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Zariski's cancellation problem for principal Ga-bundles over non-A1-uniruled non-affine schemes

非単線織非アフィンスキーム上の主Ga束に関するザリスキの消去問題

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[21] R. Kudou, *About counterexamples for generalized Zariski cancellation problem*, Comm. Algebra, **48** (2020), 2358-2368. MR4107576

[22] R. Kudou, *Zariski's cancellation problem for principal $\mathbb{G}_a$-bundles over non-$\mathbb{A}^1$-uniruled quasi-affine varieties*, Res. Math. **10** (2023), no. 1, Paper No. 2281061, 6. MR4672555

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Chapter 1

Introduction

1.1 Background

Zariski's cancellation problem for an affine variety V (ZCP for V) asks whether the existence of an isomorphism between $V \times_k \mathbb{A}^1$ and $W \times_k \mathbb{A}^1$ for an affine variety W implies that V and W are isomorphic. ZCP is known to hold for affine curves [1], $\mathbb{A}^2$ [13][23], non-$\mathbb{A}^1$-uniruled affine varieties [3][18][5][6], and line bundles over non-$\mathbb{A}^1$-uniruled affine varieties [6]. ZCP for $\mathbb{A}^n$ still remains unsolved for $n \geqq 3$ in characteristic zero. However, in positive characteristic, Gupta [16] proved that ZCP for $\mathbb{A}^n$ does not hold if $n \geqq 3$.

	$\mathrm{ch}(k) = 0$	$\mathrm{ch}(k) > 0$
$n = 1$	✓Abhyankar-Hainzer-Eakin'72	
$n = 2$	✓Fujita'79, Miyanishi-Sugie'80	✓Russel'81
$n \geq 3$	???	×Gupta'14

Table 1.1: Zariski's cancellation problem for $\mathbb{A}^n$

Counterexamples have also been constructed in characteristic 0 using 1-stably free modules over a ring (e.g., [17] [25]) or the so-called "Danielewski's fiber product trick".

Lemma 1.1.1 (Danielewski's fiber product trick [4])**.** *Let X be a k-scheme. If two affine k-schemes V and W are isomorphic to principal $\mathbb{G}_a$-bundles over X, then $V \times_k \mathbb{A}^1 \simeq_k W \times_k \mathbb{A}^1$.*

Proof. Since V (resp. W) is affine, principal $\mathbb{G}_a$-bundles over V (resp. W) are all trivial. Since $V \times_X W$ is a principal $\mathbb{G}_a$-bundle over V and W, $V \times_k \mathbb{A}^1 \simeq V \times_X W \simeq W \times_k \mathbb{A}^1$. □

If V and W above are not isomorphic, then W is a counterexample to ZCP for V. On the other hand, such a counterexample can not be constructed if X is affine, since there exists a one to one correspondence between $\mathrm{H}^1(X, \mathcal{O}_X)$ and isomorphicm classes of principal $\mathbb{G}_a$-bundles over a scheme X. This fact implies that if X is affine, then any principal $\mathbb{G}_a$-bundle over an affine scheme X is isomorphic to $X \times_k \mathbb{A}^1$. Therefore principal $\mathbb{G}_a$-bundles over non-affine schemes have been studied

Counterexamples to ZCP for principal $\mathbb{G}_a$-bundles over a noetherian integral scheme X nonseparated over $\mathbb{C}$ were constructed in the case where X is a scheme of the following form:

Definition 1.1.2. Let Y be a variety, Z a closed subvariety of Y, and $r \in \mathbb{N}$. Let $Y_0, \dots, Y_r$ be $r+1$ copies of Y. Then

$$Y_+rZ := Y \sqcup_{Y \setminus Z} \underbrace{Y \sqcup_{Y \setminus Z} \cdots \sqcup_{Y \setminus Z} Y}_{r} = Y_0 \sqcup_{Y \setminus Z} Y_1 \sqcup_{Y \setminus Z} \cdots \sqcup_{Y \setminus Z} Y_r.$$

Namely Y_+rZ is a non-separated k-scheme obtained by gluing $r+1$ copies of Y along $Y \setminus Z$.

If $Y = \mathrm{Spec}(R)$ is an affine variety and $Z = \bigcup Z_i$ is the union of principal hypersurfaces Z_i defined by $f_i \in R$ for each $i = 1, \dots, n$, then a principal $\mathbb{G}_a$-bundle over Y_+Z is defined by an element g of $R_{f_1 \cdot \cdots \cdot f_n}$, and we denote it by V_g (See section 2.1 for the construction). If $g = \prod_{i=1}^n f_i^{-m_i}$, then V_g is isomorphic to $A_{m_1, \cdots, m_n} := \mathrm{Spec}(R[s,t]/(\prod_{i=1}^n f_i^{-m_i} \cdot s - t^2 + 1))$.

Danielewski [4] proved that if $Y = \mathbb{A}^1$ and $Z = \{0\}$, then A_1 is not isomorphic to A_n for $n > 1$. Later Fieseler [11] proved that if Y is a smooth affine curve and Z is a point, then A_m is isomorphic to A_n if and only if $m = n$. In higher dimension, Dubouloz [8] proved that if $Y = \mathbb{A}^l$ and Z is the union of coordinate hyperplane, then $A_{m_1, \dots, m_l} \not\simeq A_{n_1, \dots, n_l}$ if $\{m_1, \dots, m_l\} \neq \{n_1, \dots, n_l\}$. Dryło [6][7] proved that if Y is a non-$\mathbb{A}^1$-uniruled affine variety, and Z is the union of principal hypersurfaces of Y, then $A_{m_1, \cdots, m_n} \not\simeq A_{m'_1, \cdots, m'_n}$ if there exists $i \in \{1, \dots, n\}$ such that $m_i \notin \{m'_1, \cdots, m'_n\}$.

In the case where the base scheme X of a principal $\mathbb{G}_a$-bundle is a quasi-affine variety, Counterexamples have been constructed if X is the non-zero locus $\mathrm{D}(f_1, f_2)$ of two regular functions f_1, f_2 on an affine variety $Y = \mathrm{Spec}(R)$.

	Y	r	Z
Danielewski	$\mathbb{A}^1$	1	O
Fieseler	smooth affine curve	$r \in \mathbb{N}$	point
Dubouloz	$\mathbb{A}^n$	$r \in \mathbb{N}$	coordinate hyperplanes
Dryło	non-uniruled affine variety	1	principal hypersurface

Table 1.2: Counterexamples to ZCP for principal $\mathbb{G}_a$-bundles over $X = Y_{+}rZ$

In this case, a principal $\mathbb{G}_a$-bundle over $\mathrm{D}(f_1, f_2)$ is defined by an element g of $R_{f_1 f_2}$, and we denote by V_g the principal $\mathbb{G}_a$-bundle over $\mathrm{D}(f_1, f_2)$ defined by g (see Section 3.1 for the construction). If $g = f_1^{-m} f_2^{-n}$, then V_g is isomorphic to $A_{m,n} := \mathrm{Spec}(R[s,t]/(f_1^m s + f_2^n t - 1))$. For this problem, Finston-Maubach [12] proved that if $R = \mathbb{C}[x, y, z]/(x^a + y^b + z^c)$, where a, b, c are pairwise relatively prime positive integers satisfying $1/a + 1/b + 1/c < 1$ and if $f_1 = x$, $f_2 = y$, then $A_{m,n} \simeq_{\mathbb{C}} A_{m',n'}$ for nonnegative integers m, n, m', n' if and only if $(m, n) = (m', n')$.

Such a result does not hold for general R. For example, Dubouloz-Finston-Mehta [10] proved that if $R = \mathbb{C}[x, y]$, $f_1 = x$, and $f_2 = y$, then $m + n = m' + n'$ implies $A_{m,n} \simeq_{\mathbb{C}} A_{m',n'}$. Moreover, Dubouloz-Finston [9] proved that even if $(m, n) = (m', n')$, there exists $h, h' \in R = \mathbb{C}[x, y]$ such that $V_g \not\cong V_{g'}$ for $g = h \cdot f_1^{-m} f_2^{-n}$ and $g' = h' \cdot f_1^{-m'} f_2^{-n'}$. More precisely, they showed that $A(m, n, p) = R[s, t] / (x^m s + y^n t - p(x, y))$ and $A(m', n', p') = R[s, t] / \left(x^{m'} s + y^{n'} t - p'(x, y)\right)$ for $p, p' \in R \setminus ((x)_R \cup (y)_R)$ satisfying $\deg_x p < m, \deg_y p < n$ are nonisomorphic if $\deg p = m + n - 2$ and if $\deg p' < m' + n' - 2$.

Another result related to ZCP for $\mathbb{A}^n$ was obtained by Winkelmann [26] and Finston-Jaradat [19]. They proved that $\mathbb{A}^5$ is isomorphic to a principal $\mathbb{G}_a$-bundle over a strictly quasi-affine variety, that is a quasi-affine but non-affine variety. If there exists an affine variety W that is isomorphic to a principal $\mathbb{G}_a$-bundle over such a quasi-affine variety and satisfies $W \not\cong \mathbb{A}^5$, then W is a counterexample to ZCP for $\mathbb{A}^5$.

1.2 Main Results

One of the important problem of ZCP for principal $\mathbb{G}_a$-bundles is what the condition for two principal $\mathbb{G}_a$-bundles to be isomorphic is. In previous researches, if V_g is a principal $\mathbb{G}_a$-bundle over non-affine scheme X defined by $\overline{g} \in \mathrm{H}^1(X, \mathcal{O}_X)$, the number of poles of g plays an important role for this problem. In this paper We focus on this number, and we define an invariant $\mathrm{P}(\overline{g})$ of a principal $\mathbb{G}_a$-bundle V_g over a non-affine scheme X. $\mathrm{P}(\overline{g})$ corresponds to the number of poles, and we will prove that $P(\overline{g})$ is independent of the choice of g. (See Section 2.4 in the case of $X = Y_+Z$, and Section 3.3 in the case of $X = \mathrm{D}(f_1, f_2)$). Moreover we construct new counterexamples to ZCP for principal $\mathbb{G}_a$-bundles over a non-$\mathbb{A}^1$-uniruled non-affine scheme X, especially, in the case where X is a non-separated scheme of the form Y_+rZ, and in the case where X is a quasi-affine variety of the form $\mathrm{D}(f_1, f_2)$.

