*-dimensional non–degenerate degree* 10 *subscheme which is not the complete intersection of curves* $C_1 \in |\mathcal{O}_X(1,3)|$ *and* $C_2 \in |\mathcal{O}_X(3,1)|$.

All the above cases actually occur on X.

By the Serre correspondence, we can construct a rank 2 vector bundle on a surface if the local complete intersection subscheme of pure codimension 2 satisfies the Cayley–Bacharach condition. In the proof, it is first shown that the zero–locus of a general section of an ACM bundle is pure of codimension 2. Consequently, all ACM bundles can be obtained from the zero–locus of a general section by the Serre correspondence, leading to the classification result.

In Chapter 3, we explore globally generated bundles. The works on the classification of ACM and Ulrich bundles in recent years have prompted an interest in the investigation of globally generated bundles, since Ulrich bundles are globally generated.

Of course, globally generated vector bundles themselves are significant objects in algebraic geometry. Nevertheless, even in the case of $\mathbb{P}^n$, the classification of such bundles with a small first Chern class fixed has only been explored relatively recently, as indicated by the works of [1, 29, 36, 37]. A globally generated vector bundle on $\mathbb{P}^n$ corresponds to a map from $\mathbb{P}^n$ to a grassmannian. In 2006, J. C. Sierra and L. Ugaglia [35] investigated globally generated vector bundles of rank 2 on $\mathbb{P}^n$ from this viewpoint. They further extended their investigation in 2009 to classify globally generated vector bundles of arbitrary rank on $\mathbb{P}^n$ with $c_1 = 2$ [36].

Following this line of research, E. Ballico, S. Huh, and Malaspina explored globally generated vector bundles with a fixed small first Chern class on various projective varieties, including smooth quadric threefolds [6], complete intersection Calabi-Yau threefolds [7], and Segre threefolds $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ [8] and $\mathbb{P}^1 \times \mathbb{P}^2$ [9]. Additionally, Ballico studied the case of projective space blown up at finitely many points [5]. These results are obtained by investigating nonsingular curves corresponding to globally generated vector bundles via the generalized Hartshorne-Serre correspondence.

According to the Hartshorne-Serre correspondence [3], on a nonsingular variety, if a pure codimension 2 local complete intersection subscheme satisfies certain conditions, then a vector bundles of rank $r \geq 2$ can be constructed. The given subscheme is then the degeneracy locus of $r-1$ global sections of the constructed vector bundle. Conversely, it is known that for a globally generated vector bundles of rank r, there exist $r-1$ global sections the degeneracy locus of which is a pure codimension 2 subscheme, and furthermore, it is known to become nonsingular.

The classifications of globally generated vector bundles on various threefolds by Ballico, Huh, Malaspina are inspired by the work on the classifications of ACM bundles on del Pezzo threefolds by Casnati, Faenzi, Malaspina [15, 16, 17], as mentioned by the authors [8]. More recently, Casnati and Genc [18] investigated instanton bundles on two fano threefolds, namely, $\mathbb{P}^1 \times \mathbb{P}^2$ and the projective space blown up along a line.

Inspired by these results, in Chapter 3, we consider globally generated vector bundles on the projective space blown up along a line. We follow the notation of [18]. Let X be the projective space blown up along a line, $\tilde{H}$ the pull-back of a hyperplane, E the exceptional divisor. Our main result is the classification of globally generated vector bundles on X with $c_1 = 2\tilde{H} - E (= \xi + f)$, up to trivial factor.

Theorem 1.0.3. *Let $\pi : X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^2) \to \mathbb{P}^1$ be the natural projection and let ξ and f be the classes of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}^2)}(1)$ and $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ respectively. Let $\mathcal{E}$ be a globally generated vector bundle of rank r at least 2 on X with $c_1 = \xi + f$ and $c_2 = \alpha\xi^2 + \beta\xi f$. If $\mathcal{E}$ has no trivial factor, then the possible rank r and $(s; \alpha, \beta)$ are as follows (s is the number of connected components of associated curve to $\mathcal{E}$ via the Hartshorne-Serre correspondence modulo trivial factor):*

1. $r = 2, (1; 1, 0)$;
2. $r = 2, (1; 0, 1)$: *In this case* $\mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$;
3. $r = 3, 4, (1; 1, 1)$;
4. $r = 2, 3, (2; 0, 2)$;
5. $3 \leq r \leq 6, (1; 1, 2)$.

Furthermore, there exists a globally generated vector bundle in each of these cases.

In fact, we provide a detailed description of curves associated with the globally generated vector bundles via the Hartshorne-Serre correspondence. The complete classification of nonsingular associated curves is obtained.

Throughout the whole thesis, we refer to [26] for all the unmentioned definitions, notation and results.

Chapter 2

Rank two ACM bundles on the double embedding of a quadric surface

2.1 Introduction

Let $\mathbb{P}^N$ be the N-dimensional projective space over an algebraically closed field k of characteristic 0.

If $X \subseteq \mathbb{P}^N$ is a projective variety of dimension n with respect to the embedding induced by $\mathcal{O}_X(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$, we can ask for locally free sheaves $\mathcal{E}$ on $(X, \mathcal{O}_X(h))$ such that $h^i\big(X, \mathcal{E}(th)\big) = 0$ for $t \in \mathbb{Z}, 1 \leq i \leq n-1$ which are called Arithmetically Cohen–Macaulay bundles (ACM for short) with respect to $\mathcal{O}_X(h)$.

The property of being ACM is not affected by tensoring by $\mathcal{O}_X(th)$, thus we can restrict our attention to ACM bundles which satisfy the conditions $h^0\big(X, \mathcal{E}\big) \neq 0$ and $h^0\big(X, \mathcal{E}(-h)\big) = 0$ (we briefly say that $\mathcal{E}$ is initialized) and which are indecomposable, i.e. bundles which do not split as a direct sum of bundles of lower ranks.

The varieties that have only finitely many initialized ACM bundles fall into a short list (we refer the details to [22]). For example, if $(X, \mathcal{O}_X(h)) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, then the line bundle $\mathcal{O}_{\mathbb{P}^2}$ is the only initialized indecomposable ACM bundle by the theorem of Horrocks (see [32]). The smooth quadric surface $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$ in $\mathbb{P}^3$ also supports only finitely many ACM bundles, and all the initialized indecomposable ACM bundles on it are line bundles (see again [22]).

Recently, the classification of ACM bundles of low rank on other surfaces and threefolds has been considered. For example, there is a complete classification of rank 2 ACM, indecomposable, and initialized bundles on a cubic surface ([23]), some quartic surface ([13]), del Pezzo threefolds of degree at least 3 ([4, 15, 16, 17]). More generally, many varieties with Picard number 1 are treated in [12].

By definition, the classification of ACM bundles on the fixed variety depends largely on the chosen embedding. For example, although the quadric surface $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$ in $\mathbb{P}^3$ is a classic example of a surface that has only finitely many initialized ACM bundles, there are no known results on the classification of ACM bundles of low rank on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to the general polarization $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b)$.

In this direction, Faenzi and Malaspina treated the case of $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2))$ in Theorem 3.1 of [24].

Therefore, in this paper, we consider rank 2 ACM bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to the polarization $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)$, which is a del Pezzo surface of degree 8.

In the study of ACM bundles, the existence of ACM bundles which have maximal number of global sections, namely, Ulrich bundles (see the next section) is of central interest. Since the papers [10, 11], the possible Chern classes of Ulrich bundles are intensively studied.

Our result is the following.

Main Theorem. *Let $\mathcal{E}$ be an initialized indecomposable ACM bundle of rank 2 on $(X := \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_X(2,2))$, with Chern classes $c_1 = \mathcal{O}_X(a,b)$ (we write $c_1 = (a,b)$ for short), where we may assume $a \leq b$ by symmetry, and $c_2 \in \mathbb{Z}$.*

Then one of the following assertions holds.

1. *$c_1 = (0,0), c_2 = 1$. The zero–locus of its general section is a point.*
2. *$c_1 = (0,1), c_2 = 1$. The zero–locus of its general section is a point.*
3. *$c_1 = (3,4), c_2 = 7$. The zero–locus of its general section is a 0-dimensional degree 7 subscheme contained in a pencil of hyperplanes and which is not the complete intersection of curves $C_1 \in |\mathcal{O}_X(2,1)|$ and $C_2 \in |\mathcal{O}_X(1,3)|$.*
4. *$c_1 = (3,5), c_2 = 9$. In this case $\mathcal{E}$ is an Ulrich bundle. The zero–locus of its general section is a 0-dimensional non–degenerate degree 9 subscheme.*
5. *$c_1 = (4,4), c_2 = 10$. In this case $\mathcal{E}$ is an Ulrich bundle. The zero–locus of its general section is a 0-dimensional non–degenerate degree 10 subscheme which is not the complete intersection of curves $C_1 \in |\mathcal{O}_X(1,3)|$ and $C_2 \in |\mathcal{O}_X(3,1)|$.*

All the above cases actually occur on X.

We will prove this theorem by a method similar to the one used in the papers [12], [13], [15] appropriately modified, Brill–Noether theory and case-by-case computational analysis peculiar in our case.

In particular, we focus on the uniqueness of the extension as in the beginning of 2.5.3 in order to prove the non-existence of the initialized indecomposable ACM bundles in otherwise difficult cases.

On the other hand, the key point for proving the existence of ACM bundles is Lemma 2.5.7 whose proof uses that X is Fano, and the discussion after it where we seek for geometric properties of the set of points associated via the Serre correspondence to these rank 2 ACM bundles.

2.2 Preliminaries

The general references for this and the next section are [12, 13, 15]. Let $X := \mathbb{P}^1 \times \mathbb{P}^1$ be a del Pezzo surface of degree 8, embedded in $\mathbb{P}^8$ by the very ample sheaf $\mathcal{O}_X(h) := \mathcal{O}_X(2,2)$.

Definition 2.2.1. ([30, Definition 1.2.2.]) A closed subvariety $V \subseteq \mathbb{P}^N$ is Arithmetically Cohen-Macaulay (ACM for short) if its homogeneous coordinate ring S_V is Cohen-Macaulay.

ACM varieties are characterized by the vanishing of some cohomologies.

Lemma 2.2.2. ([30, Lemma 1.2.3.]) *If* $\dim V \geq 1$*, then* $V \subseteq \mathbb{P}^N$ *is ACM if and only if* $H^i\big(\mathcal{I}_V(th)\big) = 0$ *for* $t \in \mathbb{Z}, 1 \leq i \leq \dim V$*.*

Theorem 2.2.3. ([33, Theorem 2.2.11.]) $X = \mathbb{P}^1 \times \mathbb{P}^1$ *embedded in* $\mathbb{P}^8$ *by the very ample sheaf* $\mathcal{O}_X(2,2)$ *is an ACM variety.*

Theorem 2.2.4. ([10, Corollary 3.5.]) *Let* $\mathcal{E}$ *be a vector bundle of rank* 2 *on* X*. If* $\mathcal{E}$ *is initialized and ACM, then* $h^0\big(X, \mathcal{E}\big) \leq \deg(X)\operatorname{rank}\mathcal{E} = 16$

Definition 2.2.5. We say that $\mathcal{E}$ is an Ulrich bundle if it is initialized, ACM and the equality $h^0\big(X, \mathcal{E}\big) = 16$ holds.

Let $\mathcal{E}$ be a vector bundle of rank 2 on X and let s be a global section of $\mathcal{E}$. In general its zero–locus $(s)_0 \subseteq X$ is either empty or its codimension is at most 2. Thus, if it is non–empty, we can write $(s)_0 = E \cup D$ where E has pure codimension 2 (or it is empty) and D is a divisor (or it is empty). In particular $\mathcal{E}(-D)$ has a section vanishing exactly on E, thus we can consider its Koszul complex

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|X}(c_1 - D) \longrightarrow 0. \tag{2.1}$$

Moreover we also have the following exact sequence

$$0 \longrightarrow \mathcal{I}_{E|X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0. \tag{2.2}$$

The Künneth formula for line bundles on X is the following.

$$h^i\big(X, \mathcal{O}_X(a,b)\big) = \sum_{\substack{(i_1,i_2)\in\mathbb{N}^2,\\ i_1+i_2=i}} h^{i_1}\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)\big)h^{i_2}\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)\big).$$

Lemma 2.2.6. ([33, Lemma 4.2.1.]) *There exists exactly 8 initialized ACM line bundles on X, i.e.* $\mathcal{O}_X, \mathcal{O}_X(1,0), \mathcal{O}_X(0,1), \mathcal{O}_X(1,1), \mathcal{O}_X(1,2), \mathcal{O}_X(2,1), \mathcal{O}_X(1,3), \mathcal{O}_X(3,1)$.

Remark 2.2.7. In [33], the above lemma is proved through the characterization of ACM divisors on del Pezzo surfaces as rational normal curves on X. However, in this case, by using Künneth formula, this fact can be checked directly.

