Families of rational curves and higher-dimensional algebraic geometry

有理曲線族および高次元代数幾何

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Chapter 1

Introduction

In this thesis, we investigate structures of special smooth complex projective manifolds by using deformation theory of rational curves on manifolds. This thesis consists of two main parts.

In Chapter 2, we consider embedded manifolds swept out by hypersurfaces, where a *hypersurface* means an embedded manifold which has codimension one in some linear subspace. Structure theorems for them have been obtained by several authors. E. Sato showed that n-folds swept out by linear subspaces of dimension $m \geq [\frac{n}{2}] + 1$ are scrolls ([28]). M. C. Beltrametti and P. Ionescu proved that n-folds swept out by hyperquadrics of dimension $m \geq [\frac{n}{2}] + 2$ are either scrolls or hyperquadric fibrations ([2]). K. Watanabe got that n-folds swept out by smooth cubic hypersurfaces of dimension $m \geq [\frac{n}{2}] + 3$ are either scrolls or cubic fibrations ([30]). These results motivate us to consider the case where d is large, and the following statement is naturally conjectured:

Conjecture 1.0.1. *Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold of dimension $n \geq 2d - 1$. Assume that X is swept out by smooth hypersurfaces of degree d and dimension $m \geq [\frac{n}{2}] + d$. Then either X is a scroll, or X admits a morphism $X \to Y$ whose general fibers are hypersurfaces of degree d.*

We will prove Conjecture 1.0.1 for $d = 4$:

Theorem 1.0.2. *Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold of dimension $n \geq 7$. Assume that X is swept out by smooth quartic hypersurfaces of dimension $m \geq [\frac{n}{2}] + 4$. Then either X is a scroll, or X admits a morphism $X \to Y$ whose general fibers are quartic hypersurfaces.*

Furthermore, we will provide an affirmative answer to Conjecture 1.0.1 under the stronger assumption $m \geq \frac{2n-1}{3} + d$ and also assuming Hartshorne's conjecture on complete intersections:

Theorem 1.0.3. *Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold of dimension $n \geq 3d - 1$. Assume that X is swept out by smooth hypersurfaces of degree d and dimension $m \geq \frac{2n-1}{3} + d$, and also assume Hartshorne's conjecture: If a manifold $X \subsetneq \mathbb{P}^N$ has dimension $n > \frac{2N}{3}$, then it is a complete intersection. Then either X is a scroll, or X admits a morphism $X \to Y$ whose general fibers are hypersurfaces of degree d.*

Chapter 3 deals with rationally connected manifolds, which are manifolds which contain a rational curve passing through two general points. A pair (X, H) consisting of a manifold X and an ample line bundle H on X is called a *polarized manifold*, and it is said to be *line connected*, *conic connected*, and *rationally cubic connected* if two general points of X can be joined by a rational curve of H-degree one, two, and three, respectively. It is easy to see that projective spaces $(\mathbb{P}^n, \mathscr{O}(1))$ are the only line connected manifolds. P. Ionescu, F. Russo, and V. Paterno studied conic connected manifolds, and proved that they have Picard number $\rho_X \leq 2$. G. Occhetta and V. Paterno considered rationally cubic connected manifolds. They showed that there is no upper bound on the Picard number, and got sufficient conditions for $\rho_X \leq 3$.

In this chapter, we discuss *rationally quartic connected* manifolds, namely, polarized manifolds (X, H) whose two general points can be joined by a rational curve of H-degree four. We will prove $\rho_X \leq 4$ under some assumptions:

Theorem 1.0.4. *Let X be a smooth complex projective manifold of dimension $n \geq 2$ with a fixed ample line bundle H, and assume that X is rationally connected with respect to a family $\mathscr{F}$ which satisfies $(H.\mathscr{F}) = 4$ and $(-K_X.\mathscr{F}) \geq n + 3$. Then we obtain at least one of the following:*

(a) *$\rho_X \leq 4$ and X is covered by lines;*

(b) *X is rationally cubic connected;*

(c) *X is 2-connected by conics which are numerically proportional to $\mathscr{F}$, namely, for two general points $x, y \in X$ there exist two conics C^1 and C^2 such that $[C^1] = [C^2] = \frac{1}{2}[\mathscr{F}]$, $x \in C^1$, $y \in C^2$, and $C^1 \cap C^2 \neq \emptyset$.*