In the case where X is a non-separated scheme of the form Y_+rZ, we give a necessary and sufficient condition for two principal $\mathbb{G}_a$-bundles over Y_+rZ (Proposition 2.3.2), and we proved that even if $\mathrm{P}(\overline{g_1})$ and $\mathrm{P}(\overline{g_2})$ coincide, it is not necessarily true that V_{g1} and V_{g2} are isomorphic.

Theorem 1.2.1 (Theorem 2.4.2)**.** *Let P be a closed point of $\mathbb{A}^1_* = \mathrm{Spec}k[x, x^{-1}]$ defined by $f_1 = x - 1$. Let $X = \mathbb{A}^1_{*+}P$, $g_1 = (x+1) \cdot (x-1)^{-2}$, and $g_2 = (x-1)^{-2}$. Let V_{gi} be the principal $\mathbb{G}_a$-bundle over X defined by g_i. Then $V_{g1} \times \mathbb{A}^1 \simeq V_{g2} \times \mathbb{A}^1$ and $\mathrm{P}(\overline{g_1}) = \mathrm{P}(\overline{g_2}) = 2$, but $V_{g1} \not\cong V_{g2}$.*

In the case where X is a non-$\mathbb{A}^1$-uniruled quasi-affine variety of the form $\mathrm{D}(f_1, f_2)$, we give a sufficient condition for two principal $\mathbb{G}_a$-bundles over X to be non-isomorphic.

Theorem 1.2.2 (Theorem 3.4.3)**.** *Let $\mathrm{Spec}(R)$ be a non-$\mathbb{A}^1$-uniruled affine variety. Let (f_1, f_2) be an R-regular sequence, where f_1 and f_2 are prime elements such that the ideal $(f_1, f_2)_R$ is prime. Let V_g (resp. $V_{g'}$) be the principal $\mathbb{G}_a$-bundle over $D(f_1, f_2)$ that is defined by $g = v \cdot f_1^{-m} f_2^{-n}$ (resp. $g' = w \cdot f_1^{-m'} f_2^{-n'}$) with $\mathrm{P}\left(\overline{g'}\right) = (m', n')$. Then $V_g \not\cong V_{g'}$ if (1) or (2) holds.*

(1) *$m' > m + n - 1$ or $n' > m + n - 1$*

(2) *$m', n' \leqq m + n - 1$ and $v' \notin (f_1, f_2)^{m'+n'-m-n+\delta(v)}$, where*

$$\delta(v) = \begin{cases} 0 & \text{if } v \notin (f_1, f_2) \\ 1 & \text{if } v \in (f_1, f_2). \end{cases}$$

By using this theorem, we give a counterexample to ZCP.

Corollary 1.2.3 (Corollary 3.4.4). *Let* $\mathrm{Spec}(R)$ *be a non-*$\mathbb{A}^1$*-uniruled affine variety. Let* (f_1, f_2) *be an* R*-regular sequence, where* f_1 *and* f_2 *are prime elements such that the ideal* $(f_1, f_2)_R$ *is prime. Let* m, n, m', n' *be integers. Then* $V_{f_1^{-m} f_2^{-n}} \times \mathbb{A}^1 \simeq V_{f_1^{-m'} f_2^{-n'}} \times \mathbb{A}^1$ *but* $V_{f_1^{-m} f_2^{-n}} \not\cong V_{f_1^{-m'} f_2^{-n'}}$ *if* $m + n \neq m' + n'$.

In addition, we show that even if the numbers of poles (m, n) and (m', n') coincide, there exists $h, h' \in R$ such that $V_g \not\cong V_{g'}$, where $g = h \cdot f_1^{-m} f_2^{-n}$ and $g' = h' \cdot f_1^{-m'} f_2^{-n'}$ in the case where $\mathrm{Spec}(R)$ is not $\mathbb{A}^1$-uniruled.

Corollary 1.2.4 (Corollary 3.4.5). *Let* $\mathrm{Spec}(R)$ *be a non-*$\mathbb{A}^1$*-uniruled affine variety. Let* (f_1, f_2) *be an* R*-regular sequence, where* f_1 *and* f_2 *are prime elements such that the ideal* $(f_1, f_2)_R$ *is prime. Let* m, n *be integers larger than* 1. *Let* $\phi(X, Y)$ *be an element of* $(X, Y) \backslash ((X) \cup (Y)) \subset k[X, Y]$ *satisfying* $\deg_X \phi < m, \deg_Y \phi < n$. *Then* $V_{f_1^{-m} f_2^{-n}} \times \mathbb{A}^1 \simeq V_{\phi(f_1, f_2) \cdot f_1^{-m} f_2^{-n}} \times \mathbb{A}^1$ *and* $\mathrm{P}\left(\overline{f_1^{-m} f_2^{-n}}\right) = \mathrm{P}\left(\overline{\phi\left(f_1, f_2\right) \cdot f_1^{-m} f_2^{-n}}\right)$, *but* $V_{f_1^{-m} f_2^{-n}} \not\cong V_{\phi(f_1, f_2) \cdot f_1^{-m} f_2^{-n}}$.

1.3 Notations

In this paper we work over an algebraically closed field k of characteristic zero. For an ideal I of a ring R and for an integer n, I^n denotes the ideal generated by the products of n elements of I. If $n \leq 0$, $I^n := R$. For a scheme X and for $f \in \Gamma(X, \mathcal{O}_X)$, X_f and $\mathrm{D}(f)$ denote the nonzero locus of f. For $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$, $\mathrm{D}(f_1, \ldots, f_n)$ denotes the nonzero locus of $(f_1, \ldots, f_n)$. For a ring homomorphism $\psi : R \to S$, $\mathrm{Spec}(\psi)$ denotes the morphism of schemes $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ associated to ψ. For a morphism of schemes $\phi : X \to Y$ and subschemes $X' \subseteq X$ and $Y' \subseteq Y$, if $\phi(X') \subseteq Y'$, we denote by $\phi|_{X'} : X' \to Y'$ the restriction of ϕ from X' to Y'. $\mathbb{G}_a$ denotes the additive group variety $(\mathbb{A}^1_k, +)$ over k. For a k-scheme X and for an X-scheme V with a $\mathbb{G}_a$-action on V, V is called a **principal** $\mathbb{G}_a$**-bundle over** X in the Zariski topology if there is a covering $(U_i \to X)$ for the Zariski topology on X such that $V \times_X U_i$ is isomorphic with its $\mathbb{G}_a \times_k U_i$-action to $\mathbb{G}_a \times_k U_i$ over U_i. A variety X is $\mathbb{A}^1$**-uniruled** if for general closed point x of X, there exists a nonconstant morphism $f_x : \mathbb{A}^1 \to X$ such that $x \in f_x(\mathbb{A}^1)$.

Let $\preceq$ be a partial order on $\mathbb{Z}_{\geqq 0}^{\oplus n}$ defined as follows: For $(m_i), (m'_i) \in \mathbb{Z}_{\geqq 0}^{\oplus n}$ $(m_i) \preceq (m'_i)$ if and only if $m_i \leqq m'_i$ for each i. Write $(m_i) \prec (m'_i)$ if $(m_i) \preceq (m'_i)$ and $(m_i) \neq (m'_i)$.

Chapter 2

ZCP for principal $\mathbb{G}_a$-bundles over noetherian integral scheme non-separated over k

In this chapter, let Y be an affine variety, Z the union of principal hypersurfaces Z_i defined by a prime element $f_i \in R$ for each $i = 1, \ldots, n$, and r a non-negative integer.

2.1 Construction of principal $\mathbb{G}_a$-bundles over $Y_+ rZ$

First we describe how the one-to-one correspondence between isomorphism classes of principal $\mathbb{G}_a$-bundles over $X = Y_+ rZ$ and the elements of $\mathrm{H}^1(X, \mathcal{O}_X)$ is obtained.

Since $Y_i = Y$ and $Y_i \setminus Z \simeq \mathrm{Spec}(R_{f_1 \cdots f_n})$ are affine, we can compute $\mathrm{H}^1(X, \mathcal{O}_X)$ by Čech cohomology:

$$
\begin{aligned}
\mathrm{H}^1(X, \mathcal{O}_X) &\simeq \check{\mathrm{H}}^1(\{Y_0, \ldots, Y_r\}, \mathcal{O}_X) \\
&= \mathrm{Coker}\left(\phi : \bigoplus_{i=0}^{r} R \to \mathrm{Z}_1 : (a_i) \mapsto (a_i - a_j)\right),
\end{aligned}
$$

where

$$\mathrm{Z}_1 := \mathrm{Ker}\left(\bigoplus_{0\leqq i<j\leqq r} R_{f_1\cdots f_n} \to \bigoplus_{0\leqq i<j<k\leqq r} R_{f_1\cdots f_n} : (a_{ij}) \mapsto (a_{ij} - a_{ik} + a_{jk})\right).$$

For an element $g = (g_{ij}) \in \mathrm{Z}_1$, $\overline{g}$ denotes the image of g by the natural map $\mathrm{Z}_i \to \mathrm{Coker}\phi$. The principal $\mathbb{G}_a$-bundle V_g over X defined by g is, as a total space, an $\mathbb{A}^1$-bundle over X obtained by gluing $Y_i \times \mathbb{A}^1$ and $Y_j \times \mathbb{A}^1$ along the following isomorphism between open subschemes $Y \setminus Z \times \mathbb{A}^1$ of $Y_i \times \mathbb{A}^1$ and $Y \setminus Z \times \mathbb{A}^1$ of $Y_j \times \mathbb{A}^1$:

$$G_{g_{ij}} : Y \setminus Z \times \mathbb{A}^1 \to Y \setminus Z \times \mathbb{A}^1 : (x,t) \mapsto (x, t + g_{ij}).$$

The $\mathbb{G}_a$-action on V_g is obtained by gluing equivariantly trivial $\mathbb{G}_a$-actions on $Y_i \times \mathbb{A}^1$ for $i = 0, \ldots, r$, that acts trivially on Y_i and by addition on $\mathbb{A}^1$. The image of ϕ gives the isomorphism class as principal $\mathbb{G}_a$-bundles of V_g, and we denote it by $V_{\overline{g}}$.