2.3 On the first chern class and zero–locus of general sections

Let $\mathcal{E}$ be a vector bundle of rank 2 on X and c_1, c_2 its Chern classes. For the rest of the paper, we denote by $c_1 = (a,b)$ and $D = (c,d)$ that the first Chern class in $\operatorname{Pic}(X)$ is $\mathcal{O}_X(c_1) = \mathcal{O}_X(a,b)$ and the divisor D is a general element of $|\mathcal{O}_X(c,d)|$. We may assume $a \leq b$ by symmetry. For (x,y) a first Chern class or a divisor, we say (x,y) is effective or $(x,y) \geq 0$ if $\mathcal{O}_X(x,y)$ has a global section, and this occurs if and only if $x, y \geq 0$.

Lemma 2.3.1. *Let* $\mathcal{E}$ *be an initialized ACM bundle of rank* 2 *on* X*. Then* $\mathcal{E}^\vee(2h)$ *is globally generated.*

Proof. Since $K_X = -h$ on X, we have $h^i\big(F, \mathcal{E}^\vee((2-i)h)\big) = h^{2-i}\big(F, \mathcal{E}((i-3)h)\big) = 0$, $i = 1, 2$. This implies $\mathcal{E}^\vee(2h)$ is 0–regular in the sense of Castelnuovo–Mumford, hence globally generated. ([31, Definition and Proposition of Lecture 14]) $\square$

Remark 2.3.2. In particular, $4h - c_1 = c_1(\mathcal{E}^\vee(2h))$ is effective, forcing $a, b \le 8$.

Lemma 2.3.3. *Let $\mathcal{E}$ be an indecomposable initialized ACM bundle of rank 2 on X. If $s \in H^0\big(X, \mathcal{E}\big)$, then its zero locus $(s)_0 \subseteq X$ is non–empty.*

Proof. Assume that $(s)_0 = \emptyset$. Then sequence (2.1) becomes

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(c_1) \longrightarrow 0. \tag{2.3}$$

This extension is non–split because $\mathcal{E}$ is indecomposable. Thus it gives a non–zero element of the space $\mathrm{Ext}^1\big(\mathcal{O}_X(c_1), \mathcal{O}_X\big) = H^1\big(X, \mathcal{O}_X(-c_1)\big)$, which is therefore non–zero. Twisting the above exact sequence by $\mathcal{O}_X(-c_1)$ and using the formula for rank 2 vector bundles $\mathcal{E}^\vee = \mathcal{E}(-c_1)$, we get

$$0 \longrightarrow \mathcal{O}_X(-c_1) \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Taking cohomology of the above exact sequence, we obtain a surjection $H^0\big(\mathcal{O}_X\big) \to H^1\big(\mathcal{O}_X(-c_1)\big)$. Note that because of $K_X = -h$ and Serre duality, if $\mathcal{E}$ is ACM, so is $\mathcal{E}^\vee$. This implies that $h^1\big(X, \mathcal{O}_X(-c_1)\big) = 1$. Thanks to the Künneth formula, we see that c_1 is either $(2, 0)$ or $(0, 2)$. We may assume $c_1 = (0, 2)$.

Note that on $\mathbb{P}^1$ we have the dual of the Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0.$$

The pull–back of this sequence via the second projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ yields the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(0, 1)^2 \longrightarrow \mathcal{O}_X(0, 2) \longrightarrow 0.$$

This is the same extension as (2.3) because $h^1\big(X, \mathcal{O}_X(-c_1)\big) = 1$. Hence $\mathcal{E} \cong \mathcal{O}_X(0, 1)^2$ and this is a contradiction because $\mathcal{E}$ is indecomposable. $\square$

Proposition 2.3.4. *Let $\mathcal{E}$ be an indecomposable initialized ACM bundle of rank 2 on X with $c_1 = (a, b)$, and D the possible 1-dimensional part of the zero locus of a general section of $\mathcal{E}$. Then $c_1 - D$ is effective. In particular, c_1 is effective.*

Proof. Assume that $c_1 - D$ is not effective. The cohomogy of the sequence (2.2) twisted by $\mathcal{O}_X(c_1 - D)$ gives an exact sequence $H^0\big(\mathcal{O}_X(c_1 - D)\big) = 0 \to H^0\big(\mathcal{O}_E\big) \to H^1\big(\mathcal{I}_{E|X}(c_1 - D)\big)$, which means that $\deg(E) \le h^1\big(\mathcal{I}_{E|X}(c_1 - D)\big)$, where E is the 0-dimensional part of the zero locus of a general section of $\mathcal{E}$. On the other hand, by the cohomology of (2.1), $h^1\big(\mathcal{I}_{E|X}(c_1 - D)\big) \le h^2\big(\mathcal{O}_X(D)\big) = h^0\big(\mathcal{O}_X(-D - h)\big) = 0$. Hence $\deg(E) = 0$ and $E = \emptyset$.

Therefore, by Lemma 2.3.3, we may assume $D \neq 0$. The sequence (2.1) becomes

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(c_1 - D) \longrightarrow 0.$$

Since D is effective, we can write $\mathcal{O}_X(D) = \mathcal{O}_X(c, d)$ where $c, d \ge 0$. The cohomology of the above exact sequence twisted by $\mathcal{O}_X\big(th\big)$ gives

$$H^1\big(\mathcal{E}(th)\big) = 0 \to H^1\big(\mathcal{O}_X(c_1 - D + th)\big) \to H^2\big(\mathcal{O}_X(D + th)\big) = H^0\big(\mathcal{O}_X(-D - (t+1)h)\big).$$

The last space is zero if $t \geq -1$, hence $h^1\big(\mathcal{O}_X(c_1 - D + th)\big) = 0$ for $t \geq -1$. Since $c_1 - D$ is non–effective, we see by the Künneth formula that $c_1 - D + th$ for some $t \geq -1$ is one of the initialized ACM line bundles of Lemma 2.2.6. Hence we can write $\mathcal{O}_X(c_1 - D) = \mathcal{O}_X(e, f)$ where $e, f \leq 1$ and one of them is negative since $c_1 - D$ is non–effective.

Since $\mathcal{E}$ is indecomposable, the above exact sequence defines a non–zero element of $H^1\big(\mathcal{O}_X(D - (c_1 - D))\big) = H^1\big(\mathcal{O}_X(c - e, d - f)\big)$. But the last space is zero by the Künneth formula since we know $c - e, d - f \geq -1$. This contradiction implies that $c_1 - D$ is effective. □

Before proceeding to the next proposition, we prepare some calculations. Let $\mathcal{E}$ be an indecomposable initialized ACM vector bundle of rank 2 with Chern classes c_1, c_2, then by Riemann–Roch theorem on X,

$$\chi(\mathcal{E}(th)) = \frac{1}{2}({c_1}^2 + (2t+1)c_1 h + 2t(t+1)h^2) - c_2 + 2. \tag{2.4}$$

Twisting the sequence (2.1) by $\mathcal{O}(-c_1 + th)$ and using $\mathcal{E}^\vee = \mathcal{E}(-c_1)$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_X(D - c_1 + th) \longrightarrow \mathcal{E}^\vee(th) \longrightarrow \mathcal{I}_{E|X}(th - D) \longrightarrow 0. \tag{2.5}$$

By the cohomology of the above exact sequence for $t = 0$, if we define

$$e(c_1, D) := \begin{cases} 0 & \text{if } c_1 \neq D, \\ 1 & \text{if } c_1 = D, \end{cases}$$

then $h^2\big(\mathcal{E}(-h)\big) = h^0\big(\mathcal{E}^\vee\big) = e(c_1, D)$ because $E \cup D \neq \emptyset$ (see Lemma 2.3.3) and $c_1 - D \geq 0$.

By Riemann–Roch (2.4), for $c_1 = (a, b)$,

$$\begin{aligned} \chi(\mathcal{E}(-h)) &= \frac{1}{2}({c_1}^2 - c_1 h) - c_2 + 2 \\ &= ab - a - b - c_2 + 2. \end{aligned} \tag{2.6}$$

If $\mathcal{E}$ is initialized and ACM, $\chi(\mathcal{E}(-h)) = e(c_1, D)$. Thus we have an equation

$$c_2 = ab - a - b + 2 - e(c_1, D) \tag{2.7}$$

for a rank 2 indecomposable initialized ACM vector bundle $\mathcal{E}$ on X.

By the equation (2.4) and (2.7), we have

$$\chi(\mathcal{E}) = 2a + 2b + e(c_1, D).$$

If $\mathcal{E}$ is initialized and ACM, by the cohomology of the sequence (2.5), $H^2\big(\mathcal{E}\big) = H^0\big(\mathcal{E}^\vee(-h)\big) = 0$. Thus the above equation becomes

$$h^0\big(\mathcal{E}\big) = 2a + 2b + e(c_1, D). \tag{2.8}$$

Proposition 2.3.5. *Let $\mathcal{E}$ be an indecomposable initialized ACM bundle of rank 2 on X with $c_1 = (a, b)$. Then one of the following assertions holds.*

1. $2h - c_1 \geq 0$.

2. $a + b = 8$. *In this case* $\mathcal{E}$ *is an Ulrich bundle and* $D = 0$.

Proof. First, we assume that $H^2\big(\mathcal{E}(-2h)\big) = H^0\big(\mathcal{E}^\vee(h)\big) \neq 0$. In this case, if $\mathcal{E}^\vee(th)$ is initialized, then $t \leq 1$. By Proposition 2.3.4, $c_1(\mathcal{E}^\vee(th)) = 2th - c_1$ is effective for some $t \leq 1$. Then, since c_1 is effective, we can take $t = 1$, hence (1) of the proposition holds.

From now on, we assume that $H^2\big(\mathcal{E}(-2h)\big) = 0$ and prove that the case (2) occurs. In this case $\mathcal{E}$ is 0-regular in the sense of Castelnuovo–Mumford, hence globally generated. We may assume $D = 0$ because a general section of $\mathcal{E}$ is a regular section (cf. [25, Example 12.1.11]). By Riemann–Roch,

$$\begin{aligned} \chi(\mathcal{E}(-2h)) &= \frac{1}{2}({c_1}^2 - 3c_1 h + 4h^2) - c_2 + 2 \\ &= 16 - 2(a+b) + e(c_1, D). \end{aligned} \tag{2.9}$$

By assumption, $\chi(\mathcal{E}(-2h)) = 0$. In particular, c_1 cannot be $(0,0)$ and therefore $e(c_1, D) = 0, a + b = 8$. Equation (2.8) implies $H^0\big(\mathcal{E}\big) = 16$ and $\mathcal{E}$ is Ulrich. □

2.4 Bundles with $D \neq 0$

Let $\mathcal{E}$ be an indecomposable initialized ACM bundle of rank 2 on X. Note that $c_1 - D \geq 0$ by Proposition 2.3.4. The purpose of this section is to show the following proposition.

Proposition 2.4.1. *If* $\mathcal{E}$ *is an indecomposable initialized ACM bundle of rank* 2 *on* X*, the zero–locus of its general section is pure of codimension 2.*

Proof. With reference to the proposition 2.3.5, let us assume that $2h - c_1 \geq 0$ and $D \neq 0$. We can write $c_1 = (a, b), D = (c, d), 0 \leq a, b, c, d \leq 4$. We will prove that this case does not occur. In this case $\mathcal{E}$ is not globally generated, because otherwise we could assume $D = 0$. Since $\mathcal{E}$ is initialized, by the cohomology of the sequence (2.1), the same is true for $\mathcal{O}_X(D)$. Thus one of c or d is 0 or 1.

By the cohomology of the sequence (2.1) twisted by $\mathcal{O}_X(-2h)$, we have an exact sequence $H^0\big(\mathcal{I}_{E|X}(c_1 - D - 2h)\big) \to H^1\big(\mathcal{O}_X(D - 2h)\big) \to 0$. The cohomology of the sequence (2.2) twisted by $\mathcal{O}_X(c_1 - D - 2h)$ gives an exact sequence $0 \to H^0\big(\mathcal{I}_{E|X}(c_1 - D - 2h)\big) \to H^0\big(\mathcal{O}_X(c_1 - D - 2h)\big)$. But $H^0\big(\mathcal{O}_X(c_1 - D - 2h)\big) = 0$ by the assumption that $2h - c_1 \geq 0$ and $D \neq 0$, hence we conclude $H^1\big(\mathcal{O}_X(D - 2h)\big) = 0$. Therefore the cases $D = (0, 4), (1, 4)$ (and their permutations) cannot occur.

If we moreover assume $E = \emptyset$, then the sequence (2.1) becomes

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(c_1 - D) \longrightarrow 0.$$

On the quadric surface, an effective line bundle is globally generated. Since D and $c_1 - D$ are effective, they are globally generated, hence the same is true for $\mathcal{E}$, contradicting the hypothesis. Hence we see that if we assume $D \neq 0$, then E is non–empty.