Moreover, we will provide a classification of rationally quartic connected surfaces:

Theorem 1.0.5. *Let X be a smooth complex projective surface with a fixed ample line bundle H, and assume that X is rationally connected with respect to a family $\mathscr{F}$ with $(H.\mathscr{F}) = 4$.*

(1) *If X is covered by lines, then (X, H) is isomorphic to one of the following:*

(i) $(\mathbb{P}^2, \mathscr{O}(1))$,
(ii) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(1, 3))$, (iii) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(1, 2))$, (iv) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(1, 1))$,
(v) $(\mathbb{F}_1, C_0 + 4f)$, (vi) $(\mathbb{F}_1, C_0 + 3f)$, (vii) $(\mathbb{F}_1, C_0 + 2f)$,
(viii) $(\mathbb{F}_2, C_0 + 4f)$, (ix) $(\mathbb{F}_2, C_0 + 3f)$,
(x) $(\mathbb{F}_3, C_0 + 4f)$,

where we denote by C_0 a minimal section and by f a fiber on $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(-e) \oplus \mathscr{O})$.

(2) *If X is not covered by lines and $\mathscr{F}$ is not generically unsplit (see Definition 3.2.4), then (X, H) is isomorphic to one of the following:*

(xi) $(\mathbb{P}^2, \mathscr{O}(2))$,

(xii) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(2,2))$,

(xiii) $(S_k, -K_{S_k})$ *for some* $2 \le k \le 8$,

where S_k *means a blow-up of* $\mathbb{P}^2$ *at* k *general points.*

(3) *If* X *is not covered by lines and* $\mathscr{F}$ *is generically unsplit, then* (X, H) *is isomorphic to one of the following:*

(xiv) $(\mathbb{P}^2, \mathscr{O}(4))$,

(xv) $(T_k, 4L - E_1 - \cdots - E_k)$ *for some* $1 \le k \le 15$,

(xvi) $(T_k, 4L - 2E_1 - E_2 - \cdots - E_k)$ *for some* $1 \le k \le 12$,

(xvii) $(\tilde{T}_k, 4\tilde{L} - 3\tilde{E} - 2\tilde{E}_1 - \tilde{E}_2 - \cdots - \tilde{E}_k)$ *for some* $1 \le k \le 11$,

where T_k *is a blow-up of* $\mathbb{P}^2$ *at* k *(possibly not general) points, and we denote by* L *the pullback of* $\mathscr{O}(1)$ *and by* E_i *the exceptional curve, and furthermore,* $\tilde{T}_k$ *is a blow-up of* T_k *at a point in* E_1*, and we denote by* $\tilde{L}$ *the pullback of* L*, by* $\tilde{E}$ *the exceptional curve, and by* $\tilde{E}_i$ *the strict transform of* E_i*.*

Chapter 2

On manifolds swept out by high dimensional hypersurfaces

2.1 Introduction

In this chapter, we investigate structures of embedded smooth complex projective manifolds swept out by high dimensional hypersurfaces of degree d, where a *hypersurface* means an embedded projective manifold which has codimension one in some linear subspace. They have been studied in several ways for small values of d. In case $d = 1$, E. Sato showed that projective manifolds of dimension n swept out by linear subspaces of dimension $m \geq [\frac{n}{2}] + 1$ are scrolls ([28]). In case $d = 2$, M. C. Beltrametti and P. Ionescu proved that projective manifolds of dimension $n \geq 3$ swept out by hyperquadrics of dimension $m \geq [\frac{n}{2}] + 2$ are either scrolls or hyperquadric fibrations ([2]). Remark that other results for $d = 2$ have also been obtained by Y. Kachi and E. Sato ([15]) and by B. Fu ([5]). In case $d = 3$, K. Watanabe showed that projective manifolds of dimension $n \geq 5$ swept out by smooth cubic hypersurfaces of dimension $m \geq [\frac{n}{2}] + 3$ are either scrolls or cubic fibrations ([30]). These results motivate us to consider the case where d is large, and the following statement is naturally conjectured:

Conjecture 2.1.1. *Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold of dimension $n \geq 2d - 1$. Assume that X is swept out by smooth hypersurfaces of degree d and dimension $m \geq [\frac{n}{2}] + d$. Then either X is a scroll, or X admits a morphism $X \to Y$ whose general fibers are hypersurfaces of degree d.*

We will prove Conjecture 2.1.1 for $d = 4$ (Theorem 2.3.5) by employing the classifications of Fano manifolds with high index. On the other hand, in case $d \geq 5$, this conjecture is an open problem.