2.2 Affine criterion for principal $\mathbb{G}_a$-bundles over Y_+rZ

Lemma 2.2.1 ([14])**.** *Let X be a scheme, Y an affine scheme, and $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ an open affine covering of X. Then for any morphism $f\colon X \to Y$, f is separated if and only if*

(1) *$U_\mu \cap U_\lambda$ is affine for any $\mu, \lambda \in \Lambda$;*

(2) *$\Gamma(U_\mu \cap U_\lambda, \mathcal{O}_X)$ is generated by $\Gamma(U_\mu, \mathcal{O}_X)$ and $\Gamma(U_\lambda, \mathcal{O}_X)$.*

The following proposition gives a necessary and sufficient condition for principal $\mathbb{G}_a$-bundles over Y_+rZ to be affine.

Proposition 2.2.2. *Let $Y = \mathrm{Spec}R$ be an affine variety. Let Z_i be the hypersurface of Y defined by a prime element $f_i \in R$ for each $i = 1, \ldots, n$. Let $Z := \bigcup Z_j$. Let $p : V \to Y_+rZ$ be a principal $\mathbb{G}_a$-bundle over Y_+rZ defined by $[\{g_{ij}\}] \in \check{\mathrm{H}}^1(\{Y_0, \ldots, Y_r\}, \mathcal{O}_{X_+rZ}) \simeq \mathrm{H}^1(Y_+rZ, \mathcal{O}_{Y_+rZ})$, where $g_{ij} = h_{ij} \cdot f_1^{-k_{ij,1}} \cdots f_m^{-k_{ij,m}} \in R_{f_1\cdots f_m}$, $k_{ij,l} \in \mathbb{Z}_{\geq 0}$, and $h_{ij} \in R$ such that h_{ij} can not be divided by f_l if $k_{ij,l} > 0$.*

If (a) $r = 1$ or (b) $r \geq 2$ and $\emptyset \neq Z_{l_1} \cap Z_{l_2} \not\subset \bigcup_{l\neq l_1,l_2} Z_l$ for any $l_1, l_2 = 1, \ldots, m$, then the following conditions are equivalent.

(1) $k_{ij,l} \geq 1$ *and* $(h_{ij}, f_1 \cdots f_m) = A$ *for any* $i,\ j = 0, \ldots, r$ *and* $l = 1, \ldots, m$.

(2) V *is separated.*

(3) V *is affine.*

Proof. (3) $\Rightarrow$ (2) is obvious. It is enough to show that (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3).

Let us denote $(k_{ij,1}, \ldots, k_{ij,m}) \in \mathbb{Z}^m$ by $[k_{ij}]$, $(1, \ldots, 1) \in \mathbb{Z}^m$ by $\mathbf{1}$, and $f_1^{k_{ij,1}} \cdots f_m^{k_{ij,m}}$ by $\underline{\mathrm{f}}^{[k_{ij}]}$.

First we give a necessaly and sufficient condition for V to be separated. Let $V_j := p^{-1}(Y_j)$ $(\simeq Y_j \times \mathbb{A}^1)$. Let $\mathcal{Y} := \{Y_0, \ldots, Y_r\}$ be an open covering of $Y_+ rZ$. An open subvariety $Y_i \cap Y_j$ is isomorphic to $Y \setminus Z$ for any $i, j = 0, \ldots r$. Then $V_i \cap V_j = p^{-1}(Y_i \cap Y_j)$ is isomorphic to $(Y_i \cap Y_j) \times \mathbb{A}^1$. Therefore V is separated if and only if $\Gamma(V_i \cap V_j, \mathcal{O}_V)$ is generated by the image of $\Gamma(V_i, \mathcal{O}_V)$ and $\Gamma(V_j, \mathcal{O}_V)$ for any $i, j = 0, \ldots r$ by Lemma 2.2.1. This condition is equivalent to $R_{f_1 \cdots f_m}[t] = R[g_{ij}][t]$ for any $i, j = 0, \ldots, r$ with $i \neq j$, where t is an indeterminate. Therefore V is separated if and only if $R_{\underline{\mathrm{f}}^{\mathbf{1}}} = R[g_{ij}]$ for any $i, j = 0, \ldots, r$ with $i \neq j$.

<u>(1) $\Rightarrow$ (2)</u> Suppose the condition (1). Then there exist elements $a, b \in R$ such that $1 = ah_{ij} + b\underline{\mathrm{f}}$ and $k_{ij,l} - 1 \geq 0$. Therefore $\underline{\mathrm{f}}^{-\mathbf{1}} = a\underline{\mathrm{f}}^{[k_{ij}]-\mathbf{1}} g_{ij} + b$ and $R_{\underline{\mathrm{f}}} = R[g_{ij}]$.

<u>(2) $\Rightarrow$ (1)</u> Suppose the condition (2). Then there exist $n \in \mathbb{N}$ and $a_0, \ldots a_n \in R$ such that

$$\underline{\mathrm{f}}^{-\mathbf{1}} = a_0 + a_1 g_{ij} + a_2 g_{ij}^2 + \cdots + a_n g_{ij}^n.$$

If $n = 0$, then $f_1, \ldots, f_m$ are units in R, a contradiction. Therefore $n > 0$. Multiplying both sides of the above equation by $\underline{\mathrm{f}}^{n[k_{ij}]}$, we obtain the following equation:

$$\underline{\mathrm{f}}^{n[k_{ij}]-\mathbf{1}} = a_0 \underline{\mathrm{f}}^{n[k_{ij}]} + h_{ij} s,$$

where $s = a_1 \underline{\mathrm{f}}^{(n-1)[k_{ij}]} + \cdots + a_{n-1} h_{ij}^{n-2} \underline{\mathrm{f}}^{[k_{ij}]} + a_n h_{ij}^{n-1} \in R$. Since $h_{ij} \notin (f_l)$ for any $l = 1, \ldots, m$, s can be divided by $\underline{\mathrm{f}}^{n[k_{ij}]-\mathbf{1}}$. Therefore there exists an element $s' \in A$ such that $1 = a_0 \underline{\mathrm{f}} + h_{ij} s$.

(1) $\Rightarrow$ (3) Suppose the condition (1). We first observe that if $r \neq 2$, there exists an index $j' \in \{1, \ldots, r\}$ such that $k_{0j',l} = \max_j\{k_{0j,l}\}$ for all $l = 1, \ldots, m$. Assume that there exist indices $j_1,\ j_2 \in \{1, \ldots r\}$ and l_1,

$l_2 \in \{1, \ldots, m\}$ such that $j_1 \neq j_2$, $l_1 \neq l_2$, $k_{0j_1,l_1} > k_{0j_2,l_1}$, and $k_{0j_1,l_2} < k_{0j_2,l_2}$ for contradiction. Let $\mu_l := \max\{k_{0j_1,l}, k_{0j_2,l}\}$ for each $l \in \{1, \ldots m\}$ and $[\mu] := (\mu_1, \ldots \mu_m) \in \mathbb{Z}^m_{\geq 0}$. Then It follows from the cocycle condition $g_{j_1 j_2} = g_{0j_2} - g_{0j_1}$ that

$$\underline{\mathrm{f}}^{[\mu]-[k_{j_1 j_2}]} h_{j_1 j_2} = \underline{\mathrm{f}}^{[\mu]-[k_{0j_2}]} h_{0j_2} - \underline{\mathrm{f}}^{[\mu]-[k_{0j_1}]} h_{0j_1}.$$

The right hand side of this equation is in $(f_{l_1}, f_{l_2}) \setminus ((f_{l_1}) \cup (f_{l_2})$. Therefore $\mu_{l_1} = k_{j_1 j_2, l_1}$ and $\mu_{l_2} = k_{j_1 j_2, l_2}$. Moreover, $\underline{\mathrm{f}}^{[\mu]-[k_{j_1 j_2}]} h_{j_1 j_2} \in (f_{l_1}, f_{l_2})$ implies $\underline{\mathrm{f}}^{[\mu]-[k_{j_1 j_2}]} \in (f_{l_1}, f_{l_2})$ because $h_{j_1 j_2}$ is a nonzero function on Z and $Z_{l_1} \cap Z_{l_2} \neq \emptyset$. This contradicts to the assumption $Z_{l_1} \cap Z_{l_2} \not\subset \bigcup_{l \neq l_1, l_2} Z_l$. Therefore we can choose an index $j' \in \{1, \ldots, r\}$ such that $k_{0j',l} = \max_j \{k_{0j,l}\}$ for all $l = 1, \ldots, m$.

Next, we show that there exists an affine morphism $\psi \colon V \to \mathbb{A}^1$ by induction on r. If $r = 0$ (i.e. $Y_+ rZ = Y$), then V is isomorphic to $Y \times_k \mathbb{A}^1$. Then the second projection of $Y \times_k \mathbb{A}^1$ is an affine morphism. Suppose the statement holds for $r - 1$. By the assumption, there exists $s_{ij} \in R$ such that $\overline{h_{ij} s_{ij}} = 1$ in $R/(f_1 \cdots f_m)$. Define morphisms $\phi_j \colon Y_j \to \mathbb{A}^1$ to be $\phi_j(x, t) = s_{0j}(\underline{\mathrm{f}}^{[k_{0j'}]} t + \underline{\mathrm{f}}^{[k_{0j'}]-[k_{0j}]} h_{0j})$ for each $j \in \{0, \ldots, r\}$ and define morphisms $\psi_j := \phi \circ g_j^{-1} \colon V_j \simeq Y \times_k \mathbb{A}^1 \to \mathbb{A}^1$ for each $j \in \{0, \ldots r\}$. By the cocycle condition, morphisms $\{\psi_j\}_{j=0,\ldots,r}$ glue to a morphism $\psi \colon V \to \mathbb{A}^1$. Define $H_j := g_j(Z \times_k \mathbb{A}^1) \subset V_j$. Then $\psi(H_0) = \phi_0 g_0^{-1} g_0(Z \times_k \mathbb{A}^1) = \{0\}$ and $\psi(H_{j'}) = \phi_{j'} g_{j'}^{-1} g_{j'}(Z \times_k \mathbb{A}^1) = \{1\}$. Therefore $\psi^{-1}(\mathbb{A}^1 \setminus \{0\}) \subseteq V \setminus H_0$ and $\psi^{-1}(\mathbb{A}^1 \setminus \{1\}) \subseteq V \setminus H_{j'}$. Moreover, $V \setminus H_0$ is the principal $\mathbb{G}_a$-bundle defined by the cocycle $\{g_{ij}\}_{i,j \neq 0}$, and $V \setminus H_{j'}$ is the principal $\mathbb{G}_a$-bundle defined by the cocycle $\{g_{ij}\}_{i,j \neq j'}$. Therefore $V \setminus H_0$ and $V \setminus H_{j'}$ are affine by the induction hypothesis. Therefore the restriction maps $\psi|_{V \setminus H_0} : V \setminus H_0 \to \mathbb{A}^1$ and $\psi|_{V \setminus H_{j'}} : V \setminus H_{j'} \to \mathbb{A}^1$ are affine morphisms, and hence $\psi^{-1}(\mathbb{A}^1 \setminus \{0\})$ and $\psi^{-1}(\mathbb{A}^1 \setminus \{1\})$ are affine. Therefore ψ is affine. □

Remark 2.2.3. Fieseler [11] proved this proposition in the case where Y is an affine curve, and Dubouloz [8] proved this proposition in the case where $Y = \mathbb{A}^n$ and Z is the union of coordinate hyperplanes of $\mathbb{A}^n$.