The degree of E coincides with the degree of $c_2(\mathcal{E}(-D)) = c_2 - c_1 D + D^2$. We list all the possible cases of c_1, D below. Since $c_2(\mathcal{E}(-D)) \geq 1$ is necessary, only few cases are actually possible. Note that we assumed that $a \leq b$.

c_1	D	$c_2(\mathcal{E}(-D))$	$c_1 - D - 2h$
$(2,3)$	$(0,1)$	1	$(-2,-2)$
$(2,4)$	$(0,1)$	2	$(-2,-1)$
$(3,3)$	$(0,1)$	2	$(-1,-2)$
$(3,3)$	$(1,1)$	2	$(-2,-2)$
$(3,4)$	$(0,1)$	4	$(-1,-1)$
$(3,4)$	$(0,2)$	1	$(-1,-2)$
$(3,4)$	$(1,0)$	3	$(-2,0)$
$(3,4)$	$(1,1)$	2	$(-2,-1)$
$(3,4)$	$(1,2)$	1	$(-2,-2)$
$(4,4)$	$(0,1)$	6	$(0,-1)$
$(4,4)$	$(0,2)$	2	$(0,-2)$
$(4,4)$	$(1,1)$	4	$(-1,-1)$
$(4,4)$	$(1,2)$	2	$(-1,-2)$

The cohomology of the sequence (2.1) twisted by $\mathcal{O}_X(-h)$ yields

$$0 \to H^1\big(\mathcal{I}_{E|X}(c_1 - D - h)\big) \to H^2\big(\mathcal{O}_X(D-h)\big) = H^0\big(\mathcal{O}_X(-D)\big) = 0.$$

Hence $H^1\big(\mathcal{I}_{E|X}(c_1 - D - h)\big) = 0$. By the cohomology of the sequence (2.2) twisted by $\mathcal{O}_X(c_1 - D - 2h)$, we have an exact sequence

$$0 \to H^2\big(\mathcal{I}_{E|X}(c_1 - D - 2h)\big) \to H^2\big(\mathcal{O}_X(c_1 - D - 2h)\big) \to 0.$$

Among the possible cases of c_1 and D, if $(c_1, D) \neq \big((2,3),(0,1)\big), \big((3,3),(1,1)\big), \big((3,4),(1,2)\big)$, then $H^2\big(\mathcal{O}_X(c_1 - D - 2h)\big) = 0$. Hence in these cases, $H^2\big(\mathcal{I}_{E|X}(c_1 - D - 2h)\big) = 0$.

The vanishing of these cohomologies implies $\mathcal{I}_{E|X}(c_1 - D)$ is 0–regular in the sense of Castelnuovo–Mumford, hence $\mathcal{I}_{E|X}(c_1 - D)$ is globally generated, hence the same holds for $\mathcal{E}$, a contradiction.

If $(c_1, D) = \big((2,3),(0,1)\big), \big((3,3),(1,1)\big), \big((3,4),(1,2)\big)$, then $\mathcal{I}_{E|X}(c_1 - D) = \mathcal{I}_{E|X}(2,2)$. Since in these cases the degree of E is at most 2, $\mathcal{I}_{E|X}(2,2)$ is generated by global sections (a zero dimensional subscheme of degree 2 in $\mathbb{P}^3$ is necessarily a complete intersection of a line and a conic). By the sequence (2.1), $\mathcal{E}$ is also globally generated, a contradiction.

We have seen that all the possible cases of pairs of c_1 and D are excluded. Thus, combined with Proposition 2.3.5, the proposition is proved.

□

2.5 Bundles with $2h - c_1 \geq 0$

Let $\mathcal{E}$ be an indecomposable initialized ACM bundle of rank 2 on X with $2h - c_1 \geq 0$. In this case, $D = 0$ and $E \neq \emptyset$ (by Lemma 2.3.3 and Proposition 2.4.1) and it follows that $c_2 \geq 1$. We list all the possible cases of c_1, c_2 below. For each c_1, we can compute c_2 by the formula (2.7). Note that $c_1 \geq 0$ by Proposition 2.3.4.

c_1	$(0,0)$	$(0,1)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(2,2)$	$(2,3)$	$(2,4)$	$(3,3)$	$(3,4)$	$(4,4)$
c_2	1	1	1	1	1	1	2	3	4	5	7	10

In the following, we investigate indecomposable initialized ACM bundles $\mathcal{E}$ with such Chern classes by using Theorem 2.5.2 below.

Definition 2.5.1. Let $\mathcal{G}$ be a coherent sheaf on X. We say that a locally complete intersection subscheme $E \subseteq X$ of dimension 0 is Cayley–Bacharach (CB for short) with respect to $\mathcal{G}$ if, for each $E' \subseteq E$ of degree $\deg(E) - 1$, the natural morphism $H^0\big(X, \mathcal{I}_{E|X} \otimes \mathcal{G}\big) \to H^0\big(X, \mathcal{I}_{E'|X} \otimes \mathcal{G}\big)$ is an isomorphism.

The following theorem is in [28, Theorem 5.1.1], see also [13, Theorem 4.2].

Theorem 2.5.2. *Let $E \subseteq X$ be a locally complete intersection subscheme of dimension 0. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on X with $\det(\mathcal{E}) = \mathcal{L}$ and having a section s such that $E = (s)_0$ if and only if E is CB with respect to $\mathcal{L}(-h)$.*

2.5.1 The case $c_1 = (0,0), c_2 = 1$

In this case, E is a point. A point is trivially CB with respect to $\mathcal{O}_X(c_1 - h) = \mathcal{O}_X(-2,-2)$. Thus there exists a rank 2 vector bundle $\mathcal{E}$ and an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|X} \longrightarrow 0.$$

By the cohomology of the above exact sequence twisted by $\mathcal{O}_X(-h)$, $\mathcal{E}$ is initialized.

By the cohomology of the above exact sequence twisted by $\mathcal{O}_X(th)$, if $t \geq 0$,

$$0 \to H^1\big(\mathcal{E}(th)\big) \to H^1\big(\mathcal{I}_{E|X}(th)\big) \to H^2\big(\mathcal{O}_X(th)\big) = H^0\big(\mathcal{O}_X(-(t+1)h)\big) = 0.$$

By the cohomology of (2.2) twisted by $\mathcal{O}_X(th)$, if $t \geq 0$, $H^1\big(\mathcal{I}_{E|X}(th)\big) = 0$ because E is a point. Therefore $H^1\big(\mathcal{E}(th)\big) = 0, t \geq 0$. By Serre duality and $\mathcal{E}^\vee = \mathcal{E}$, $H^1\big(\mathcal{E}(-(t+1)h)\big) = H^1\big(\mathcal{E}(th)\big) = 0, t \geq 0$. Hence $H^1\big(\mathcal{E}(th)\big) = 0$ for $t \in \mathbb{Z}$, and $\mathcal{E}$ is ACM.

If $\mathcal{E}$ is decomposable, we can write $\mathcal{E} \cong \mathcal{O}_X(A) \oplus \mathcal{O}_X(B)$. If we write $A = (x,y), B = (z,w)$, then by comparing Chern classes, $x+z = 0, y+w = 0, xw+yz = 1$. Since there are no integer solutions for these equations, $\mathcal{E}$ is indecomposable.

Therefore, we have proved the following.

Theorem 2.5.3. *There exist indecomposable initialized ACM vector bundles of rank 2 with Chern classes $c_1 = (0,0), c_2 = 1$ on X. The zero–locus of their general section is a point.*

2.5.2 The case $c_1 = (0,1), c_2 = 1$

In this case, E is a point. As in the previous case, a point is trivially CB with respect to $\mathcal{O}_X(c_1 - h) = \mathcal{O}_X(-2,-1)$. Thus there exists a rank 2 vector bundle $\mathcal{E}$ and an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|X}(0,1) \longrightarrow 0.$$

By the cohomology of the above exact sequence twisted by $\mathcal{O}_X(-h)$, $\mathcal{E}$ is initialized.

By the cohomology of the above exact sequence twisted by $\mathcal{O}_X(th)$, if $t \geq 0$,

$$0 \to H^1\big(\mathcal{E}(th)\big) \to H^1\big(\mathcal{I}_{E|X}(c_1 + th)\big) \to H^2\big(\mathcal{O}_X(th)\big) = H^0\big(\mathcal{O}_X(-(t+1)h)\big) = 0.$$

By the cohomology of (2.2) twisted by $\mathcal{O}_X(th)$, if $t \geq 0$, $H^1\big(\mathcal{I}_{E|X}(c_1 + th)\big) = 0$ because E is a point. Therefore $H^1\big(\mathcal{E}(th)\big) = 0, t \geq 0$.

Recall the exact sequence (2.5) in this case.

$$0 \longrightarrow \mathcal{O}_X(-c_1 + th) \longrightarrow \mathcal{E}^\vee(th) \longrightarrow \mathcal{I}_{E|X}(th) \longrightarrow 0. \tag{2.10}$$

Note that $\mathcal{O}_X(-c_1)$ is an ACM line bundle in this case. Hence, by the cohomology of this exact sequence, If $t \geq 0$,

$$0 \to H^1\big(\mathcal{E}^\vee(th)\big) \to H^1\big(\mathcal{I}_{E|X}(th)\big) \to H^2\big(\mathcal{O}_X(-c_1+th)\big) = H^0\big(\mathcal{O}_X(c_1-(t+1)h)\big) = 0$$

Again, because E is a point, $H^1\big(\mathcal{I}_{E|X}(th)\big) = 0$ if $t \geq 0$. Hence $H^1\big(\mathcal{E}^\vee(th)\big) = 0, t \geq 0$.

By Serre dualiy, we have $H^1\big(\mathcal{E}(-(t+1)h)\big) = H^1\big(\mathcal{E}^\vee(th)\big) = 0$. Hence $H^1\big(\mathcal{E}(th)\big) = 0$ for $t \in \mathbb{Z}$, and $\mathcal{E}$ is ACM.

If $\mathcal{E}$ is decomposable, we can write $\mathcal{E} \cong \mathcal{O}_X(A) \oplus \mathcal{O}_X(B)$. Since we have seen that $\mathcal{E}$ is ACM, its direct summands $\mathcal{O}_X(A), \mathcal{O}_X(B)$ are ACM line bundles. But as in the previous subsection, by computation, we see $(A, B) = \big((0,1),(1,-1)\big)$ up to permutations, and $\mathcal{O}_X(1,-1)$ is not ACM, a contradiction. Therefore $\mathcal{E}$ is indecomposable.

We have proved the following.

Theorem 2.5.4. *There exist indecomposable initialized ACM vector bundles of rank 2 with Chern classes $c_1 = (0,1), c_2 = 1$ on X. The zero–locus of their general section is a point.*

2.5.3 Uniqueness of the non-split extension

In the remaining cases, if there exists a rank 2 indecomposable initialized ACM vector bundle $\mathcal{E}$, then there is the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|X}(c_1) \longrightarrow 0. \tag{2.11}$$

Where $c_1 = (a, b), a \geq 1, b \geq 1$ and $c_2 = \deg(E) = ab - a - b + 2$ by Formula (2.7) and the vanishing of D.

Since $\mathcal{E}$ is initialized, by the cohomology of the above exact sequence twisted by $\mathcal{O}_X(-h)$, we have $H^0\big(\mathcal{I}_{E|X}(c_1 - h)\big) = 0$. By the cohomology of (2.2) twisted by $\mathcal{O}_X(c_1 - h)$,

$$0 \to H^0\big(\mathcal{O}_X(c_1 - h)\big) \to H^0\big(\mathcal{O}_E\big) \to H^1\big(\mathcal{I}_{E|X}(c_1 - h)\big) \to 0.$$

The first space is of dimension $h^0\big(\mathcal{O}_X(a-2, b-2)\big) = (a-1)(b-1) = ab-a-b+1$ (This is also true when a or b is 1). The second is $h^0\big(\mathcal{O}_E\big) = ab - a - b + 2$. Therefore the last space is of dimension 1. This implies that $\dim Ext^1(\mathcal{I}_{E|X}(c_1), \mathcal{O}_X) \cong h^1\big(\mathcal{I}_{E|X}(c_1 + K_X)\big) = 1$. Thus $\mathcal{E}$ is the unique non–split extension of the above exact sequence. This fact is used several times throughout the paper.

2.5.4 The cases where $\mathcal{E}$ is decomposable

First, we consider the case $c_1 = (1,1), c_2 = 1$. If there exists a rank 2 indecomposable initialized ACM vector bundle $\mathcal{E}$ with these Chern classes, then there is the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|X}(1,1) \longrightarrow 0.$$

By the argument in the previous subsection, $\mathcal{E}$ is the unique non–split extension of the above exact sequence.

On the other hand, a point E is always a complete intersection of lines $L_1 \in |\mathcal{O}_X(1,0)|, L_2 \in |\mathcal{O}_X(0,1)|$, and the rank 2 vector bundle $\mathcal{O}_X(1,0) \oplus \mathcal{O}_X(0,1)$ has the same Chern classes as $\mathcal{E}$.

Therefore $\mathcal{O}_X(1,0) \oplus \mathcal{O}_X(0,1)$ sits in the same exact sequence as above, hence $\mathcal{E} \cong \mathcal{O}_X(1,0) \oplus \mathcal{O}_X(0,1)$ and $\mathcal{E}$ is decomposable. We conclude that there is no indecomposable initialized ACM vector bundle of rank 2 with these Chern classes.