However, we provide an affirmative answer to Conjecture 2.1.1 under the stronger assumption $m \geq \frac{2n-1}{3} + d$ and also assuming Hartshorne's conjecture. Here the statement of Hartshorne's conjecture is the following:

Conjecture 2.1.2 (R. Hartshorne)**.** *Let $X \subsetneq \mathbb{P}^N$ be a smooth projective manifold of dimension $n \geq 3$. If $n > \frac{2N}{3}$, then X is a complete intersection.*

In fact, we only need to assume the following conjecture which is a weaker version of Conjecture 2.1.2:

Conjecture 2.1.3. *Let $X \subsetneq \mathbb{P}^N$ be a smooth projective manifold of dimension $n \geq 5$. Let $\mathscr{L}$ be a covering family of lines on X, and let $x \in X$ be a general point. If $\dim \mathscr{L}_x > \frac{2(n-1)}{3} (= \frac{2}{3} \dim \mathbb{P}(T_x X^*))$, then $\mathscr{L}_x \subsetneq \mathbb{P}(T_x X^*)$ is a complete intersection.*

We will prove the following theorem:

Theorem 2.1.4. *Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold of dimension $n \geq 3d-1$. Assume that there exists a smooth hypersurface of degree d and dimension $m \geq \frac{2n-1}{3} + d$ passing through a general point of X. Furthermore, assume that Conjecture 2.1.3 is true. Then X admits a morphism $\varphi : X \to Y$ such that*

- *φ is a contraction of an extremal ray;*
- *the relative dimension of φ is at least m;*
- *either φ is a scroll, or general fibers of φ are hypersurfaces of degree d.*

In our proof, in order to obtain a contraction φ, we need the theory of families of lines. In addition, in order to determine fibers of φ, we use Hartshorne's conjecture and the theory of second fundamental forms, which were employed in [5] and [13].

2.2 Preliminaries

Throughout this chapter, we consider $X \subset \mathbb{P}^N$ an n-dimensional smooth complex projective manifold. A morphism $\varphi : X \to Y$ is called a *scroll* when it is a projective space bundle $\mathbb{P}(\mathscr{E}) \to Y$ for some vector bundle $\mathscr{E}$ on Y and its fibers are embedded linearly in $\mathbb{P}^N$.

We denote by $F^1(X)$ the Hilbert scheme of lines on X. For a point $x \in X$, we also denote by $F^1_x(X)$ the Hilbert scheme of lines on X passing through x. An irreducible component of $F^1(X)$ is called a *family of lines* on X. For a family of lines $\mathscr{L}$, an *$\mathscr{L}$-line* means a line which is a member of $\mathscr{L}$. Let $\mathrm{Univ}(X)$ be the universal family of $\mathrm{Hilb}(X)$, and let $p : \mathrm{Univ}(X) \to \mathrm{Hilb}(X)$ and $q : \mathrm{Univ}(X) \to X$ be the associated morphisms. For a subset $\mathscr{V} \subset \mathrm{Hilb}(X)$, $q(p^{-1}(\mathscr{V}))$ is denoted by $\mathrm{Locus}(\mathscr{V})$. A family of lines $\mathscr{L}$ is said to be *covering* if $\mathrm{Locus}(\mathscr{L}) = X$. For a covering family of lines $\mathscr{L}$ and for a general point $x \in X$, we denote by $\mathscr{L}_x$ the scheme of $\mathscr{L}$-lines passing through x, which is called the *variety of minimal rational tangents* (at x w.r.t. $\mathscr{L}$).