2.3 Necessary and sufficient condition for two principal $\mathbb{G}_a$-bundles to be nonisomorphic

Lemma 2.3.1. *Let Y be a non-$\mathbb{A}^1$-uniruled affine variety and let X be a k-scheme equiped with a dominant morphism $X \to Y$ such that there exists a covering $(X_i \to X)$, where X_i is a variety of dimension $\dim(Y)$ for each i. Let $p\colon V \to X$ and $q\colon W \to X$ be Zariski locally trivial $\mathbb{A}^n$-bundles over X. Then an isomorphism $\Phi\colon V \to W$ descends to an automorphism $\phi\colon X \to X$ such that $\phi \circ p = q \circ \Phi$.*

Proof. We can take a covering $(X_i \to X)_{i \in I}$ of X so that p and q are trivial over X_i for each i. Since Y is not $\mathbb{A}^1$-uniruled, Dryło's lemma [6, Lemma 2] implies that the fibers of p are contracted by the morphism $q \circ \Phi\colon V \to X$. Therefore the composition of any section $s_i\colon X_i \to p^{-1}X_i$ and $q \circ \Phi$ is independent of the choice of s_i for each i. The compositions $q \circ \Phi \circ s_i$ and $q \circ \Phi \circ s_j$ coincide on $X_i \cap X_j$ for the same reason. Therefore $\phi\colon X \to X$ exists. The inverse of ϕ can be constructed in the same way. □

Proposition 2.3.2. *Let $Y = \operatorname{Spec}(R)$ be a non-$\mathbb{A}^1$-uniruled affine variety. Let Z_i be the hypersurface of Y defined by $f_i \in R$ for each $i = 1, \dots, m$. Let $Z := \bigcup Z_j$. For $k = 1, 2$, let V_k be a principal $\mathbb{G}_a$-bundle over $X = Y_+ rZ$ defined by $g_k \in \mathrm{H}^1(X, \mathcal{O}_X)$.Then V_1 and V_2 are isomorphic if and only if g_1 and g_2 are in the same orbit of the action by $\operatorname{Aut}(Y_+ rZ) \times \Gamma(Y, \mathcal{O}_Y^\times)$.*

Proof. The computation of this proof in the case of $r = 1$ is almost the same as the proof of the sufficient condition for two principal $\mathbb{G}_a$-bundles to be non-isomorphic by R. Dryło [7]. Suppose that $\Phi\colon V_1 \to V_2$ is an isomorphism. By Lemma 2.3.1, there exists a unique automorphism $\phi\colon Y_+ rZ \to Y_+ rZ$ satisfies $\phi \circ p_1 = p_2 \circ \Phi$, where $p_k : V_k \to Y_+ rZ$ is the canonical projection of principal $\mathbb{G}_a$-bundles for $k = 1, 2$. Let $Y_i' := \phi(Y_i)$ and $\mathcal{Y}' := \{Y_0', \dots, Y_r'\}$, which is an open covering of $Y_+ rZ$. Suppose that V_1 is defined by $\{g_{ij}\} \in \mathrm{Z}^1(\mathcal{Y}, \mathcal{O}_{Y_+ rZ})$ and V_2 is defined by $\{g_{ij}'\} \in \mathrm{Z}^1(\mathcal{Y}', \mathcal{O}_{Y_+ rZ})$. Then the following diagram is commutative for each i, $j = 0, \dots, r$ $(i \neq j)$;

$$\begin{tikzcd}
(Y_i \cap Y_j) \times_k \mathbb{A}^1 \arrow[r, "g_i"] \arrow[d, "\alpha_{ij}"] & p_1^{-1}(Y_i \cap Y_j) \arrow[r, "\Phi"] \arrow[d, "\mathrm{id}"] & p_2^{-1}(Y_i' \cap Y_j') \arrow[r, "g_i'^{-1}"] \arrow[d, "\mathrm{id}"] & (Y_i' \cap Y_j') \times_k \mathbb{A}^1 \arrow[d, "\alpha_{ij}'"] \\
(Y_i \cap Y_j) \times_k \mathbb{A}^1 \arrow[r, "g_j"] \arrow[dr, "\mathrm{pr}_1"'] & p_1^{-1}(Y_i \cap Y_j) \arrow[r, "\Phi"] \arrow[d, "p_1"] & p_2^{-1}(Y_i' \cap Y_j') \arrow[r, "g_j'^{-1}"] \arrow[d, "p_2"] & (Y_i' \cap Y_j') \times_k \mathbb{A}^1 \arrow[dl, "\mathrm{pr}_1"] \\
& Y_i \cap Y_j \arrow[r, "\phi"] & Y_i' \cap Y_j' &
\end{tikzcd}$$

where $\alpha_{ij}(x,t) = (x, t + g_{ij}(x))$, $\alpha'_{ij}(x',t) = (x', t + g'_{ij}(x'))$. For an isomorphism $f : A[t] \to B[t]$ of domains such that $f|_A : A \to B$ is an isomorphism, $f(t)$ should be equals to $at+b$, where $a \in B^\times$ and $b \in B$ by the computation of the degree of $f(t)$. Therefore the commutativity of this diagram implies that there exists $a_i \in \Gamma(Y_i, \mathcal{O}^\times_{Y_+rZ}) = R^\times$ and $b_i \in \Gamma(Y_i, \mathcal{O}_{Y_+rZ})$ for each $i = 0, \ldots, r$ such that $g_i'^{-1} \circ \Phi \circ g_i(x,t) = (\phi(x), a_i t + b_i)$, $g_j'^{-1} \circ \Phi \circ g_j(x,t) = (\phi(x), a_j t + b_j)$. Therefore

$$a_i(x)t + b_i(x) + g'_{ij}(\phi(x)) = a_j(x)(t + g_{ij}(x)) + b_j(x).$$

Therefore we can glue $\{a_i\}$. Let a be an element of $\Gamma(Y_+rZ, \mathcal{O}^\times_{Y_+rZ}) \simeq \Gamma(Y, \mathcal{O}^\times_Y)$ such that $a|_{X_i} = a_i$. Then

$$g'_{ij}(\phi(x)) - g_{ij}(x)a = b_j(x) - b_i(x),$$

Therefore cocyles $\{g'_{ij}(\phi(x))\}$ and $\{g_{ij}(x)a\}$ define principal $\mathbb{G}_a$-bundles isomorphic to each other. $\square$

Next we study the automorphisms group of Y_+rZ. Let Y be a variety, Z a closed subset of Y, and r an integer. We will use the following notations for the proof of Proposition 2.3.5

- $\mathfrak{S}_{r+1}$: the symmetric group of degree $r+1$.
- N_Z: the number of connected components of Z
- $Z_1, \ldots, Z_{N_Z}$: the connected components of Z
- $Y_0, \ldots, Y_r$: open subsets of Y_+rZ defined in Definition 1.1.2.
- $u_i : Y_i \hookrightarrow Y_+rZ$: the inclusion morphism for each $i \in \{0, \ldots, r\}$

- $e_i : Y \simeq Y_i$: the natural isomorphism for each $i \in \{0, \ldots, r\}$.
- $h : Y_+rZ \to Y$: the morphism obtained by gluing $\{{e_i}^{-1}\}_{i\in\{0,\ldots,r\}}$.
- $Z_{,i} := e_i(Z)$ for each $i \in \{0, \ldots, r\}$.
- $Z_{k,i} := e_i(Z_k)$ for each $i \in \{0, \ldots, r\}$ and for each $k \in \{1, \ldots, N_Z\}$.
- $(r+1)Z := \bigcup_{i\in\{0,\ldots,r\}} Z_{,i}$ $(\subset Y_+rZ)$.
- $Y_{\setminus Z} := Y_+rZ \setminus (r+1)Z$ $(\simeq Y \setminus Z)$
- $\mathrm{End}(Y) :=$ the monoid of endomorphisms of Y
- $\mathrm{Aut}(Y) :=$ the group of automorphisms of Y
- $\mathrm{End}_Z(Y) := \{\Phi \in \mathrm{End}(Y) |\ \Phi(Z) \subseteq Z\}$
- $\mathrm{Aut}_Z(Y) := \{\Phi \in \mathrm{Aut}(Y) |\ \Phi(Z) = Z\}$.

The following two lemmas by J. Ax [2] and S. Kaliman [20] show that an injective endomorphism of an algebraic variety is an isomorphism.