In the same way, we see that in the cases listed below, any initialized ACM vector bundle of rank 2 with these Chern classes must be decomposable.

c_1	c_2	$\mathcal{E}$
$(1,1)$	1	$\mathcal{O}_X(1,0) \oplus \mathcal{O}_X(0,1)$
$(1,2)$	1	$\mathcal{O}_X(0,1) \oplus \mathcal{O}_X(1,1)$
$(1,3)$	1	$\mathcal{O}_X(0,1) \oplus \mathcal{O}_X(1,2)$
$(1,4)$	1	$\mathcal{O}_X(0,1) \oplus \mathcal{O}_X(1,3)$
$(2,2)$	2	$\mathcal{O}_X(1,1) \oplus \mathcal{O}_X(1,1)$
$(2,3)$	3	$\mathcal{O}_X(1,1) \oplus \mathcal{O}_X(1,2)$
$(2,4)$	4	$\mathcal{O}_X(1,2) \oplus \mathcal{O}_X(1,2)$
$(3,3)$	5	$\mathcal{O}_X(1,2) \oplus \mathcal{O}_X(2,1)$

2.5.5 Toward the construction of vector bundles using Brill–Noether theory

From now on, we will prove Lemma 2.5.7 to construct indecomposable initialized ACM vector bundles of rank 2 via Theorem 2.5.2. For the proof, we use Brill–Noether theory. A similar method appeared in [20], [19]. Let C be a smooth curve. Recall that the Brill–Noether locus

$$W^r_d(C) = \{L \in \mathrm{Pic}^d(C) | h^0(C, L) \geq r+1\}$$

has a scheme structure as a union of projective varieties.

We need two lemmas.

Lemma 2.5.5. ([2, Lemma 3.5 in Chapter IV]) *Suppose $g - d + r \geq 0$. Then no component of $W^r_d(C)$ is entirely contained in $W^{r+1}_d(C)$.*

Lemma 2.5.6. ([2, Lemma 3.3 in Chapter IV]) *Suppose $r \geq d - g$. Then every component of $W^r_d(C)$ has dimension greater or equal to the Brill–Noether number*

$$\rho(g, r, d) := g - (r+1)(g - d + r).$$

Let V be the image of the map $\sigma : W^1_{d-1}(C) \times C \to W^1_d(C)$ defined by $(\mathcal{L}, p) \to \mathcal{L}(p)$. The members of $W^1_d(C)$ which have a base point are contained in V.

Lemma 2.5.7. *If $d := ab - a - b + 2 \geq 3$ and $C \in |\mathcal{O}_X(a,b)|$, then the $(W^1_d(C) \setminus W^2_d(C)) \cap (W^1_d(C) \setminus V)$ is open and non–empty, and its elements $\mathcal{L} \cong \mathcal{O}_C(E)$ are such that*

1. *E is CB with respect to $\mathcal{O}_X(C-h)$.*
2. *$h^0\big(\mathcal{I}_{E|X}(C-h)\big) = 0$.*

Proof. In the following, let E be a divisor on a smooth curve $C \in |\mathcal{O}_X(a,b)|$, and $\deg(E) := d = ab - a - b + 2$ (cf. Formula (2.7)). Note that the genus of C is $g = ab - a - b + 1$, hence $g = d - 1$.

We have that $g - d + 1 = 0$, hence by Lemma 2.5.5, no component of $W^1_d(C)$ is entirely contained in $W^2_d(C)$. Thus, the subset $W^1_d(C) \setminus W^2_d(C)$ is non–empty, open and dense in $W^1_d(C)$, and its element $\mathcal{L} \cong \mathcal{O}_C(E)$ satisfies $h^0\big(\mathcal{O}_C(E)\big) = 2$.

V is the image of the map $\sigma : W^1_{d-1}(C) \times C \to W^1_d(C)$ defined by $(\mathcal{L}, p) \to \mathcal{L}(p)$. Since $1 = d - g$, Lemma 2.5.6 implies that every component of $W^1_d(C)$ has dimension at least $\rho(g,1,d) = g = d - 1$.

If $\mathcal{L} \in W^1_{d-1}(C)$, by Riemann–Roch and Serre duality, $h^0\big(\mathcal{L}\big) = h^0\big(\mathcal{L}^\vee \otimes \omega_C\big) + d - 1 - g + 1 = h^0\big(\mathcal{L}^\vee \otimes \omega_C\big) + 1$. The degree of the line bundle $\mathcal{L}^\vee \otimes \omega_C$ is equal to $-(d-1) + 2g - 2 = d - 3$. Thus $W^1_{d-1}(C) \cong W^0_{d-3}(C)$, and the latter is of dimension $d-3$, because $W^0_{d-3}(C) \setminus W^1_{d-3}(C)$ is isomorphic to an open subset of the $(d-3)$-fold symmetric product of C. Therefore $W^1_{d-1}(C) \times C$ is of dimension $d-2$, and its image V is properly contained in every component of $W^1_d(C)$.

Thus the intersection of the open dense subsets $(W^1_d(C) \setminus W^2_d(C)) \cap (W^1_d(C) \setminus V)$ is non–empty, and its element $\mathcal{L} \cong \mathcal{O}_C(E)$ is base point free which satisfies $h^0\big(\mathcal{O}_C(E)\big) = 2$.

We will prove that $\mathcal{L} \cong \mathcal{O}_C(E)$ satisfies (1) and (2) of Lemma 2.5.7. We have the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{I}_{E|X} \longrightarrow \mathcal{O}_C(-E) \longrightarrow 0, \tag{2.12}$$

because $\mathcal{I}_{C|X} \cong \mathcal{O}_X(-C), \mathcal{I}_{E|C} \cong \mathcal{O}_C(-E)$. Twisting by $\mathcal{O}_X(C-h)$, we obtain

$$0 \longrightarrow \mathcal{O}_X(-h) \longrightarrow \mathcal{I}_{E|X}(C-h) \longrightarrow \mathcal{O}_C(-E) \otimes \mathcal{O}_X(C-h)|_C \longrightarrow 0.$$

Taking cohomology, we have

$$\begin{aligned}
h^0\big(\mathcal{I}_{E|X}(C-h)\big) &= h^0\big(\mathcal{O}_C(-E) \otimes \mathcal{O}_X(C-h)|_C\big) \\
&= h^0\big(\mathcal{O}_C(-E) \otimes \mathcal{O}_X(C+K_X)|_C\big) \\
&= h^0\big(\mathcal{O}_C(-E) \otimes K_C\big) \text{ (By the adjunction formula on C)} \\
&= h^1\big(\mathcal{O}_C(-E) \otimes K_C\big) - d + 2g - 2 - g + 1 \text{ (By Riemann–Roch)} \\
&= h^0\big(\mathcal{O}_C(E)\big) - d + g - 1 \text{ (By Serre duality)} \\
&= h^0\big(\mathcal{O}_C(E)\big) - 2. \ (g = d-1)
\end{aligned}$$

If E defines a base point free complete linear system of dimension 1 on C, then $h^0\big(\mathcal{O}_C(E)\big) = 2$ and for each $E' \subseteq E$ of degree $\deg(E) - 1$, $h^0\big(\mathcal{O}_C(E')\big) = h^0\big(\mathcal{O}_C(E)\big) - 1$. By the above equalities, we deduce $h^0\big(\mathcal{I}_{E|X}(C-h)\big) = h^0\big(\mathcal{I}_{E'|X}(C-h)\big) = 0$ and E is CB with respect to $\mathcal{O}_X(C-h)$. Hence (1) and (2) hold. □

2.5.6 The case $c_1 = (3,4), c_2 = 7$

Let $\mathcal{E}$ be an indecomposable initialized ACM bundle of rank 2 with $c_1 = (3,4), c_2 = 7$. By Proposition 2.4.1, the zero–locus E of its general section is pure of codimension 2.

We will see that $h^1\big(\mathcal{E}(-2h)\big) = 0$ is equivalent to $h^0\big(\mathcal{I}_{E|X}(h)\big) = 2$. By Riemann–Roch (cf. (2.9)),

$$\chi(\mathcal{E}(-2h)) = 16 - 2(3+4) = 2.$$

Since $\mathcal{E}$ is initialized and by Serre duality, $\chi(\mathcal{E}(-2h)) = h^0\big(\mathcal{E}^\vee(h)\big) - h^1\big(\mathcal{E}^\vee(h)\big)$.

By the cohomology of the exact sequence (2.5), when $t = 1$,

$$0 \to H^0\big(\mathcal{E}^\vee(h)\big) \to H^0\big(\mathcal{I}_{E|X}(h)\big) \to 0.$$

Hence $h^1\big(\mathcal{E}(-2h)\big) = h^1\big(\mathcal{E}^\vee(h)\big) = h^0\big(\mathcal{E}^\vee(h)\big) - 2 = h^0\big(\mathcal{I}_{E|X}(h)\big) - 2$, so that $h^1\big(\mathcal{E}(-2h)\big) = 0$ is equivalent to $h^0\big(\mathcal{I}_{E|X}(h)\big) = 2$. Hence, since $\mathcal{E}$ is ACM in this case, $h^0\big(\mathcal{I}_{E|X}(h)\big) = 2$ holds and E is contained in a pencil of hyperplanes.

If E is a complete intersection of curves $C_1 \in |\mathcal{O}_X(2,1)|$ and $C_2 \in |\mathcal{O}_X(1,3)|$, then $\mathcal{O}_X(2,1) \oplus \mathcal{O}_X(1,3)$ sits in the same exact sequence as (2.11). Hence $\mathcal{E} \cong \mathcal{O}_X(2,1) \oplus \mathcal{O}_X(1,3)$, contradicting the hypothesis.

We have seen that the statement (3) of the Main Theorem holds in this case.

In what follows, we will prove the existence of indecomposable initialized ACM vector bundles of rank 2 with these Chern classes. The result is as follows.

Theorem 2.5.8. *There exist indecomposable initialized ACM vector bundles of rank 2 with Chern classes $c_1 = (3,4), c_2 = 7$ on X. The zero–locus E of their general section defines a base point free linear system of dimension 1 on a curve $C \in |\mathcal{O}_X(3,4)|$ and satisfies $h^0\big(\mathcal{I}_{E|X}(2,2)\big) = 2, h^0\big(\mathcal{I}_{E|X}(2,1)\big) = 0$.*

Proof. By Lemma 2.5.7 (1), there exists a vector bundle $\mathcal{E}$ and an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|X}(3,4) \longrightarrow 0,$$

Where $E \subset C \in |\mathcal{O}_X(3,4)|$ is of degree 7. We will prove that $\mathcal{E}$ is initialized ACM and indecomposable. By Lemma 2.5.7 (2) and the cohomology of the above sequence, $\mathcal{E}$ is initialized.

The cohomology of the above exact sequence twisted by $\mathcal{O}_X(th)$ gives $0 \to H^1\big(\mathcal{E}(th)\big) \to H^1\big(\mathcal{I}_{E|X}(c_1 + th)\big) \to 0, t \geq 0$. On the other hand, the cohomology of the exact sequence (2.12) twisted by $\mathcal{O}_X(c_1 + th)$ gives, for $t \geq 0$,

$$\begin{aligned} h^1\big(\mathcal{I}_{E|X}(c_1 + th)\big) &= h^1\big(\mathcal{O}_C(-E) \otimes \mathcal{O}_X(c_1 + th)|_C\big) \\ &= h^0\big(\mathcal{O}_C(E) \otimes \mathcal{O}_X(-c_1 - th)|_C \otimes \omega_C\big). \end{aligned}$$

Note that $\deg(E) = 7$, the genus of $C = 6$. Thus the degree of the line bundle in the last cohomology is $7 - 24 - 6t - 8t + 12 - 2 = -7 - 14t$. For $t \geq 0$, the last formula is negative. Hence, $H^1\big(\mathcal{E}(th)\big) = 0$ for $t \geq 0$.

Note that $\mathcal{O}_X(3,4)$ is an ACM line bundle. In this case, by the cohomology of the exact sequence (2.5), We have $0 \to H^1\big(\mathcal{E}^\vee(th)\big) \to H^1\big(\mathcal{I}_{E|X}(th)\big)$.

The cohomology of the exact sequence (2.12) twisted by $\mathcal{O}_X(th)$ gives, for $t \geq 1$,

$$\begin{aligned} h^1\big(\mathcal{I}_{E|X}(th)\big) &= h^1\big(\mathcal{O}_C(-E) \otimes \mathcal{O}_X(th)|_C\big) \\ &= h^0\big(\mathcal{O}_C(E) \otimes \mathcal{O}_X(-th)|_C \otimes \omega_C\big). \end{aligned}$$

The degree of the line bundle in the last cohomology is $7-6t-8t+12-2=17-14t$.

The last formula is negative if $t \geq 2$. Hence, $H^1\big(\mathcal{E}^\vee(th)\big) = H^1\big(\mathcal{E}(-(t+1)h)\big) = 0$ for $t \geq 2$

We compute $h^1\big(\mathcal{E}(-2h)\big)$ in what follows. By the same argument as in the beginning of this subsection, $h^1\big(\mathcal{E}(-2h)\big) = 0$ if and only if $h^0\big(\mathcal{I}_{E|X}(h)\big) = 2$.