Proposition 2.2.1 ([10, Theorem 1.5, Theorem 2.5])**.** *Suppose that $X \subset \mathbb{P}^N$ is covered by lines. Let $\mathscr{L}$ be a covering family of lines, and x a general point of X. Assume that $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathscr{O}_X(1) \rangle$, where $\mathscr{O}_X(1)$ is the restriction of the tautological line bundle $\mathscr{O}_{\mathbb{P}^N}(1)$ to X. If $\dim \mathscr{L}_x \geq \frac{n-1}{2}$, then $\mathscr{L}_x \subset \mathbb{P}(T_x X^*)$ is smooth, irreducible, and non-degenerate.*

Proposition 2.2.2 ([30, Proposition 2.2]). *Suppose that $X \subset \mathbb{P}^N$ is covered by lines, and let x be a general point of X. Assume that $F^1_x(X)$ is irreducible. Then there exists a unique covering family of lines $\mathscr{L}$. In particular, $\mathscr{L}_x = F^1_x(X)$.*

We denote by ρ_X the Picard number of X. Suppose that X is Fano, namely, the anticanonical divisor $-K_X$ is ample. We denote by $i(X)$ the greatest positive integer i such that $-K_X = iH$ for some ample divisor H, which is called the *index* of X. We also denote by $l(X)$ the minimum of intersection numbers of $-K_X$ with rational curves on X, which is called the *pseudo-index* of X.

Proposition 2.2.3 ([31]). *Suppose that X is Fano. If $l(X) \geq \frac{n+3}{2}$, then $\rho_X = 1$.*

Proposition 2.2.4 ([17]). *Let X be a Fano manifold. Then $i(X) \leq n+1$. Furthermore,*

(1) *if $i(X) = n+1$, then X is isomorphic to $\mathbb{P}^n$;*

(2) *if $i(X) = n$, then X is isomorphic to a quadric hypersurface.*

Proposition 2.2.5 ([6], [7]). *Let X be a Fano manifold with index $i(X) = n-1$ whose Picard group is generated by a very ample divisor. Then X is isomorphic to one of the following:*

(1) *a cubic hypersurface,*

(2) *a complete intersection of two quadric hypersurfaces,*

(3) *a linear section of the Grassmann variety $G(2, \mathbb{C}^5) \subset \mathbb{P}^9$.*

Proposition 2.2.6 ([22]). *Let X be a Fano manifold with index $i(X) = n-2$ whose Picard group is generated by a very ample divisor. Then X is isomorphic to one of the following:*

(1) *a quartic hypersurface,*

(2) *a complete intersection of a quadric hypersurface and a cubic hypersurface,*

(3) *a complete intersection of three quadric hypersurfaces,*

(4) *a linear section of a quadric section of the cone $C \subset \mathbb{P}^{10}$ over the Grassmann variety $G(2, \mathbb{C}^5) \subset \mathbb{P}^9$,*

(5) *a linear section of the spinor variety S_4 which is an irreducible component of the Fano variety of 4-planes in Q^8,*

(6) *a linear section of the Grassmann variety $G(2, \mathbb{C}^6) \subset \mathbb{P}^{14}$,*

(7) *a linear section of the symplectic Grassmann variety $SG(3, \mathbb{C}^6) \subset \mathbb{P}^{13}$,*

(8) *the G_2-variety which is the variety of isotropic 5-planes for a non-degenerate skew-symmetric 4-linear form on $\mathbb{C}^7$.*

Let $NE(X)$ be the cone of effective 1-cycles on X. For a family of lines $\mathscr{L}$, we denote by $[\mathscr{L}]$ the numerical class of an $\mathscr{L}$-line, and denote by $(D.\mathscr{L})$ the intersection number of a divisor D and an $\mathscr{L}$-line.

Proposition 2.2.7 ([23, Theorem 3.3]). *Suppose that $X \subset \mathbb{P}^N$ is covered by lines, and let $\mathscr{L}$ be a covering family of lines. Assume that $(-K_X.\mathscr{L}) \geq \frac{n+1}{2}$. Then $\mathbb{R}_{\geq 0}[\mathscr{L}]$ is an extremal ray of $NE(X)$.*

Let $S(X) \subset \mathbb{P}^N$ be the secant variety of X, which is the closure of the union of secant lines. The *secant defect* of X is defined as the number $\delta(X) := 2n+1-\dim S(X)$. Clearly $\delta(X) \geq 0$.