Lemma 2.3.3 ([2])**.** *Let X be a scheme of finite type over a scheme Y. Let $\phi : X \to X$ be a Y-morphism. If ϕ is injective then ϕ is surjective.*

Lemma 2.3.4 ([20])**.** *Let $\phi : X \to X$ be a surjective endomorphism of a variety X over a field k of characteristic zero. Then ϕ is an automorphism.*

Proposition 2.3.5. *The following sequence of non-abelian groups is a right split exact sequence.*

$$1 \longrightarrow {\mathfrak{S}_{r+1}}^{\oplus N_Z} \xrightarrow{S} \mathrm{Aut}(Y_+rZ) \xrightarrow{T} \mathrm{Aut}_Z(Y) \longrightarrow 1$$

In other words, for any automorphism Φ of Y_+rZ, there exist the unique element $\sigma \in {\mathfrak{S}_{r+1}}^{\oplus N_Z}$ and the unique automorphism $\phi \in \mathrm{Aut}_Z(Y)$ such that $\Phi = R(\phi) \circ S(\sigma)$, where R is a group homomorphism from $\mathrm{Aut}_Z(Y)$ to $\mathrm{Aut}(Y_+rZ)$ such that T is a section of R.

Proof. Let Φ be an automorphism of Y_+rZ. First we show that for each $i \in \{0, \dots, r\}$, the image of $Z_{k,i}$ by Φ is equal to $Z_{k',i'}$ for some $k' \in \{1, \dots N_Z\}$ and for some $i' \in \{0, \dots, r\}$, where k' is independent of the choice of i. Let $\phi_i := h \circ \Phi \circ e_i : Y \to Y$ for each i. Then $\phi_1, \dots, \phi_r$ are endomorphisms of Y coincide on the open subset $Y \setminus Z$ with each others. Since Y is separated, $\phi_i = \phi_j$ as a morphism of varieties for any $i,\, j \in \{0, \dots, r\}$. Therefore the images of $Z_{k,i}$ and $Z_{k,j}$ by $h \circ \Phi : Y_+rZ \to Y$ coincide for any $i,\, j \in \{0, \dots, r\}$. Therefore the assertion holds.

Next we construct a map $T : \mathrm{Aut}(Y_+rZ) \to \mathrm{Aut}_Z(Y)$. Let

$$T' : \mathrm{Aut}(Y_+rZ) \to \mathrm{End}(Y) : \Phi \mapsto h \circ \Phi \circ u_0 \circ e_0.$$

Then the image of T' is in $\mathrm{End}_Z(Y)$. Since $T'(\Phi)$ is an injective endomorphism, Lemma 2.3.3 and Lemma 2.3.4 imply that $T'(\Phi)$ is an automorphism. Therefore we can restrict the codomain of T' to $\mathrm{Aut}_Z(Y)$. Let $T : \mathrm{Aut}(Y_+rZ) \to \mathrm{Aut}_Z(Y) : \Phi \mapsto T'(\Phi)$. The map T is a group homomorphism because for any $\Phi_1,\, \Phi_2 \in \mathrm{Aut}(Y_+rZ)$, $T(\Phi_1)\circ T(\Phi_2)$ and $T(\Phi_1 \circ \Phi_2)$ coincide on $Y \setminus Z$, and therefore coincide on Y. The group homomorphism T is surjective because for $\phi \in \mathrm{Aut}_Z(Y)$, we can glue $\{u_i \circ e_i \circ \phi \circ {e_i}^{-1} : Y_i \to Y_+rZ\}$ to an isomorphism $R(\phi) : Y_+rZ \to Y_+rZ$, which satisfies $T(R(\phi)) = \phi$ by the construction.

Next we construct a map $S : {\mathfrak{S}_{r+1}}^{\oplus N_Z} \to \mathrm{Aut}(Y_+rZ)$. For $\sigma = (\sigma_1, \dots, \sigma_{N_Z}) \in {\mathfrak{S}_{r+1}}^{\oplus N_Z}$, let $Y_{i,\sigma} := Y_{\setminus Z} \cup \bigcup_{i \in \{0,\dots,r\}} Z_{k,\sigma_k(i)}$ and let $e_{i,\sigma} : Y_i \to Y_{i,\sigma}$ be the canonical isomorphism, which is an identity on $X_{\setminus Z}$. then we can glue $\{e_{i,\sigma}\}$ to a endomorphism $S(\sigma)$ of X_+rZ, which is an isomorphism by construction. In this way, we can construct a map S from ${\mathfrak{S}_{r+1}}^{\oplus N_Z}$ to $\mathrm{Aut}(Y_+rZ)$. The map S is also an injective group homomorphism because $S(\sigma)$ corresponds to the permutation of $Z_{0,i}, \dots, Z_{r,i}$ by σ for each i.

Finally we show that the above sequence is exact. Since the automorphism $S(\sigma)$ of Y_+rZ is an identity map on $Y_{\setminus Z}$, the automorphism $T(S(\sigma))$ is the identity map on Y. Therefore $\mathrm{Im}(S) \subseteq \mathrm{Ker}(T)$. Conversely, suppose that for $\Phi \in \mathrm{Aut}(Y_+rZ)$, $T(\Phi)$ equals to id_Y. Then $h \circ \Phi \circ e_i = \mathrm{id}_Y$ for each i and $\Phi(Z_{k,i}) = Z_{k',i'(k,i)}$. Let $\sigma_k : \{0, \dots, r\} \to \{0, \dots, r\} : i \mapsto i'(k,i)$ for each $k \in \{1, \dots, N_Z\}$. Since Φ is an automorphism of Y_+rZ, $\sigma_k \in \mathfrak{S}_{r+1}$. Let $\sigma = (\sigma_1, \dots, \sigma_{N_Z})$. Then $S(\sigma) = \Phi$ by the construction. $\square$

2.4 Counterexamples to ZCP for principal $\mathbb{G}_a$-bundles over Y_+Z

Lemma 2.4.1. *Let $Y = \mathrm{Spec}(R)$ be an affine variety. Let Z be the union of principal hypersurfaces Z_i defined by a prime element f_i for each $i = 1, \dots, n$ such that $(f_1, \dots, f_n)$ is an R-regular sequence. Let $X = Y_+Z$. Let $g = h \cdot f_1^{-m_1} \cdots f_n^{-m_n} \in R_{f_1 \cdots f_n}$, where $h \in R$ such that $f_i \nmid h$ if $m_i > 0$. Then $(m_1, \dots, m_n)$ is the minimum element of the following set for the order $\preceq$:*

$$S_{\overline{g}} := \left\{ (m_1', \dots, m_n') \in \mathbb{Z}_{\geqq 0}^{\bigoplus n} | \overline{h' \cdot f_1^{-m_1'} \cdots f_n^{-m_n'}} = \overline{g} \ in\ \mathrm{H}^1(X, \mathcal{O}_X) \right\}.$$

Proof. For any $g' \in R_{f_1 \cdots f_n}$ such that $\overline{g'} = \overline{g}$, there exists $b \in R$ such that $g' = (h + b \cdot f_1^{m_1} \cdots f_n^{m_n}) \cdot f_1^{-m_1} \cdots f_n^{-m_n}$. Then $f_i \nmid (h + b \cdot f_1^{m_1} \cdots f_n^{m_n})$ if $m_i > 0$, and therefore $(m_1, \dots, m_n)$ is the minimum element of the above set. □

Now we denote by $\mathrm{P}(\overline{g})$ **the minimum element of** $\mathrm{S}_{\overline{g}}$ for the order $\preceq$. The above lemma implies that $P(\overline{g})$ is an invariant of principal $\mathbb{G}_a$-bundles over $X = Y_+Z$.

Theorem 2.4.2. *Let P be a closed point of $\mathbb{A}^1_* = \mathrm{Spec}(k[x, x^{-1}])$ defined by $f_1 = x - 1$. Let $X = \mathbb{A}^1_{*+}P$, $g_1 = (x+1) \cdot (x-1)^{-2}$, and $g_2 = (x-1)^{-2}$. Let V_{gi} be the principal $\mathbb{G}_a$-bundle over X defined by g_i. Then $V_{g1} \times \mathbb{A}^1 \simeq V_{g2} \times \mathbb{A}^1$ and $\mathrm{P}(\overline{g_1}) = \mathrm{P}(\overline{g_2}) = 2$, but $V_{g1} \not\cong V_{g2}$.*

Proof. The group of automorphisms of Y_+P can be expressed by using an element of $\mathfrak{S}_2 \simeq \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$ and an element of $\mathrm{Aut}_P(Y)$. An automorphism of $Y = \mathbb{A}^{1*}$ which fixes $P = \{x - 1 = 0\}$ should be an automorphism of Y which sends x to x (denoted by ϕ_1) or which sends x to x^{-1} (denoted by ϕ_{-1}). The automorphism $S(\overline{0})$ is an identity of Y_+P and $S(\overline{1})$ is an automorphism of Y_+P which is an identity on $Y \backslash P$ but replace $P_0 \in Y_0$ and $P_1 \in Y_1$.

By proposition 2.2.2, V_{g1} and V_{g2} are affine. Suppose that V is isomorphic to W for contradiction. By Proposition 2.3.2, there exist a unit $u' \in k[x, x^{-1}]$ and an automorphism Φ of Y_+P such that $g_2 = u' \cdot g_1(\Phi)$. The automorphism Φ is a composition of $S(\bar{i})$ and $R(\phi_j)$ for some $i \in \{0, 1\}$ and for some $j \in \{1, -1\}$. Since the automorphism $S(1)$ corresponds to replace open sets Y_0

and Y_1, $g_1(S(\bar{i})) = (-1)^i g_1$. Therefore $g_1(R(\phi_{-1})) = [\frac{2+(x^{-1}-1)}{(x^{-1}-1)^2}] = [\frac{2+3(x-1)}{(x-1)^2}]$. Therefore

$$g_1(\Phi) = [(-1)^i \frac{2+k(x-1)}{(x-1)^2}]$$

for some $k \in \{1, 3\}$. Since u and u' are units of $k[x, x^{-1}]$, there exist $c, c' \in k^*$ and $m, m' \in \mathbb{Z}$ such that $u = c \cdot x^m$ and $u' = c' \cdot x^{m'}$. We may assume that $m, m' \geq 0$ since for any unit $a \in k[x, x^{-1}]$, $g_2 = u' \cdot g_1(\Phi)$ if and only if $ag_2 = au' \cdot g_1(\Phi)$. If $n \neq 2$, then $g_2 - u' \cdot g_1(\Phi)$ can not vanish. Therefore we may assume that $n = 2$. Then

$$\begin{aligned} & g_2 - u' \cdot g_1(\Phi) \qquad (2.1) \\ = \ & [\frac{1}{(x-1)^2}\{(-1)^i(2+k(x-1))(c+cm(x-1)) - c' - c'm'(x-1)\}] \\ = \ & [\frac{1}{(x-1)^2}\{(-1)^i 2c - c' + ((-1)^i kc + (-1)^i 2cm - a'm')(x-1)\}]. \end{aligned}$$

Therefore $g_2 = u' \cdot g_1(\Phi)$ if and only if

$$\begin{cases} (-1)^i 2c - c' = 0, \\ (-1)^i kc + (-1)^i 2cm - a'm' = 0. \end{cases}$$

This condition implies $k + 2m - 2m' = 0$, but this contradicts to $k = 1$ or 3.