Lemma 2.5.9. *There exists a zero dimensional subscheme E of degree 7 on a curve $C \in |\mathcal{O}_X(3,4)|$ such that for E the conditions (1), (2) of Lemma 2.5.7 hold and $h^0\big(\mathcal{I}_{E|X}(2,2)\big) = 2$.*

Proof. By the cohomology of the exact sequence (2.12) twisted by $\mathcal{O}_X(2,2)$,

$$0 \to H^0\big(\mathcal{I}_{E|X}(2,2)\big) = H^0\big(\mathcal{O}_C(-E) \otimes \mathcal{O}_X(2,2)|_C\big) \to 0.$$

We write the line bundle $\mathcal{O}_C(h) := \mathcal{O}_X(2,2)|_C$. This is an effective divisor of degree 14 on C, hence $\mathcal{O}_C(-E) \otimes \mathcal{O}_C(h)$ is of degree 7.

By Riemann–Roch on C, if $\mathcal{L}$ is a line bundle of degree 7 on C, $h^0\big(\mathcal{L}^\vee \otimes \mathcal{O}_C(h)\big) \geq 7-6+1 = 2$. Therefore, $i : \mathcal{L} \to \mathcal{L}^\vee \otimes \mathcal{O}_C(h)$ maps $W^1_d(C)$ isomorphically onto itself. Moreover, i is an involution. Thus $i^{-1}(W^1_7(C) \setminus W^2_7(C))$ is a non–empty open dense subset. Hence the intersection $(W^1_7(C) \setminus W^2_7(C)) \cap (W^1_7(C) \setminus V) \cap i^{-1}(W^1_7(C) \setminus W^2_7(C))$ is non–empty and its element contains a divisor with the desired properties. □

Proof of Theorem 2.5.8, continued. Let E be the zero dimensional subscheme on a curve C that has the properties described in the lemma and $\mathcal{E}$ the corresponding vector bundle obtained via Theorem 2.5.2 from E. We have shown $h^1\big(\mathcal{E}(th)\big) = 0$ for $t \neq -1$.

Finally, we prove $h^1\big(\mathcal{E}(-h)\big) = 0$. For later use, we will show that if $\mathcal{E}$ is a rank 2 initialized vetor bundle with Chern classes $c_1 = (a,b) \geq 0, c_2 = ab - a - b + 2$, and the zero–locus of its general section is pure of codimension 2, then $h^1\big(\mathcal{E}(-h)\big) = 0$. By the exact sequence (2.5) in this case, we have that $h^2\big(\mathcal{E}(-h)\big) = h^0\big(\mathcal{E}^\vee\big) = 0$. Since $\mathcal{E}$ is initialized, By Riemann–Roch (cf. (2.6)),

$$\begin{aligned}
\chi(\mathcal{E}(-h)) &= -h^1\big(\mathcal{E}(-h)\big) \qquad (2.13)\\
&= \frac{1}{2}({c_1}^2 - 2c_2 - c_1 h) + 2\\
&= ab - a - b - c_2 + 2\\
&= 0.
\end{aligned}$$

Hence $h^1\big(\mathcal{E}(-h)\big) = 0$.

We have proved that the rank 2 vector bundle $\mathcal{E}$ we constructed is initialized and ACM. We will prove that $\mathcal{E}$ is indecomposable. If it is decomposable, by comparing Chern classes, the unique possibility is $\mathcal{E} \cong \mathcal{O}_X(2,1) \oplus \mathcal{O}_X(1,3)$. In this case, E is a complete intersection of divisors $A \in |\mathcal{O}_X(2,1)|$ and $B \in |\mathcal{O}_X(1,3)|$.

Therefore, if we can show that E as in Lemma 2.5.9 moreover satisfies $H^0\big(\mathcal{I}_{E|X}(2,1)\big) = 0$, then there does not exist a divisor $A \in |\mathcal{O}_X(2,1)|$ which contains E, and E can not be a complete intersection of divisors $A \in |\mathcal{O}_X(2,1)|$ and $B \in |\mathcal{O}_X(1,3)|$. Then the vector bundle $\mathcal{E}$ obtained via Theorem 2.5.2 from E is indecomposable.

Thus we have to seek for a divisor E of degree 7 on C which satisfies the following conditions.

1. E is CB with respect to $\mathcal{O}_X(1,2)$.
2. $h^0\big(\mathcal{I}_{E|X}(1,2)\big) = 0$.
3. $h^0\big(\mathcal{I}_{E|X}(2,2)\big) = 2$.
4. $h^0\big(\mathcal{I}_{E|X}(2,1)\big) = 0$.

Notice that (1) and (2) are as in Lemma 2.5.7 and (3) comes from Lemma 2.5.9.

We start from a zero dimensional subscheme E of X which satisfies (4). Since $h^0\big(\mathcal{O}_X(2,1)\big) = 6$, there exists a zero dimensional subscheme E of degree 7 which is not contained in any divisor $A \in |\mathcal{O}_X(2,1)|$, i.e., satisfying (4), and since $h^0\big(\mathcal{O}_X(3,4)\big) = 20$, there is a smooth curve $C \in |\mathcal{O}_X(3,4)|$ which contains E. By Riemann-Roch, if a divisor E on C is of degree 7, $H^0\big(\mathcal{O}_C(E)\big) \geq 7-6+1 = 2$. Hence $\mathcal{O}_C(E)$ is contained in $W^1_7(C)$. By the cohomology of the exact sequence (2.12) twisted by $\mathcal{O}_X(2,1)$, we obtain $H^0\big(\mathcal{I}_{E|X}(2,1)\big) = H^0\big(\mathcal{O}_C(-E)\otimes\mathcal{O}_X(2,1)|_C\big)$, hence the latter space is also zero.

Therefore, we have seen that there is an element $\mathcal{L}_0 \in W^1_7(C)$ which satisfies $H^0\big(\mathcal{L}_0^\vee \otimes \mathcal{O}_X(2,1)|_C\big) = 0$. Since vanishing of cohomology is an open condition, it then follows that if $\mathcal{W}$ is any irreducible component of $W^1_7(C)$ containing $\mathcal{L}_0$, the divisor $E \in |\mathcal{L}|$ in the general member $\mathcal{L}$ of $\mathcal{W}$ satisfies the condition (4). Then, by the proof of Lemma 2.5.9, since $\mathcal{L}$ is general, E satisfies also the conditions (1), (2), (3). Hence this E has the desired properties. $\square$

2.5.7 The case $c_1 = (4,4), c_2 = 10$

In this case, $\mathcal{E}$ is an Ulrich bundle by Proposition 2.3.5. This will be treated in the next section.

2.6 Ulrich bundles

Ulrich bundles on del Pezzo surfaces were investigated in [14]. The classification result is already given by a different method, but we prove it through the same method as in the previous section.

In this section, we use the alternative characterization of Ulrich Bundles, which is used as the standard definition of Ulrich sheaves in most papers.

Proposition 2.6.1. ([21, Proposition 2.2.]) *Let $\mathcal{E}$ be a vector bundle of rank 2 on X. Then the following are equivalent.*

1. *$\mathcal{E}$ is Ulrich.*
2. *$H^1\big(\mathcal{E}(-h)\big) = H^2\big(\mathcal{E}(-2h)\big) = H^1\big(\mathcal{E}(-2h)\big) = H^0\big(\mathcal{E}(-h)\big) = 0$.*

By Proposition 2.3.5, if $c_1 = (a,b)$, then $a+b = 8$. By Lemma 2.3.3, $c_2 \geq 1$ is necessary. Note that $c_2 = ab - a - b + 2$ by Formula (2.7) and the vanishing of D. We list the possible Chern classes of a rank 2 indecomposable Ulrich bundle $\mathcal{E}$ below.

c_1	$(1,7)$	$(2,6)$	$(3,5)$	$(4,4)$
c_2	1	6	9	10

Let $\mathcal{E}$ be an Ulrich bundle with the prescribed Chern classes. By Proposition 2.4.1, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|X}(c_1) \longrightarrow 0. \tag{2.14}$$

By the discussion of subsection 2.5.3, $\mathcal{E}$ is the unique non–split extension of the above exact sequence.

2.6.1 The case $c_1 = (1, 7), c_2 = 1$

Let $\mathcal{E}$ be an initialized vector bundle of rank 2 with the prescribed Chern classes and suppose that the exact sequence as above exists. In this case, $\mathcal{E}$ has the same Chern classes as $\mathcal{O}_X(1, 6) \oplus \mathcal{O}_X(0, 1)$. The zero dimensional subscheme E of degree 1 is always realized as a complete intersection of a divisor in $|\mathcal{O}_X(1, 6)|$ and a divisor in $|\mathcal{O}_X(0, 1)|$. Therefore, $\mathcal{O}_X(1, 6) \oplus \mathcal{O}_X(0, 1)$ sits in the same extension as $\mathcal{E}$, hence $\mathcal{E} \cong \mathcal{O}_X(1, 6) \oplus \mathcal{O}_X(0, 1)$. Note that in the discussion of subsection 2.5.3, the assumption that $\mathcal{E}$ is initialized is the only needed ($\mathcal{E}$ need not be ACM). Hence, there is no indecomposable Ulrich vector bundle of rank 2 with these Chern classes.

2.6.2 The case $c_1 = (2, 6), c_2 = 6$

Let $\mathcal{E}$ be an initialized vector bundle of rank 2 with the prescribed Chern classes and suppose that the exact sequence as above exists. As in the previous subsection, we see $\mathcal{E} \cong \mathcal{O}_X(1, 3) \oplus \mathcal{O}_X(1, 3)$. Hence, there is no indecomposable Ulrich vector bundle of rank 2 with these Chern classes.

2.6.3 The case $c_1 = (3, 5), c_2 = 9$

Let $\mathcal{E}$ be an indecomposable Ulrich bundle of rank 2 with $c_1 = (3, 5), c_2 = 9$. By Proposition 2.4.1, the zero–locus E of its general section is pure of codimension 2.

By the cohomology of the exact sequence (2.5) for $t = 1$,

$$0 \to H^0\big(\mathcal{E}^\vee(h)\big) \to H^0\big(\mathcal{I}_{E|X}(h)\big) \to 0.$$

By Proposition 2.6.1, $h^0\big(\mathcal{E}^\vee(h)\big) = h^2\big(\mathcal{E}(-2h)\big) = 0$, so we have $h^0\big(\mathcal{I}_{E|X}(h)\big) = 0$, hence E is non–degenerate. This is the statement (4) of the Main Theorem.

In what follows, we will prove the existence of indecomposable Ulrich vector bundles of rank 2 with these Chern classes. The result is as follows.

Theorem 2.6.2. *There exist indecomposable initialized ACM vector bundles of rank 2 with Chern classes $c_1 = (3, 5), c_2 = 9$ on X, and every such vector bundle is Ulrich. The zero–locus E of their general section defines a base point free linear system of dimension 1 on a curve $C \in |\mathcal{O}_X(3, 5)|$ and satisfies $h^0\big(\mathcal{I}_{E|X}(2, 2)\big) = 0$.*

Proof. By Lemma 2.5.7 (1), there is a rank 2 vector bundle $\mathcal{E}$ and an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|X}(3, 5) \longrightarrow 0.$$

We will check the vanishing of cohomologies. By Lemma 2.5.7 (2), $\mathcal{E}$ is initialized. Hence $h^0\big(\mathcal{E}(-h)\big) = 0$.

$h^1\big(\mathcal{E}(-h)\big) = 0$ is also true by the computation (2.13).

By Riemann–Roch (cf. (2.9)),

$$\chi(\mathcal{E}(-2h)) = -h^1\big(\mathcal{E}(-2h)\big) + h^2\big(\mathcal{E}(-2h)\big) = 16 - 2(3+5) = 0.$$

Hence $h^1\big(\mathcal{E}(-2h)\big) = h^2\big(\mathcal{E}(-2h)\big)$.

Finally, we will compute $h^2\big(\mathcal{E}(-2h)\big) = h^0\big(\mathcal{E}^\vee(h)\big) = 0$. By the discussion at the beginning of this subsection, this is equivalent to $h^0\big(\mathcal{I}_{E|X}(h)\big) = 0$

Thus we have to seek for a divisor E of degree 9 on C which satisfies the following conditions.

1. E is CB with respect to $\mathcal{O}_X(1,3)$.
2. $h^0\big(\mathcal{I}_{E|X}(1,3)\big) = 0$.
3. $h^0\big(\mathcal{I}_{E|X}(2,2)\big) = 0$.

Notice that (1) and (2) are as in Lemma 2.5.7.

To achieve this, we need the following lemma.