Proposition 2.2.8 ([11, Theorem 3.14]). *Suppose that $X \subset \mathbb{P}^N$ is a Fano manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathscr{O}_X(1) \rangle$. If $i(X) > \frac{2n}{3}$, then $\delta(X) > 0$.*

Suppose that $X \subsetneq \mathbb{P}^N$ is non-degenerate and $n \geq 2$, and let $x \in X$ be a general point. We will define the second fundamental form $|II_{x,X}|$ as in [14, Remark 3.2.11] and [27, Definition 1.5]. Consider the projection $\pi_x : X \dashrightarrow \mathbb{P}^{N-n-1}$ from T_xX onto a disjoint linear subspace $\mathbb{P}^{N-n-1} \subset \mathbb{P}^N$. The map π_x is associated to the linear system of hyperplane sections cut out by hyperplanes containing T_xX, or equivalently, by the hyperplane sections singular at x. Let $\phi : \mathrm{Bl}_x(X) \to X$ be the blow-up of X at x, $E := \mathbb{P}(T_xX^*) \subset \mathrm{Bl}_x(X)$ the exceptional divisor, and H a hyperplane section of $X \subsetneq \mathbb{P}^N$. The restriction of the induced rational map $\widetilde{\pi}_x : \mathrm{Bl}_x(X) \dashrightarrow \mathbb{P}^{N-n-1}$ to E is given by a linear system in $|\phi^*(H) - 2E|_{|E} \subset |-2E_{|E}| = |\mathscr{O}_{\mathbb{P}(T_xX^*)}(2)| = \mathbb{P}(S^2(T_xX))$.

Definition 2.2.9. The *second fundamental form* $|II_{x,X}| \subset \mathbb{P}(S^2(T_xX))$ is the non-empty linear system of quadric hypersurfaces in $\mathbb{P}(T_xX^*)$ defining the restriction of $\widetilde{\pi}_x$ to E.

Clearly $\dim |II_{x,X}| \leq N-n-1$. The base locus on E of the second fundamental form $|II_{x,X}|$ consists of *asymptotic directions*, namely, of directions associated to lines having a contact of order at least three with X at x.

Proposition 2.2.10 ([27, Theorem 2.3(1)], see also [13, Proposition 1.2]). *Suppose that $X \subsetneq \mathbb{P}^N$ is non-degenerate and $n \geq 2$. If $\delta(X) > 0$, then $\dim |II_{x,X}| = N-n-1$ for a general point $x \in X$.*

We will use the following proposition several times:

Proposition 2.2.11 ([32, I Proposition 2.16]). *Let $X \subset \mathbb{P}^N$ be a non-degenerate smooth projective manifold. Let Z be a closed subvariety of X such that $\dim Z > [\frac{N-1}{2}]$. Then $\mathrm{codim}_{\mathbb{P}^N}(X) \leq \mathrm{codim}_{\langle Z \rangle}(Z)$, where $\langle Z \rangle$ means the linear span of Z.*

2.3 Main results

Notation 2.3.1. Let S be a smooth hypersurface of degree d and dimension $m > d$. Then, for a general point $x \in S$, $F^1_x(S) \subset \mathbb{P}(T_xS^*)$ is a smooth complete intersection of degrees $(d, d-1, d-2, \ldots, 2)$ by [10, 1.4.2]. In particular, it is irreducible. Thus, there exists a unique covering family of lines on S by Proposition 2.2.2. We denote it by $\mathscr{L}^S$.

From now on, we consider the case where $d \geq 2$. First, under the weaker assumption that $m \geq [\frac{n}{2}] + d$, we prove the following two assertions.

Lemma 2.3.2. *Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold of dimension $n \geq 2d - 1$. Assume that there exists a smooth hypersurface of degree $d \geq 2$ and dimension $m \geq [\frac{n}{2}] + d$ passing through a general point of X. Then there exists a covering family of lines $\mathscr{L}$, and for a general point $x \in X$, there exists a smooth hypersurface $S_x \subset X$ of degree d and dimension m passing through x such that $\mathscr{L}^{S_x} \subset \mathscr{L}$ (see Notation 2.3.1) and $F^1_x(S_x) \subset \mathscr{L}_x$.*