$\square$

Chapter 3

ZCP for principal $\mathbb{G}_a$-bundles over quasi-affine varieties

3.1 Construction of principal $\mathbb{G}_a$-bundles over $\mathrm{D}(f_1, f_2)$

First we describe how the one-to-one correspondence between isomorphism classes of principal $\mathbb{G}_a$-bundles over X and the elements of $\mathrm{H}^1(X, \mathcal{O}_X)$ is obtained. Since $\mathrm{D}(f_1)$, $\mathrm{D}(f_2)$, and $\mathrm{D}(f_1 f_2)$ are affine, we can compute $\mathrm{H}^1(X, \mathcal{O}_X)$ by Čech cohomology;

$$\begin{aligned}
\mathrm{H}^1(X, \mathcal{O}_X) &\simeq \check{\mathrm{H}}^1(\{\mathrm{D}(f_1), \mathrm{D}(f_2)\}, \mathcal{O}_X) \\
&= \mathrm{Coker}\left(\phi : R_{f_1} \bigoplus R_{f_2} \to R_{f_1 f_2} : (a, b) \mapsto a - b\right) \\
&= R_{f_1 f_2}/(R_{f_1} + R_{f_2}).
\end{aligned}$$

For an element $g \in R_{f_1 f_2}$, $\overline{g}$ denotes the image of g by the natural map $R_{f_1 f_2} \to \mathrm{Coker}\phi$. The principal $\mathbb{G}_a$-bundle V_g over X defined by g is, as a total space, an $\mathbb{A}^1$-bundle over X obtained by gluing $\mathrm{D}(f_1) \times \mathbb{A}^1$ and $\mathrm{D}(f_2) \times \mathbb{A}^1$ along the following isomorphism between open subschemes $\mathrm{D}(f_1 f_2) \times \mathbb{A}^1$ of $\mathrm{D}(f_1) \times \mathbb{A}^1$ and $\mathrm{D}(f_1 f_2) \times \mathbb{A}^1$ of $\mathrm{D}(f_2) \times \mathbb{A}^1$:

$$G_g : D(f_1 f_2) \times \mathbb{A}^1 \to \mathrm{D}(f_1 f_2) \times \mathbb{A}^1 : (x, t) \mapsto (x, t + g).$$

The $\mathbb{G}_a$-action on V_g is obtained by gluing equivariantly trivial $\mathbb{G}_a$-actions on $\mathrm{D}(f_i) \times \mathbb{A}^1$ for $i = 1, 2$, that acts trivially on $\mathrm{D}(f_i)$ and by addition on $\mathbb{A}^1$. The image of ϕ gives the isomorphism class as principal $\mathbb{G}_a$-bundles of V_g, and we denote it by $V_{\overline{g}}$.

3.2 Sufficient condition for principal $\mathbb{G}_a$-bundles over $\mathrm{D}(f_1, f_2)$ to be affine

Dubouloz-Finston-Mehta [10, Section 2] showed that nontrivial principal $\mathbb{G}_a$-bundles over $\mathbb{A}^2_*$ are affine, where $\mathbb{A}^2_*$ is a complement of a one point in $\mathbb{A}^2$. In general, a nontrivial principal $\mathbb{G}_a$-bundle over a quasi-affine variety is not necessarily affine, but their result suggest that there exist many nontrivial affine principal $\mathbb{G}_a$-bundles over $\mathrm{D}(f_1, f_2)$. In this section, we extend their result to principal $\mathbb{G}_a$-bundles over $\mathrm{D}(f_1, f_2)$.

Lemma 3.2.1 ([24][15, Theorem 5.2.1])**.** *A scheme X is affine if and only if there is a finite set of elements $f_1, \cdots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that X_{f_i} are affine, and $f_1, \cdots, f_n$ generates the unit ideal in $\Gamma(X, \mathcal{O}_X)$.*

Lemma 3.2.2. *Let X be a quasi-affine variety. Then the principal $\mathbb{G}_a$-bundle V over X defined by $g \in \mathrm{H}^1(X, \mathcal{O}_X)$ is affine if there exists $b \in \Gamma(X, \mathcal{O}_X)$ such that the principal $\mathbb{G}_a$-bundle V' defined by $b \cdot g \in \mathrm{H}^1(X, \mathcal{O}_X)$ is affine.*

Proof. Let $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that $\mathcal{U} = \{X_{f_i}\}_{i \in I}$ ($I = \{1, \ldots, n\}$) is an affine open covering of X. We may suppose that V and W are defined by Čech 1-cocycles $\{g_{ij}\}_{i,j \in I}$ and $\{b \cdot g_{ij}\}_{i,j \in I}$ of $\mathcal{O}_X$ relative to the open covering $\mathcal{U}$. Then $\Gamma(V, \mathcal{O}_V)$ and $\Gamma(W, \mathcal{O}_W)$ can be represented as follows:

$$\begin{aligned} \Gamma(V, \mathcal{O}_V) &= \{\{\phi_i(t)\}_{i \in I} |\, \phi_i(t) \in \Gamma(X_{f_i}, \mathcal{O}_X)[t], \phi_i(t + g_{ij}) = \phi_j(t)\} \\ \Gamma(W, \mathcal{O}_W) &= \left\{\{\phi_i(t)\}_{i \in I} |\phi_i(t) \in \Gamma(X_{f_i}, \mathcal{O}_X)[t], \phi_i(t + b \cdot g_{ij}) = \phi_j(t)\right\} \end{aligned}$$

Suppose that W is affine. By lemma 3.2.1, there exists $\{\phi_{i,k}(t)\}_{i,k \in I} \in \Gamma(W, \mathcal{O}_W)$ such that

$$\{f_1 \cdot \phi_{i,1}(t) + \cdots + f_n \cdot \phi_{i,n}(t)\}_{i \in I} = \{1\}_{i \in I} = 1.$$

Let $\psi_{i,k}(t) = \phi_{i,k}(b \cdot t)$ for each i and k. Then $\{\psi_{i,k}(t)\}_{i \in I} \in \Gamma(V, \mathcal{O}_V)$ for each k and

$$\{f_1 \cdot \psi_{i,1}(t) + \cdots + f_n \cdot \psi_{i,n}(t)\}_{i \in I} = \{f_1 \cdot \phi_{i,1}(b \cdot t) + \cdots + f_n \cdot \phi_{i,n}(b \cdot t)\}_{i \in I} = 1.$$

□

For a polynomial $\phi(x,y) = \sum_{i,j} a_{ij} \cdot x^i y^j$ in $R[x,y]$, let $\mathrm{Supp}(\phi)$ be the subset of $\mathbb{Z}_{\geqq 0}^{\oplus 2}$ consisting of elements (i,j) with $a_{ij} \neq 0$. Let $\mathrm{Min}(\phi)$ be the set consisting of minimal elements of $\mathrm{Supp}(\phi)$ for the order $\preceq$.

Proposition 3.2.3. *Let R be an integral domain and (f_1, f_2) an R-regular sequence, where f_1 and f_2 are prime elements. Let m, n be nonnegative integers. For $v \in R$, $g = v \cdot f_1^{-m} f_2^{-n}$. Then V_g is affine if there is $\phi(x,y) \in R[x,y]$ such that $v = \phi(f_1, f_2)$ and there is $(I,J) \in \mathrm{Min}(\phi)$ such that $(I,J) \prec (m,n)$, and $a_{IJ} \in R^*$.*

Proof. By the assumption, the following equation holds:

$$f_1^{m-I-1} \cdot f_2^{n-J-1} \cdot g \;=\; a_{IJ} \cdot f_1^{-1} \cdot f_2^{-1} + \sum_{(i,j)\neq(I,J)} a_{ij} \cdot f_1^{i-I-1} \cdot f_2^{j-J-1}.$$

The right-hand side of this equation is equal to $a_{IJ} \cdot f_1^{-1} \cdot f_2^{-1}$ in $\check{\mathrm{H}}^1(\{\mathrm{D}(f_1), \mathrm{D}(f_2)\}, \mathcal{O}_{D(f_1,f_2)})$. Since the principal $\mathbb{G}_a$-bundle over $D(f_1, f_2)$ defined by $a_{IJ} \cdot f_1^{-1} \cdot f_2^{-1}$ is isomorphic to $\mathrm{Spec}(R[s,t]/(f_1 s + f_2 t - a_{IJ}))$, Lemma 3.2.2 implies that V_g is affine. □

3.3 Invariant of principal $\mathbb{G}_a$-bundles over $X = \mathrm{D}(f_1, f_2)$

Lemma 3.3.1. *Let R be an integral domain, (f_1, f_2) be an R-regular sequence, where f_1 and f_2 are prime elements, and $\overline{g} \in \mathrm{H}^1(X, \mathcal{O}_X)$. Then the set $\mathrm{S}_{\overline{g}} := \{(m,n) \in \mathbb{Z}_{\geqq 0}^{\oplus 2} | \exists h \in R, \overline{h \cdot f_1^{-m} f_2^{-n}} = \overline{g}\}$ has a minimum element for the order $\preceq$.*

Proof. Since there exist only finite elements smaller than (m,n) in $\mathbb{Z}_{\geq 0}^{\oplus 2}$, it is sufficient to show that for $(m,n), (m'n') \in \mathrm{S}_{\overline{g}}$ with $m > m'$ and $n < n'$, there exists $(M,N) \in \mathrm{S}_{\overline{g}}$ such that $(M,N) \preceq (m,n), (m',n')$. Let $h, h' \in R$ such that $\overline{g} = \overline{h \cdot f_1^{-m} \cdot f_2^{-m}} = \overline{h' \cdot f_1^{-m'} f_2^{-n'}}$ in $\mathrm{H}^1(\mathrm{D}(f_1, f_2), \mathcal{O}_{\mathrm{D}(f_1,f_2)})$. We may suppose that $f_1, f_2 \nmid h, h'$. Then there exist $c_1, c_2 \in R$ and $o_1, o_2 \in \mathbb{Z}_{\geq 0}$ such that

$$h \cdot f_1^{-m} f_2^{-n} - c_1 \cdot f_1^{-o_1} = h' \cdot f_1^{-m'} f_2^{-n'} + c_2 \cdot f_2^{-o_2},$$

and $f_i \nmid c_i$ if $o_i > 0$. Since $f_1 \nmid (h \cdot f_1^{o_1 - m} + c_1 \cdot f_2^n)$, the above equation implies $o_1 \leqq m$, and $o_2 \leqq n'$ holds. Therefore, there exists $(M,N) \in \mathrm{S}_{\overline{g}}$ such that $(M,N) \preceq (m,n), (m',n')$. □

Now we denote by $\mathrm{P}(\overline{g})$ **the minimum element of** $\mathrm{S}_{\overline{g}}$ for $\overline{g} \in \mathrm{H}^1(\mathrm{D}(f_1, f_2), \mathcal{O}_{\mathrm{D}(f_1,f_2)})$.