Lemma 2.6.3. *The curve $C \in |\mathcal{O}_X(3,5)|$ is a non–degenerate curve in $\mathbb{P}^8$*

Proof. By the exact sequence

$$0 \longrightarrow \mathcal{I}_{X|\mathbb{P}^8} \longrightarrow \mathcal{I}_{C|\mathbb{P}^8} \longrightarrow \mathcal{I}_{C|X} \longrightarrow 0,$$

since X is non–degenerate and ACM, twisting by $\mathcal{O}_X(h)$ and taking cohomology, we have

$$h^0\big(\mathcal{I}_{C|\mathbb{P}^8}(h)\big) = h^0\big(\mathcal{I}_{C|X}(h)\big) = h^0\big(\mathcal{O}_X(-C+h)\big) = h^0\big(\mathcal{O}_X(-1,-3)\big) = 0$$

This implies that C is non–degenerate. □

Proof of Theorem 2.6.2, continued. By the above lemma, we can take a zero dimensional subscheme $E \subset C$ of degree 9 which spans $\mathbb{P}^8$. Then, since E is not contained in a hyperplane, $H^0\big(\mathcal{I}_{E|X}(2,2)\big) = 0$.

We now use again a little Brill-Noether theory on C. By Riemann-Roch, if a divisor E in C is of degree 9, $h^0\big(\mathcal{O}_C(E)\big) \geq 9-8+1 = 2$. Hence $\mathcal{O}_C(E)$ is contained in $W^1_9(C)$. By the cohomology of the exact sequence (2.12) twisted by $\mathcal{O}_X(h)$, we obtain $H^0\big(\mathcal{I}_{E|X}(2,2)\big) = H^0\big(\mathcal{O}_C(-E+h)\big)$, hence the latter space is also zero.

Therefore, we have seen that there is an element $\mathcal{L}_0 \in W^1_9(C)$ which satisfies $H^0\big(\mathcal{L}_0^\vee(h)\big) = 0$. Since vanishing of cohomology is an open condition, it then follows that if $\mathcal{W}$ is any irreducible component of $W^1_9(C)$ containing $\mathcal{L}_0$, then for the general member $\mathcal{L}$ in $\mathcal{W}$ the divisor $E \in |\mathcal{L}|$ satisfies the above three conditions.

For such a divisor E and a vector bundle $\mathcal{E}$ obtained via Theorem 2.5.2 from E, we thus obtain $H^2\big(\mathcal{E}(-2h)\big) = 0$.

Therefore we conclude that $\mathcal{E}$ is an Ulrich bundle by Proposition 2.6.1.

If it is decomposable, we can write $\mathcal{E} \cong \mathcal{O}_X(A) \oplus \mathcal{O}_X(B)$, where the line bundles $\mathcal{O}_X(A)$ and $\mathcal{O}_X(B)$ are ACM. But, by comparing Chern classes, the possibilities are $(A,B) = \big((1,4),(2,1)\big), \big((0,3),(3,2)\big)$ up to permutations. Since these line bundles are not both ACM, this is impossible and the vector bundle $\mathcal{E}$ is indecomposble. □

2.6.4 The case $c_1 = (4,4), c_2 = 10$

Let $\mathcal{E}$ be an indecomposable initialized ACM bundle of rank 2 with $c_1 = (4,4), c_2 = 10$. By Proposition 2.4.1, the zero–locus E of its general section is pure of codimension 2.

By the same argument as in the beginning of the previous subsection, $h^0\big(\mathcal{I}_{E|X}(h)\big) = 0$ and E is non–degenerate.

If E is a complete intersection of curves $C_1 \in |\mathcal{O}_X(1,3)|$ and $C_2 \in |\mathcal{O}_X(3,1)|$, then $\mathcal{O}_X(1,3) \oplus \mathcal{O}_X(3,1)$ sits in the same exact sequence as (2.14). Hence $\mathcal{E} \cong \mathcal{O}_X(1,3) \oplus \mathcal{O}_X(3,1)$, contradicting the hypothesis. Hence the statement (5) of the Main Theorem holds.

In what follows, we will prove the existence of indecomposable Ulrich vector bundles of rank 2 with these Chern classes. The result is as follows.

Theorem 2.6.4. *There exist indecomposable initialized ACM vector bundles of rank 2 with Chern classes $c_1 = (4,4), c_2 = 10$ on X and every such vector bundle is Ulrich. The the zero–locus E of their general section defines a base point free linear system of dimension 1 on a curve $C \in |\mathcal{O}_X(4,4)|$ and satisfies $h^0\big(\mathcal{I}_{E|X}(1,3)\big) = 0$.*

Proof. By Lemma 2.5.7 (1), there is a rank 2 vector bundle $\mathcal{E}$ and an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{E|X}(4,4) \longrightarrow 0.$$

We will check the vanishing of cohomologies. By Lemma 2.5.7 (2), $\mathcal{E}$ is initialized. Hence $H^0\big(\mathcal{E}(-h)\big) = 0$.

$H^1\big(\mathcal{E}(-h)\big) = 0$ is also true by the computation (2.13).

Since $\mathcal{E}^\vee \cong \mathcal{E}(-2h)$, $H^2\big(\mathcal{E}(-2h)\big) = H^0\big(\mathcal{E}(-h)\big) = 0$ and $H^1\big(\mathcal{E}(-2h)\big) = H^1\big(\mathcal{E}(-h)\big) = 0$. By Proposition 2.6.1, $\mathcal{E}$ is an Ulrich bundle.

If it is decomposable, by comparing Chern classes, the unique possibility is $\mathcal{E} \cong \mathcal{O}_X(1,3) \oplus \mathcal{O}_X(3,1)$. Therefore, if E is not a complete intersection of a divisor in $|\mathcal{O}_X(1,3)|$ and a divisor in $|\mathcal{O}_X(3,1)|$, then $\mathcal{E}$ is indecomposable.

Thus we have to seek for a divisor E of degree 10 on C which satisfies the following conditions.

1. E is CB with respect to $\mathcal{O}_X(2,2)$.
2. $h^0\big(\mathcal{I}_{E|X}(2,2)\big) = 0$.
3. $h^0\big(\mathcal{I}_{E|X}(1,3)\big) = 0$.

Notice that (1) and (2) are as in Lemma 2.5.7.

This is possible, and the discussion is almost the same as the one in subsection 2.5.6, hence we do not repeat it here. □

Chapter 3

Globally generated vector bundles on a projective space blown up along a line

3.1 Introduction

Let $\mathbb{P}^n$ be the n-dimensional projective space over an algebraically closed field k of characteristic 0. Globally generated vector bundles on projective varieties are fundamental objects in algebraic geometry. However, even in the case of $\mathbb{P}^n$, their classification for small first Chern class has only been studied fairly recently [1, 29, 36, 37]. Subsequently, Ballico, Huh and Malaspina investigated globally generated vector bundles on projective varieties such as smooth quadric threefolds [6], complete intersection Calabi-Yau threefolds [7], Segre threefolds [8, 9]. Moreover, Ballico studied the case of projective space blown up at finitely many points [5].

In this article, we consider similar questions on the projective space blown up along a line (we follow the notation of [18]). Let X be the projective space blown up along a line, $\tilde{H}$ the pull-back of a hyperplane, E the exceptional divisor. Our main result is the classification of globally generated vector bundles on X with $c_1 = 2\tilde{H} - E (= \xi + f$, cf. the next section), up to trivial factor.

Main Theorem. *Let $\pi : X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^2) \to \mathbb{P}^1$ be the natural projection and let ξ and f be the classes of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}^2)}(1)$ and $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ respectively. Let $\mathcal{E}$ be a globally generated vector bundle of rank r at least 2 on X with $c_1 = \xi + f$ and $c_2 = \alpha\xi^2 + \beta\xi f$. If $\mathcal{E}$ has no trivial factor, then the possible rank r and $(s; \alpha, \beta)$ are as follows (s is the number of connected components of associated curve to $\mathcal{E}$ via the Hartshorne-Serre correspondence modulo trivial factor):*

1. $r = 2, (1; 1, 0)$;
2. $r = 2, (1; 0, 1)$: *In this case* $\mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$;
3. $r = 3, 4, (1; 1, 1)$;
4. $r = 2, 3, (2; 0, 2)$;
5. $3 \le r \le 6, (1; 1, 2)$.

Furthermore, there exists a globally generated vector bundle in each of these cases.

In fact, we present a description of the possible c_2 as represented by the associated curves (cf. Section 3).

Remark 3.1.1. The classification of globally generated vector bundles on $\mathbb{P}^n$ with $c_1 = 2$ is the main result of [36], and the classification of globally generated vector bundles on $\mathbb{P}^3$ blown up at a point with $c_1 = 2\tilde{H} - E$ is one of the main results of [5] (cf. Theorem 3.2. in loc. cit).

3.2 Preliminaries

Let $\sigma : X \to \mathbb{P}^3$ be the projective space blown up along a line R, and let $\tilde{H}, E, \xi$, and f be as in Introduction. Thus $\omega_X \cong \mathcal{O}_X(-3\xi - f)$, and we have an isomorphism

$$A(X) \cong \mathbb{Z}[\xi, f]/(f^2, \xi^3 - \xi^2 f),$$

so that $\xi^3 = \xi^2 f$ are the classes of a point.

Trivially $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is globally generated. On the other hand, since $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^2$ is globally generated, it follows that the same holds for $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}^2)}(1)$. In fact, by [18, Remark 4.7.], $\mathcal{O}_X(a\xi + bf)$ is globally generated if and only if $a, b \geq 0$. Moreover, $\mathcal{O}_X(\xi) \cong \sigma^*\mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_X(\tilde{H})$.

Since the normal bundle of the blown up line R inside $\mathbb{P}^3$ is $\mathcal{N}_{R|\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(1)^2$, it follows that $E = \sigma^{-1}(R) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and σ induces an isomorphism $X \setminus \sigma^{-1}(R) \cong \mathbb{P}^3 \setminus R$. Let $H \subseteq \mathbb{P}^3$ be a plane through R. On the one hand, $\sigma^{-1}(H)$ is in the class of ξ. On the other hand, $\sigma^{-1}(H)$ is the union of E with the strict transform of H. Such a strict transform is in the linear system $|f|$, hence E is the unique element in $|\xi - f|$.

Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X with the first Chern class c_1. Then it fits into the exact sequence

$$0 \to \mathcal{O}_X^{r-1} \to \mathcal{E} \to \mathcal{I}_C(c_1) \to 0, \tag{3.1}$$

where C is a smooth curve on X by [29, Section 2. G.]. The associated curve C represents the second Chern class c_2 of $\mathcal{E}$.

For the construction of vector bundles as in (3.1), we use [3, Theorem 1.1.] for $L := \mathcal{O}_X(c_1)$. Note that for a smooth curve C, $\bigwedge_C := \bigwedge^2 N \otimes L^\vee|_C \cong \omega_C \otimes \omega_X^\vee \otimes L^\vee|_C$. By the discussion in the beginning of [3] (cf. also [7, Theorem 2.8.]), for the existence of a globally generated vector bundle of rank r, $\bigwedge_C$ must be globally generated (and trivial if $r = 2$), and an $(r-1)$-dimensional vector subspace of $h^0(\bigwedge_C)$ corresponds to a globally generated vector bundle of rank r without trivial factor, i.e. with no factor isomorphic to $\mathcal{O}_X$. Conversely if $\bigwedge_C$ is globally generated, we can construct a vector bundle $\mathcal{E}$ by [3, Theorem 1.1.] and $\mathcal{E}$ is globally generated if and only if $\mathcal{I}_C(c_1)$ is globally generated.

Proposition 3.2.1. ([34, Proposition 1]) *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on a reduced irreducible projective variety V over k such that $h^0(\mathcal{E}(-c_1)) \neq 0$. Then we have*

$$\mathcal{E} \cong \mathcal{O}_V^{r-1} \oplus \mathcal{O}_V(c_1).$$

By this proposition, when $c_1 = 0$ the only globally generated vector bundle of rank r on X is $\mathcal{O}_X^r$.

Proposition 3.2.2. *Let $\mathcal{E}$ be a globally generated vector bundle of rank at least 2 on X with $c_1 = a\xi, a \geq 1$, $\sigma : X \to \mathbb{P}^3$ the blow up map. Then there exists a globally generated vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ such that $\mathcal{E} = \sigma^*\mathcal{F}$.*

Proof. Since $E \in |\xi - f|$, we have $\xi \cdot E = \xi(\xi - f) = \xi^2 - \xi f$. By [18, Remark 3.1.], this is a class of a line, say $N \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$, in $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, since a general plane $H \subseteq \mathbb{P}^3$ intersects R at a point, $\xi^2 - \xi f$ is the class of a fiber of $\sigma|_E : E \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = a\xi$ and C be its associated curve. Assume that $C \cap E$ is non-empty. Since $\mathcal{I}_{C\cap E}(a\xi)|_E \cong \mathcal{I}_{C\cap E|\mathbb{P}^1\times\mathbb{P}^1}(a, 0)$ is globally generated, $C \cap E$ is a disjoint union of lines. Hence C contains a line $N \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ as a connected component.

Now, consider the line bundle $\omega_X^\vee \otimes L^\vee|_N = \mathcal{O}_X((3-a)\xi + f)|_N$, which has degree 1 on N. Since N is rational, $\bigwedge_N$ is not globally generated. This contradicts the fact that C is the associated curve of $\mathcal{E}$.

We conclude that $C \cap E = \emptyset$. Then $C \cong \sigma(C)$. $\mathcal{I}_{C|X}(a\xi)$ is globally generated if and only if $\mathcal{I}_{\sigma(C)|\mathbb{P}^3}(a)$ is globally generated, and $\bigwedge_C \cong \omega_C \otimes \omega_X^\vee \otimes L^\vee|_C = \omega_C \otimes \mathcal{O}_X(4\tilde{H} - E) \otimes \mathcal{O}_X(-a\tilde{H})|_C \cong \omega_{\sigma(C)} \otimes \mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{O}_{\mathbb{P}^3}(-a)|_{\sigma(C)} \cong \bigwedge_{\sigma(C)}$.

Therefore, there exists a globally generated vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ with $c_1 = \mathcal{O}_{\mathbb{P}^3}(a)$, such that $\mathcal{E} \cong \sigma^*\mathcal{F}$. □

Proposition 3.2.3. *Let $\mathcal{E}$ be a globally generated vector bundle of rank r at least 2 on X with $c_1 = bf, b \geq 1$, $\pi : X \to \mathbb{P}^1$ the projective bundle map. Then there exists a globally generated vector bundle $\mathcal{F}$ on $\mathbb{P}^1$ such that $\mathcal{E} = \pi^*\mathcal{F}$.*

Proof. Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = bf$, and $F := \pi^{-1}(p) \cong \mathbb{P}^2$ be a fiber. Then the restricted bundle $\mathcal{E}|_F$ is a globally generated vector bundle on $\mathbb{P}^2$ with $c_1 = 0$. By Proposition 3.2.1, $\mathcal{E}|_F \cong \mathcal{O}_{\mathbb{P}^2}^r$. The base-change theorem [32, page 11] implies $\pi_*\mathcal{E}$ is locally free of rank r.

The natural map $\pi^*\pi_*\mathcal{E} \to \mathcal{E}$ is surjective since $\mathcal{E}$ is globally generated and, being a surjective morphism between two vector bundles of the same rank, it is an isomorphism. By the theorem of Grothendieck [32, Theorem 2.1.1.], $\pi_*\mathcal{E}$ is a direct sum of line bundles and hence globally generated since its pull-back must be globally generated. □

Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = a\xi + bf$. If $c_1 = a\xi$, $\mathcal{E}$ is a pull-back of a globally generated vector bundle on $\mathbb{P}^3$ by σ (Proposition 3.2.2). If $c_1 = bf$, $\mathcal{E}$ is a pull-back of a globally generated vector bundle on $\mathbb{P}^1$ by π (Proposition 3.2.3). Thus, in these cases, the classification of globally generated vector bundles on X is reduced to the classification of globally generated vector bundles on $\mathbb{P}^n$.

3.3 Proof of the Main Theorem

Lemma 3.3.1. *Let $\mathcal{E}$ be a globally generated vector bundle on X with $c_1 = \xi + bf$, $b \geq 0$ and $\pi : X \to \mathbb{P}^1$ be the projective bundle map. Let the associated curve of $\mathcal{E}$ be*

C . If C has a connected component which is not in a fiber of π, then C is connected and rational.

Proof. Let $F := \pi^{-1}(p) \cong \mathbb{P}^2$ be a fiber. Since $\mathcal{I}_C(\xi + bf)$ is globally generated, $\mathcal{I}_C(\xi + bf)|_F \cong \mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$ is globally generated.

Let C_i be a connected component of C which is not in a fiber of π. Since $\mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$ is globally generated, $\deg(C_i \cap F) = 1$. Hence $\pi|_{C_i} : C_i \to \mathbb{P}^1$ is of degree 1 and C_i is rational. If there exists another component C_j not contained in a fiber, then there exists a fiber F such that $C \cap F$ is 0-dimensional and $\deg(C \cap F) \geq 2$. Then, a line through two points of $C \cap F$ is in the base locus of $\mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$. This contradicts the fact that $\mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$ is globally generated. Hence at most one component is not in a fiber of π. If there exists another component C_j contained in a fiber F, then $C \cap F$ contains C_j and a point of $C_i \cap F$ not on C_j. Again, this contradicts the fact that $\mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$ is globally generated, and so $C = C_i$ is connected. □

Now, let $\mathcal{E}$ be a globally generated vector bundle of rank r at least 2 on X with $c_1 = \xi + f$, $C = \bigcup_{i=1}^s C_i$ its associated curve.

Remark 3.3.2. (cf. [8, Remark 2.4, 2.7.], [9, Remark 2.7.]) By assumption $\mathcal{I}_C(\xi+f)$ is globally generated. Let us take two general divisors $M_1, M_2 \in |\mathcal{I}_C(\xi + f)|$. Set $Y := M_1 \cap M_2$. By the Bertini theorem we have $Y = C \cup D$ with either $D = \emptyset$ or D a reduced curve containing no component of C and smooth outside $C \cap D$. Each connected component C_i of C appears with multiplicity one in Y, because affixing points $p_i \in C_i$ for every i, we may find a divisor $T \in |\mathcal{I}_C(\xi + f)|$ not containing the tangent line of C_i at p_i. Y is also connected since we have $h^0(\mathcal{O}_Y) = 1$ by vanishing of cohomologies (by [18, Proposition 4.1.]) and a Koszul complex standard exact sequence. The adjunction formula gives $\omega_Y \cong \mathcal{O}_Y(-\xi + f)$ and so we have

$$\begin{aligned} 2p_a(Y) - 2 &= \deg(\xi + f)(\xi + f)(-\xi + f) \\ &= \deg(-\xi^3 - \xi^2 f) \\ &= -2 \end{aligned}$$

Hence $p_a(Y) = 0$ and Y is rational.

Lemma 3.3.3. *If C has a connected component in E, then C is contained in E and its class is ξ^2 or ξf.*

Proof. By assumption, $\mathcal{I}_{C\cap E}(\xi + f)|_E$ is globally generated. By the adjunction theorem on E, $\omega_E = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2) = \mathcal{O}_X(-4\tilde{H}+2E)|_E = \mathcal{O}_X(-2\xi-2f)|_E$. Hence $\mathcal{I}_{C\cap E}(\xi + f)|_E \cong \mathcal{I}_{C\cap E}(2\tilde{H} - E)|_E \cong \mathcal{I}_{C\cap E,\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ and a curve in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ has the class $(\xi + f)(\xi - f) = \xi^2$.

Next, let us review the explicit description of X, the projective space $\mathbb{P}^3$ blown up along a line R. Let x_0, x_1, x_2, x_3 be the coordinates of $\mathbb{P}^3$ and let the line R be defined by the vanishing of x_0, x_1. Let y_0, y_1 be the coordinates of $\mathbb{P}^1$. Then, X is the closed subscheme in $\mathbb{P}^3 \times \mathbb{P}^1$ defined by the equation $x_0y_1 = x_1y_0$.

Let $H \subseteq \mathbb{P}^3$ be a plane. If H does not contain R, it intersects R at a point and its strict transform (pull-back) in X intersects E in a line of class $\xi^2 - \xi f$, which is the fiber of $\sigma|_E : E \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ at the point $H \cap R$.

If H contains R, we can write H as $a_0x_0 + a_1x_1 = 0$, where $a_0, a_1 \in k$, and the class of its strict transform in X is f. Since X is defined by the equation $x_0y_1 = x_1y_0$, the strict transform of H in X intersects E at the point $(x_0, x_1, x_2, x_3; y_0, y_1) = (0, 0, x_2, x_3; -a_1, a_0)$, which is a point on a line in the section of $\sigma|_E : E \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, with class ξf. Each strict transform of a hyperplane containing R intersects E in each line in the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$.

Let $Q \subseteq \mathbb{P}^3$ be an irreducible quadric cone containing R. Then the strict transform of Q in X intersects E in the union of the line in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ corresponding to the fiber of $\sigma|_E : E \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ at the vertex of Q and the line in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$ corresponding to the tangent plane of Q along R.

Let $Q \subseteq \mathbb{P}^3$ be an irreducible smooth quadric surface containing R. We can write Q as $a_{00}x_0^2 + a_{01}x_0x_1 + a_{11}x_1^2 + f(x_2, x_3)x_0 + g(x_2, x_3)x_1 = 0$, where $f(x_2, x_3)$ and $g(x_2, x_3)$ are linear polynomials in x_2, x_3. Hence the strict transform of Q in X intersects E in the coordinates $(x_0, x_1, x_2, x_3; y_0, y_1) = (0, 0, x_2, x_3; -g(x_2, x_3), f(x_2, x_3))$. Since Q is smooth, f and g are nonzero, so this intersection is a rational curve in the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$.

Suppose C contains a line $N \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ with class $\xi^2 - \xi f$ in E. The degree of the line bundle $\mathcal{O}_X(2\xi)$ restricted to N is zero. Then $\bigwedge_N \cong \omega_N \otimes \omega_X^\vee \otimes L^\vee|_N \cong \omega_N \otimes \mathcal{O}_X(2\xi)|_N$ is not globally generated, a contradiction.

Assume $N := C \cap E \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$ is a line with class ξf in E. The element of $|\mathcal{I}_C(\xi+f)|$ is the strict transform of a union of two planes, one of them containing R, or a quadric cone containing R. If $|\mathcal{I}_C(\xi + f)|$ consists only of the strict transforms of two planes, then the strict transform of the plane defined by N is in the base locus of $\mathcal{I}_C(\xi + f)$. We conclude that there is a quadric cone containing $\sigma(C) \cup R$. Assume that $C \neq N$. There is a complete intersection Y containing C, whose class is $\xi^2 + 2\xi f$ (we may assume $Y \cap E = N$). The residual curve to N in Y is $D = \xi^2 + \xi f$. $\sigma(D)$ is a conic or a union of lines and since Y is connected and C is smooth, the only possibility is that C is the disjoint union of N and the strict transform of a line disjoint from R (and the residual curve to C in Y is a line that intersects each component of C). This is a contradiction since every line in a quadric cone containing R must intersect R. We conclude that $C = N = \xi f$ in this case.

Assume $N := C \cap E \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ is a rational curve with class ξ^2 in E. Since $\xi^2 \cdot f \neq 0$, N is not contained in a fiber of $\pi : X \to \mathbb{P}^1$. By Lemma 3.3.1, we conclude that $C = N = \xi^2$ in this case. □

Proposition 3.3.4. *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X with $c_1 = \xi + f, c_2 = \xi^2$. If the associated curve C is contained in E and $\mathcal{E}$ has no trivial factor, then $\mathcal{E}$ exists if and only if $r = 2$.*

Proof. $|\mathcal{I}_C(\xi + f)|$ contains the strict transforms of the smooth quadrics defined by equations of the form $a_{00}x_0^2 + a_{01}x_0x_1 + a_{11}x_1^2 + f(x_2, x_3)x_0 + g(x_2, x_3)x_1 = 0$, where a_{00}, a_{01} and a_{11} are arbitrary and $f(x_2, x_3)$ and $g(x_2, x_3)$ are fixed. These equations show that $x_0 = x_1 = 0$, hence C is cut out by global sections. Therefore, $\mathcal{I}_C(\xi + f)$ is globally generated. Since $\bigwedge_C \cong \omega_C \otimes \mathcal{O}_X(2\xi)|_C \cong \mathcal{O}_C$ and $h^0(\mathcal{O}_C) = 1$, by [3, Theorem 1.1.], this case gives a rank r globally generated bundle if and only if $r = 2$. □

Proposition 3.3.5. *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X with $c_1 = \xi + f, c_2 = \xi f$. If the associated curve C is contained in E and $\mathcal{E}$ has no trivial factor, then $\mathcal{E}$ exists if and only if $r = 2, \mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$.*

Proof. Since $\bigwedge_C \cong \omega_C \otimes \mathcal{O}_X(2\xi)|_C \cong \mathcal{O}_C$ and $h^0(\mathcal{O}_C) = 1$, this curve gives a rank 2 vector bundle $\mathcal{E}$. We will prove that $\mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$. Since $h^0(\mathcal{O}_C(f)) = h^0(\mathcal{O}_C) = 1$ and $h^0(\mathcal{O}_X(f)) = 2$ by [18, Proposition 4.1], so $h^0(\mathcal{I}_C(f)) > 0$. By the exact sequence (3.1), we have $h^0(\mathcal{E}(-\xi)) > 0$ and so there is a non-zero map $m : \mathcal{O}_X(\xi) \to \mathcal{E}$. We have $h^0(\mathcal{E}(-2\xi)) = h^0(\mathcal{E}(-\xi - f)) = 0$ since $h^0(\mathcal{I}_C(-\xi + f)) = h^0(\mathcal{I}_C) = 0$, and so $\mathcal{O}_X(\xi)$ is saturated in $\mathcal{E}$. Hence the cokernel of m is torsion free, i.e. $\mathrm{coker}(m) \cong \mathcal{I}_T(f)$ with either $T = \emptyset$ or T is a locally complete intersection curve. Since $c_2(\mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)) = \xi f = C$, we have $T = \emptyset$. Since $h^1(\mathcal{O}_X(\xi - f)) = 0$, this implies $\mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$. □

Proposition 3.3.6. *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X with $c_1 = \xi + f$. If the associated curve $C = \bigcup_{i=1}^{s} C_i = \alpha\xi^2 + \beta\xi f$ has no connected component contained in E, and $\mathcal{E}$ has no trivial factor, then $\mathcal{E}$ exists if and only if r and $(s; \alpha, \beta)$ are as in the Main theorem.*

Proof. Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = \xi + f$, and assume that its associated curve C has no connected component contained in E. Since we assume $\mathcal{E}$ has no trivial factor, C is non-empty. Let $Y = \xi^2 + 2\xi f$ be a complete intersection of two general divisors in $|\mathcal{I}_C(\xi + f)|$. $Y = C \cup D$ with either $D = \emptyset$ or D a reduced curve containing no component of C and smooth outside $C \cap D$. We may assume Y does not have a component in E.