Lemma 3.3.2. *Let R be an integral domain and (f_1, f_2) an R-regular sequence, where f_1 and f_2 are prime elements. Let m, n be nonnegative integers. For $h \in R$, let $g = h \cdot f_1^{-m} f_2^{-n}$. Then $\mathrm{P}(\overline{g}) = (m, n)$ if and only if $h \notin (f_1^m, f_2) \cup (f_1, f_2^n)$.*

Proof. It is enough to show that $\mathrm{P}(\overline{g}) \preceq (m-1, n)$ if and only if $h \in (f_1, f_2^n)$, and $\mathrm{P}(\overline{g}) \preceq (m, n-1)$ if and only if $h \in (f_1^m, f_2)$.

Suppose $h \in (f_1, f_2^n)$, i.e. there exist $a, b \in R$ such that $h = af_1 + bf_2^n$. Then $g - bf_1^{-m} = a \cdot f_1^{-m+1} f_2^{-n}$. Therefore $\mathrm{P}(\overline{g}) \preceq (m-1, n)$. Conversely, suppose $\mathrm{P}(\overline{g}) \preceq (m-1, n)$. Then there exists $a, b \in R$ and integers $i \geq m, j \geq n$ such that

$$\overline{g} = \overline{(hf_1^{i-m} f_2^{j-n} - af_2^i - bf_1^j) \cdot f_1^{-i} f_2^{-j}},$$

$f_1 \nmid a$ (resp. $f_2 \nmid b$) if $i > m$ (resp. $j > n$), and $hf_1^{i-m} f_2^{j-n} - af_2^i - bf_1^j \in (f_1)^{i-m+1}, (f_2)^{j-n}$. If $i > m$, then $af_2^j \in (f_1)^{i-m+1}$, and this is a contradiction. If $j > n$, then $bf_1^i \in (f_2)^{j-n}$, and this is a contradiction. Therefore $i = m$ and $j = n$. In this case, $h - af_2^n - bf_1^m \in (f_1)$. Therefore $h \in (f_1, f_2^n)$. In the same way, we can show that $\mathrm{P}(\overline{g}) \preceq (m, n-1)$ if and only if $h \in (f_1^m, f_2)$. $\square$

Lemma 3.3.3. *Let R be an integral domain and (f_1, f_2) an R-regular sequence, where f_1 and f_2 are prime elements. Let m, n be nonnegative integers. Let $\phi(X, Y) = \sum_{i,j} a_{ij} X^i Y^j = \sum_j b_j(X) Y^j = \sum_i c_i(Y) X^i \in R[X, Y]$ such that $X, Y \nmid \phi(X, Y)$. If $\phi(X, Y)$ satisfies the following two conditions:*

(1) *$0 \leqq \exists J \leqq n-1$ s.t. $a_{0J} \notin (f_1, f_2)$, $X \nmid b_J(X)$, and $\forall j < J$, $(X | b_j(X)$ or $f_2 | a_{0j})$;*

(2) *$0 \leqq \exists I \leqq m-1$ s.t. $a_{I0} \notin (f_1, f_2)$, $Y \nmid c_I(Y)$, and $\forall i < I$, $(Y | c_i(Y)$ or $f_1 | a_{i0})$;*

then $\mathrm{P}\left(\overline{\phi(f_1, f_2) \cdot f_1^{-m} f_2^{-n}}\right) = (m, n)$.

Proof. Suppose that

$$(m', n') := \mathrm{P}\left(\overline{\phi(f_1, f_2) \cdot f_1^{-m} f_2^{-n}}\right) \prec (m, n)$$

for contradiction. We assume that $m' < m$. Then there exists $c_1 \in R$ such that $f_1 | \phi(f_1, f_2) + c_1 \cdot f_2^n$, and there exists $c \in R$ such that

$$\begin{aligned} & c \cdot f_1 - b_0(f_1) - b_1(f_1) \cdot f_2^1 - \cdots - b_{I-1}(f_1) \cdot f_2^{I-1} \\ = \quad & f_2^I \cdot \left\{ a_{I0} + (b_I(f_1) - a_{I0}) + b_{I+1}(f_1) \cdot f_2 + \cdots + b_n(f_1) \cdot f_2^{n-I} \right\}. \end{aligned}$$

Then $a_{I0} \in (f_1, f_2)$ since $b_I(f_1) - a_{I0}$ can be divided by f_1, and this is a contradiction. $\square$

Proposition 3.3.4. *Let R be an integral domain and (f_1, f_2) an R-regular sequence, where f_1 and f_2 are prime elements. Let m and n be nonnegative integers. Suppose $\phi(X, Y) = \sum_{i,j} a_{ij} X^i Y^j \in k[X, Y] \setminus ((X) \cup (Y))$ satisfies $\deg_X \phi < m, \deg_Y \phi < n$. Then $\mathrm{P}\left(\overline{\phi(f_1, f_2) \cdot f_1^{-m} f_2^{-n}}\right) = (m, n)$.*

Proof. It is enough to show the existence of $(I', J') \in \mathbb{Z}_{\geq 0}^{\oplus 2}$ satisfying conditions (1) and (2) of Lemma 3.3.3. Since $\phi(X, Y) \in k[X, Y]$, $a_{ij} \notin (f_1, f_2)$ if $a_{ij} \neq 0$ for any (i, j). An integer j satisfying $X \nmid b_j(X)$ also exists since $f_1 \nmid \phi(f_1, f_2)$. Let J' be the minimum integer of j. Then J' satisfies the condition (1). An integer I' that satisfies (2) also exists for the same reason. $\square$

3.4 Sufficient condition for two principal $\mathbb{G}_a$-bundles to be nonisomorphic

Lemma 3.4.1. *Let $\mathrm{Spec}(R)$ be an affine variety. Let (f_1, f_2) be an R-regular sequence, where f_1 and f_2 are prime elements. Let V_g (resp. $V_{g'}$) be the principal $\mathbb{G}_a$-bundle defined by $g = v \cdot f_1^{-m} f_2^{-n}$ (resp. $g' = w \cdot f_1^{-m'} f_2^{-n'}$) over $X = D(f_1, f_2)$, where $v, w \in R$. Then $V_g \simeq_X V_{g'}$ if and only if there exists $a \in R^*$ such that $\overline{a \cdot g} = \overline{g'}$ in $\mathrm{H}^1(X, \mathcal{O}_X)$.*

Proof. Suppose that an isomorphism $\Phi : V_g \simeq_X V_{g'}$ exists. Then there exists $\phi_i \in \mathrm{Aut}(R_{f_i}[t])$ for each i such that $G_{g'} \circ \phi_1 = \phi_2 \circ G_g$, i.e., there exists $a_i \in R_{f_i}^*$ and $b_i \in R_{f_i}$ for each i such that $a_1 t + b_1 + g' = a_2(t + g) + b_2$. This equation implies that $a_1 = a_2 \in R^*$ and $g' - a_2 g = b_2 - b_1$. Conversely, $\overline{a \cdot g} = \overline{g'}$ implies the existence of $a \in R^*$ and $b_i \in R_{f_i}$, which satisfies the above condition. Therefore, $V_g \simeq_X V_{g'}$. $\square$

Lemma 3.4.2. *Let* $\mathrm{Spec}(R)$ *be an affine variety and* f_1, f_2 *elements in* $R \setminus \{0\}$. *Let* V_g *be a principal* $\mathbb{G}_a$*-bundle over* $X = \mathrm{D}(f_1, f_2)$ *defined by* $g = v \cdot f_1^{-m} f_2^{-n}$. *Let* ψ *be an automorphism of* R *such that* $\psi(f_1, f_2)_R = (f_1, f_2)_R$. *Then there exists* $h \in (f_1, f_2)_R^{m+n-2} \setminus \{0\}$ *such that the pushforward of* V_g *by* $\phi := \mathrm{Spec}(\psi)|_X : X \to X$ *is the principal* $\mathbb{G}_a$*-bundle over* X *defined by* $g_\phi := \psi^{-1}(v) \cdot h \cdot f_1^{-m-n+1} f_2^{-m-n+1}$.

Proof. Let $f_i' := \psi^{-1}(f_i)$ for short. Let $\mathcal{U}' = \{Y_{f_1'}, Y_{f_2'}\}$, which is an open affine covering of X. Then the pushforward of V_g by ϕ is defined by $\overline{g'} = \overline{\psi^{-1}(v) \cdot f_1'^{-m} \cdot f_2'^{-n}} \in \check{\mathrm{H}}^1(\mathcal{U}', \mathcal{O}_X)$.