We have $(\alpha\xi^2 + \beta\xi f)f = \alpha\xi^2 f$. This implies that the line bundle $\mathcal{O}_X(f)$ restricted to C has degree α. Since $\mathcal{O}_X(f)$ is globally generated, it follows that $\alpha \geq 0$. The same argument applied to D implies that $\alpha \leq 1$. By assumption and $(\alpha\xi^2 + \beta\xi f)(\xi - f) = \beta\xi^2 f$, $C \cap E$ is a 0-dimensional scheme of degree β. It follows that $0 \leq \beta \leq 2$.

Assume $(\alpha, \beta) = (1, 0)$. Then $\sigma(C)$ is a line disjoint from R. Therefore, $\sigma(C) \cup R$ is cut out by quadrics since $\sigma(C) \cup R$ is the intersection of the unions of a plane containing $\sigma(C)$ and a plane containing R. This implies $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ is globally generated, hence the same for its pull-back $\mathcal{I}_C(2\tilde{H} - E) = \mathcal{I}_C(\xi + f)$. Since C is connected, rational and $\deg(\omega_C \otimes \mathcal{O}_X(2\xi)|_C) = 0$, we have $\bigwedge_C \cong \mathcal{O}_C$ and $h^0(\bigwedge_C) = 1$. By [3, Theorem 1.1.], this case gives a rank r globally generated vector bundle if and only if $r = 2$.

Assume $(\alpha, \beta) = (0, 1)$. Then $\sigma(C)$ is a line intersecting R at a point. Therefore, $\sigma(C) \cup R$ is a reducible conic and so $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ is globally generated. This implies $\mathcal{I}_C(2\tilde{H} - E)$ is also globally generated. Since C is rational and $\deg(\omega_C \otimes \mathcal{O}_X(2\xi)|_C) = 0$, we have $\bigwedge_C \cong \mathcal{O}_C$ and $h^0(\bigwedge_C) = 1$. Hence this case gives a rank r globally generated vector bundle if and only if $r = 2$. In this case, $\sigma(C)$ is in a plane containing R, hence C is contained in a fiber f of π. So we have $h^0(\mathcal{I}_C(f)) > 0$ and we can show $\mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$ as in the proof of Proposition 3.3.5.

Assume $(\alpha, \beta) = (1, 1)$ and $s = 1$. Then $\sigma(C)$ is a conic intersecting R at a point. Therefore, $\sigma(C) \cup R$ is cut out by quadrics since $\sigma(C) \cup R$ is the intersection of the unions of a plane containing $\sigma(C)$ and a plane containing R, and the quadric cones containing $\sigma(C)$ and R. This implies that $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ is globally generated, therefore $\mathcal{I}_C(2\tilde{H} - E)$ is also globally generated. Since C is rational and $\deg(\omega_C \otimes \mathcal{O}_X(2\xi)|_C) = 2$, so $\bigwedge_C$ is globally generated, non-trivial and we have $h^0(\bigwedge_C) = 3$. Hence this case gives a rank r globally generated vector bundle if and only if $3 \leq r \leq 4$.

Assume $(\alpha, \beta) = (1, 1)$ and $s = 2$. Then $\sigma(C)$ is a disjoint union of two lines, intersecting R at a point of one of its components. We write $\sigma(C) = L_1 \cup L_2$, where L_1, L_2 are lines and L_2 intersects R. Let $\langle L_2, R\rangle$ be the plane spanned by L_2 and R. Since L_1 intersects $\langle L_2, R\rangle$ at a point in $\langle L_2, R\rangle \setminus (L_2 \cup R)$, every quadric vanishing along $L_1 \cup L_2 \cup R$ must vanish on $\langle L_2, R\rangle$. Therefore $\langle L_2, R\rangle$ is contained in the base locus of $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ and so its strict transform is contained in the base locus of $\mathcal{I}_C(2\tilde{H} - E)$. Thus $\mathcal{I}_C(2\tilde{H} - E)$ is not globally generated.

Assume $(\alpha, \beta) = (0, 2)$ and $s = 1$. $\sigma(C)$ is a conic intersecting R at two points. The plane spanned by $\sigma(C)$ also contains R, so every quadric vanishing along $\sigma(C) \cup R$ must vanish on this plane. Thus, the strict transform of this plane is contained in the base locus of $\mathcal{I}_C(2\tilde{H} - E)$.

Assume $(\alpha, \beta) = (0, 2)$ and $s = 2$. $\sigma(C)$ is a disjoint union of two lines, each of which intersects R at a point. We write $\sigma(C) = L_1 \cup L_2$, where L_1, L_2 are the two lines. Then $\sigma(C) \cup R$ is cut out by quadrics since it is the intersection of the unions of a plane containing L_1 and the plane $\langle L_2, R\rangle$, and the unions of a plane containing L_1 and the plane $\langle L_2, R\rangle$. This implies $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ is globally generated, hence $\mathcal{I}_C(2\tilde{H} - E)$ is also globally generated. Since C has two components and each component is rational, $\bigwedge_C \cong \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1}^2$ is globally generated, trivial and we have $h^0(\bigwedge_C) = 2$. Hence this case gives a rank r globally generated vector bundle if and only if $2 \leq r \leq 3$.

Assume $(\alpha, \beta) = (1, 2)$. In this case, $C = Y$ and $\sigma(C)$ is a twisted cubic having R as a bisecant line. C is rational and $\mathcal{I}_C(2\tilde{H} - E)$ is globally generated. Since $\deg(\omega_C \otimes \mathcal{O}_X(2\xi)|_C) = 4$, $\bigwedge_C$ is globally generated, non-trivial and we have $h^0(\bigwedge_C) = 5$. Hence this case gives a rank r globally generated vector bundle if and only if $3 \leq r \leq 6$. □

Bibliography

[1] C. Anghel, I. Coandă, N. Manolache, *Globally Generated Vector Bundles with Small c_1 on Projective Spaces*, Mem. Amer. Math. Soc. Vol. 253. No. 1209. (2018).

[2] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, *Geometry of Algebraic Curves. Vol I*, Grundlehren Math. Wiss. Vol. 267, Springer-Verlag, New York (1985).

[3] E. Arrondo, *A home-made Hartshorne-Serre correspondence*, Rev. Mat. Complut. 20 (2) (2007), 423-443.

[4] E. Arrondo, L. Costa, *Vector bundles on Fano 3–folds without intermediate cohomology*, Commun. Algebra 28 (2000), 3899–3911.

[5] E. Ballico, *Spanned vector bundles on blowing ups of $\mathbb{P}^3$ at finitely many points and with very low c_1*, Gulf J. Math. 3 (3) (2015).

[6] E. Ballico, S. Huh, F. Malaspina, *Globally generated vector bundles of rank 2 on a smooth quadric threefold*, J. Pure Appl. Algebra 218 (2) (2014), 197–207.

[7] E. Ballico, S. Huh, F. Malaspina, *Globally generated vector bundles on complete intersection Calabi-Yau threefolds*, Adv. Theor. Math. Phys. 20 (5) (2016), 1125-1166.

[8] E. Ballico, S. Huh, F. Malaspina, *Globally generated vector bundles on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with low first Chern classes*, J. Algebra 450 (2016), 487-521.

[9] E. Ballico, S. Huh, F. Malaspina, *Globally generated vector bundles on the Segre threefold with Picard number two*, Math. Nachr. 289 (5-6) (2016), 523-536.

[10] M. Casanellas, R. Hartshorne, *ACM bundles on cubic surfaces*, J. Eur. Math. Soc. 13 (2011), 709-731.

[11] M. Casanellas, R. Hartshorne, F. Geiss, F. O. Schreyer, *Stable Ulrich bundles*, Internat. J. Math. 23 (8) (2012), 1250083–1250133.

[12] G. Casnati, *On rank two aCM bundles*, Comm. Algebra 45 (2017), 4139-4157.

[13] G. Casnati, *Rank two aCM bundles on general determinantal quartic surfaces in $\mathbb{P}^3$*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 63 (2017), 51-73.

[14] G. Casnati, *Rank 2 stable Ulrich bundles on anticanonically embedded surfaces*, Bull. Aust. Math. Soc. 95 (2017), 22–37.

[15] G. Casnati, D. Faenzi, F. Malaspina, *Rank two aCM bundles on the del Pezzo threefold with Picard number 3*, J. Algebra 429 (2015), 413–446.

[16] G. Casnati, D. Faenzi, F. Malaspina, *Rank two aCM bundles on the del Pezzo fourfold of degree 6 and its general hyperplane section*, J. Pure Appl. Algebra 22 (2018), 585–609.

[17] G. Casnati, M. Filip, F. Malaspina, *Rank two aCM bundles on the del Pezzo threefold of degree 7*, Rev. Mat. Complut. 30 (2017), 129–165.

[18] G. Casnati, O. Genc, *Instanton bundles on two Fano threefolds of index 1*, Forum Math. 32 (5) (2020), 1315-1336.

[19] G. Casnati, R. Notari, *Examples of rank two aCM bundles on smooth quartic surfaces in $\mathbb{P}^3$*, Rend. Circ. Mat. Palermo (2) 66 (1) (2017), 19–41.

[20] E. Coskun, R. S. Kulkarni, Y, Mustopa, *Pfaffian quartic surfaces and representations of Clifford algebras*, Doc. Math. 17 (2012), 1003–1028.

[21] E. Coskun, R. S. Kulkarni, Y Mustopa, *The geometry of Ulrich bundles on del Pezzo surfaces*, J. Algebra 375 (2013), 280–301.

[22] D. Eisenbud, J. Herzog, *The classification of homogeneous Cohen–Macaulay rings of finite representation type*, Math. Ann. 280 (1988), 347–352.

[23] D. Faenzi, *Rank 2 arithmetically Cohen–Macaulay bundles on a nonsingular cubic surface*, J. Algebra 319 (2008), 143–186.

[24] D. Faenzi, F. Malaspina, *Surfaces of minimal degree of tame representation type and mutations of Cohen–Macaulay modules*, Adv. Math. 310 (2017), 663–695.

[25] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin (1998).

[26] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. 52. Springer-Verlag, New York (1977).

[27] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proceedings of the London Mathematical Society 3.4 (1964), 689-713.

[28] D. Huybrechts, M, Lehn, *The Geometry of Moduli Spaces of Sheaves. Second edition*, Cambridge University Press (2010).

[29] N, Manolache, *Globally generated vector bundles on $\mathbb{P}^3$ with $c_1 = 3$*, arXiv (2012), arXiv:1202.5988.

[30] J. C. Migliore, *Introduction to Liaison Theory and Deficiency Modules*, Progr. Math. 165, Birkhäuser, Boston (1998).

[31] D. Mumford, *Lectures on Curves on an Algebraic Surface*, Ann. of Math. Stud. Vol. 59, Princeton University Press, Princeton (1966).

[32] C. Okonek, M. Schneider, H. Spindler, *Vector Bundles on Complex Projective Spaces*, Progr. Math. 3, Birkhäuser, Boston (1980).

[33] J. Pons-Llopis, F. Tonini, *ACM bundles on del Pezzo surfaces*, Matematiche (Catania) 64 (2009), 177–211.

[34] J. C. Sierra, *A degree bound for globally generated vector bundles*, Math. Z. 262 (2009), 517–525.

[35] J. C. Sierra, L. Ugaglia, *On double Veronese embeddings in the Grassmannian* $\mathbb{G}(1, N)$, Math. Nachr. 279.7 (2006), 798-804.

[36] J. C. Sierra, L. Ugaglia, *On globally generated vector bundles on projective spaces*, J. Pure Appl. Algebra 213 (11) (2009), 2141–2146.

[37] J. C. Sierra, L. Ugaglia, *On globally generated vector bundles on projective spaces II*, J. Pure Appl. Algebra 218 (1) (2014), 174–180.

List of papers by Takuya NEMOTO

(1) T. Nemoto, *Rank two ACM bundles on the double embedding of a quadric surface*, Comm. Algebra 50 (12) (2022), 5096-5112.

(2) T. Nemoto, *Globally generated vector bundles on a projective space blown up along a line*, Gulf J. Math. 15 (2) (2023), 152-159.