Let

$$\begin{array}{rcl} \alpha_1 & := & \sum_{i=0}^{n-1} {}_{m+n-1}\mathrm{C}_i \cdot (a \cdot f_1')^{n-1-i} \cdot (b \cdot f_2')^i, \\ \alpha_2 & := & \sum_{i=n}^{m+n-1} {}_{m+n-1}\mathrm{C}_i \cdot (a \cdot f_1')^{m+n-1-i} \cdot (b \cdot f_2')^{i-n}, \\ \beta_1 & := & \sum_{i=0}^{n-1} {}_{m+n-1}\mathrm{C}_i \cdot (c \cdot f_1')^{n-1-i} \cdot (d \cdot f_2')^i, \\ \beta_2 & := & \sum_{i=n}^{m+n-1} {}_{m+n-1}\mathrm{C}_i \cdot (c \cdot f_1')^{m+n-1-i} \cdot (d \cdot f_2')^{i-n}, \end{array}$$

and h $:= \alpha_1\beta_2 - \alpha_2\beta_1$. Then $f_1^{m+n-1} = f_1'^m \cdot \alpha_1 + f_2'^n \cdot \alpha_2$ and $f_2^{m+n-1} = f_1'^m \cdot \beta_1 + f_2'^n \cdot \beta_2$.

Therefore,

$$\psi^{-1}(v) \cdot f_1'^{-m} \cdot f_2'^{-n} = \psi^{-1}(v) \cdot \alpha_1 \cdot f_2'^{-n} \cdot f_1^{-m-n+1} + \psi^{-1}(v) \cdot \alpha_2 \cdot f_1'^{-m} \cdot f_1^{-m-n+1} \text{ on } Y_{f_1},$$

$$\psi^{-1}(v) \cdot f_1'^{-m} \cdot f_2'^{-n} = \psi^{-1}(v) \cdot \beta_1 \cdot f_2'^{-n} \cdot f_2^{-m-n+1} + \psi^{-1}(v) \cdot \beta_2 \cdot f_1'^{-m} \cdot f_2^{-m-n+1} \text{ on } Y_{f_2}.$$

Hence, the principal $\mathbb{G}_a$-bundle $V_{g'} \times_X Y_{f_1'}$ over $Y_{f_1'}$ is defined by

$$\begin{array}{rl} & \psi^{-1}(v) \cdot \beta_2 \cdot f_1'^{-m} \cdot f_2^{-m-n+1} - \psi^{-1}(v) \cdot \alpha_2 \cdot f_1'^{-m} \cdot f_1^{-m-n+1} \\ = & \psi^{-1}(v) \cdot h \cdot f_1^{-m-n+1} f_2^{-m-n+1}. \end{array}$$

In the same way, $V_{g'} \times_X Y_{f_2'}$ over $Y_{f_2'}$ is defined by the above element. $\square$

Theorem 3.4.3. *Let* $\mathrm{Spec}(R)$ *be a non-*$\mathbb{A}^1$*-uniruled affine variety. Let* (f_1, f_2) *be an* R*-regular sequence, where* f_1 *and* f_2 *are prime elements such that the ideal* $(f_1, f_2)_R$ *is prime. Let* V_g *(resp.* $V_{g'}$*) be the principal* $\mathbb{G}_a$*-bundle over* $X = D(f_1, f_2)$ *defined by* $g = v \cdot f_1^{-m} f_2^{-n}$ *(resp.* $g' = w \cdot f_1^{-m'} f_2^{-n'}$*), where* $\mathrm{P}(\overline{g'}) = (m', n')$. *Then* $V_g \not\cong V_{g'}$ *if (1) or (2) holds.*

(1) $m' > m + n - 1$ *or* $n' > m + n - 1$

(2) $m', n' \leqq m+n-1$ *and* $v' \notin (f_1, f_2)^{m'+n'-m-n+\delta(v)}$, *where*

$$\delta(v) = \begin{cases} 0 & \text{if } v \notin (f_1, f_2) \\ 1 & \text{if } v \in (f_1, f_2). \end{cases}$$

Proof. Let $p : V_g \to X$ (resp. $p' : V_{g'} \to X$) be the structure morphism of the principal $\mathbb{G}_a$-bundle V_g over X (resp. $V_{g'}$ over X). Suppose that there exists an isomorphism $\Phi : V_g \simeq V_{g'}$ for contradiction. By Lemma 2.3.1, there exists an automorphism ϕ of X such that $\phi \circ p = p' \circ \Phi$. Since (f_1, f_2) is R-regular, there exists an automorphism ψ of R such that $\phi = \mathrm{Spec}(\psi)|_X : X \to X$. Then $\psi^{-1}(f_1, f_2)_R = (f_1, f_2)_R$ since $(f_1, f_2)_R$ is a prime ideal in R. By Lemma 3.4.2, the pushforward of the principal $\mathbb{G}_a$-bundle V_g by Φ is defined by g_ϕ. Then V_{g_ϕ} is isomorphic to $V_{g'}$ as a scheme over X. Therefore, there exists a unit a in R such that $\overline{a \cdot g'} = \overline{g_\phi}$ in $\mathrm{H}^1(X, \mathcal{O}_X)$ by Lemma 3.4.1.

If $m' > m+n-1$ or $n > m+n-1$, then $\mathrm{P}\left(\overline{g'}\right) \prec (m', n')$, which is a contradiction. Therefore $V_g \not\cong V_{g'}$.

If $m', n' \leqq m+n-1$ and $v' \notin (f_1, f_2)^{m'+n'-m-n+\delta(v)}$, then we can take $c_1, c_2 \in R$ that satisfy $g' = g_\phi + c_1 \cdot f_1^{-m-n+1} + c_2 \cdot f_2^{-m-n+1}$. Then,

$$a \cdot v' \cdot f_1^{m+n-m'-1} f_2^{m+n-n'-1} = \psi^{-1}(v) \cdot h + c_1 \cdot f_2^{m+n-1} + c_2 \cdot f_1^{m+n-1}$$

for some $h \in (f_1, f_2)^{m+n-2}$. Since $\psi^{-1}(v) \in (f_1, f_2)^{\delta(v)}$ and $h \in (f_1, f_2)^{m+n-2}$, the right-hand side of the above equation is in $(f_1, f_2)^{m+n-2+\delta(v)}$. Then $a \cdot v' \in (f_1, f_2)^{m'+n'-m-n+\delta(v)}$ since (f_1, f_2) is an R-regular sequence, which is a contradiction. Therefore, $V_g \not\cong V_{g'}$. □

Corollary 3.4.4. *Let* $\mathrm{Spec}(R)$ *be a non-*$\mathbb{A}^1$*-uniruled affine variety. Let* (f_1, f_2) *be an* R*-regular sequence, where* f_1 *and* f_2 *are prime elements such that the ideal* $(f_1, f_2)_R$ *is prime. Let* m, n, m', n' *be integers. Then* $V_{f_1^{-m} f_2^{-n}} \times \mathbb{A}^1 \simeq V_{f_1^{-m'} f_2^{-n'}} \times \mathbb{A}^1$ *but* $V_{f_1^{-m} f_2^{-n}} \not\cong V_{f_1^{-m'} f_2^{-n'}}$ *if* $m+n \neq m'+n'$.

Proof. Since $V_{f_1^{-m} f_2^{-n}}$ and $V_{f_1^{-m'} f_2^{-n'}}$ are affine, Danielewski's fiber product trick implies $V_{f_1^{-m} f_2^{-n}} \times \mathbb{A}^1 \simeq V_{f_1^{-m'} f_2^{-n'}} \times \mathbb{A}^1$. Suppose that $m'+n' > m+n$. In the case where $m' > m+n-1$ or $n' > m+n-1$, Theorem 3.4.3 implies that $V_{f_1^{-m} f_2^{-n}} \not\cong V_{f_1^{-m'} f_2^{-n'}}$. In the case where $m', n' \leqq m+n-1$, then $1 \notin (f_1, f_2)^{m'+n'-m-n}$. Therefore, Theorem 3.4.3 implies that $V_{f_1^{-m} f_2^{-n}} \not\cong V_{f_1^{-m'} f_2^{-n'}}$. □

Corollary 3.4.5. *Let* $\mathrm{Spec}(R)$ *be a non-*$\mathbb{A}^1$*-uniruled affine variety. Let* (f_1, f_2) *be an* R*-regular sequence, where* f_1 *and* f_2 *are prime elements such that the ideal* $(f_1, f_2)_R$ *is prime. Let* m, n *be integers larger than* 1*. Let* $\phi(X,Y)$ *be an element of* $(X,Y) \setminus ((X) \cup (Y)) \subset k[X,Y]$ *satisfying* $\deg_X \phi < m, \deg_Y \phi < n$*. Then* $V_{f_1^{-m} f_2^{-n}} \times \mathbb{A}^1 \simeq V_{\phi(f_1,f_2)\cdot f_1^{-m} f_2^{-n}} \times \mathbb{A}^1$ *and* $\mathrm{P}\left(\overline{f_1^{-m} f_2^{-n}}\right) = \mathrm{P}\left(\overline{\phi\left(f_1, f_2\right) \cdot f_1^{-m} f_2^{-n}}\right)$*, but* $V_{f_1^{-m} f_2^{-n}} \not\simeq V_{\phi(f_1,f_2)\cdot f_1^{-m} f_2^{-n}}$*.*

Proof. Let $\phi'(X,Y) := 1 \in k[X,Y]$. Then ϕ and ϕ' satisfy the condition of Proposition 3.2.3 and Proposition 3.3.4. Therefore $V_{f_1^{-m} f_2^{-n}}$ and $V_{\phi(f_1,f_2)\cdot f_1^{-m} f_2^{-n}}$ are affine, Danielewski's fiber product trick implies $V_{f_1^{-m} f_2^{-n}} \times \mathbb{A}^1 \simeq V_{\phi(f_1,f_2)\cdot f_1^{-m} f_2^{-n}} \times \mathbb{A}^1$, and

$$\mathrm{P}\left(\overline{f_1^{-m} f_2^{-n}}\right) = \mathrm{P}\left(\overline{\phi\left(f_1, f_2\right) \cdot f_1^{-m} f_2^{-n}}\right) = (m, n)\,.$$

Since (f_1, f_2) is an R-regular sequence, $1 \notin (f_1, f_2) = (f_1, f_2)^{m+n-m-n+1}$. Therefore, $f_1^{-m} f_2^{-n}$ and $\phi(f_1, f_2) \cdot f_1^{-m} f_2^{-n}$ satisfy condition (2) of Theorem 3.4.3. Therefore $V_{f_1^{-m} f_2^{-n}} \not\simeq V_{\phi(f_1,f_2)\cdot f_1^{-m} f_2^{-n}}$. $\square$

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