Proof. First, we remark that the inequalities $m \geq [\frac{n}{2}]+d$ and $n \geq m$ yield $m \geq 2d-1$. Since $d \geq 2$, in particular, we have $m > d$. We denote by $F^{m,d}(X)$ the Hilbert scheme of hypersurfaces of degree d and dimension m which are contained in X. Let $\mathscr{S}$ be the open subscheme of $F^{m,d}(X)$ parametrizing smooth subvarieties. Let $\{\mathscr{L}^i\}_i$ be the irreducible components of $F^1(X)$, and set $\mathscr{S}^i := \{[S] \in \mathscr{S} | \mathscr{L}^S \subset \mathscr{L}^i\}$. Since each $\mathscr{L}^S$ is irreducible, $\mathscr{S}$ is equal to the union of $\{\mathscr{S}^i\}_i$. By assumption, we know that $\overline{\mathrm{Locus}(\mathscr{S})} = X$. This implies that $\overline{\mathrm{Locus}(\mathscr{S}^i)} = X$ for some i. Now, the uniqueness of $\mathscr{L}^S$ gives an open dense subset $U^S \subset S$ such that $\mathscr{L}^S_x = F^1_x(S)$ for any point $x \in U^S$. Then we have $\overline{\bigcup_{[S]\in\mathscr{S}^i} U^S} = X$. Therefore, for a general point $x \in X$, there exists a member $[S_x] \in \mathscr{S}^i$ such that $x \in U^{S_x}$. Then $\mathscr{L}^{S_x} \subset \mathscr{L}^i$ and $F^1_x(S_x) = \mathscr{L}^{S_x}_x \subset \mathscr{L}^i_x$, as desired. □

Proposition 2.3.3. *Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold of dimension $n \geq 2d-1$. Assume that there exists a smooth hypersurface of degree $d \geq 2$ and dimension $m \geq [\frac{n}{2}] + d$ passing through a general point of X. Then X admits a contraction of an extremal ray $\varphi : X \to Y$ whose general fiber F satisfies the following conditions:*

(i) *F is a Fano manifold with index $i(F) \geq \dim F - d + 2$;*

(ii) $\mathrm{Pic}(F) \cong \mathbb{Z}\langle \mathscr{O}_F(1) \rangle$*;*

(iii) *F is also swept out by smooth hypersurfaces of degree d and dimension m;*

(iv) *there exists a covering family $\mathscr{F}$ of lines on F such that for a general point $x \in F$ $\mathscr{F}_x \subset \mathbb{P}(T_x F^*)$ is smooth, irreducible, and non-degenerate.*

Furthermore,

(a) *if F is a linear space, then φ is a scroll;*

(b) *if F is a hypersurface, then* $\deg F = d$.

Proof. According to Lemma 2.3.2, we get a covering family of lines $\mathscr{L}$, and for a general point $x \in X$, we have a smooth hypersurface $S_x \subset X$ of degree d and dimension m passing through x such that $\mathscr{L}^{S_x} \subset \mathscr{L}$ (see Notation 2.3.1) and $F^1_x(S_x) \subset \mathscr{L}_x$.

From deformation theory ([29, Theorem 4.3.5(i)]), we see that $\dim F^1_x(S_x) \geq m - d$ and $\dim \mathscr{L}_x = (-K_X.\mathscr{L}) - 2$. Thus the intersection number $r := (-K_X.\mathscr{L})$ satisfies

$$r = \dim \mathscr{L}_x + 2 \geq \dim F^1_x(S_x) + 2 \geq m - d + 2 \geq [\frac{n}{2}] + 2 > \frac{n+1}{2}.$$

Hence $\mathbb{R}_{\geq 0}[\mathscr{L}]$ is a K_X-negative extremal ray of $NE(X)$ by Proposition 2.2.7. By the contraction theorem, we obtain the extremal contraction $\varphi : X \to Y$ associated to $\mathbb{R}_{\geq 0}[\mathscr{L}]$.

Let F be a general fiber of φ, and set $f := \dim F$. Let H be a hyperplane section of $X \subset \mathbb{P}^N$. Then, since $K_X + rH$ is the pullback of a divisor by φ, we see that $-K_F = -K_X|_F = rH$. Thus the pseudo-index $l(F)$ satisfies

$$l(F) \geq r \geq [\frac{n}{2}] + 2 \geq \frac{f+3}{2}.$$

According to Proposition 2.2.3, we conclude that $\rho(F) = 1$ and $\mathrm{Pic}(F) \cong \mathbb{Z}\langle \mathscr{O}_F(1) \rangle$.

On the other hand, we see that $S_x \subset F$ for a general point $x \in F$. Indeed, for any point $y \in S_x$, x and y can be connected by a chain of $\mathscr{L}^{S_x}$-lines $\{l_i\}_i$. Since $\mathscr{L}^{S_x} \subset \mathscr{L}$, each line l_i is contracted to a point by φ, and hence $y \in F$. So F is also swept out by smooth hypersurfaces of degree d and dimension m. By applying Lemma 2.3.2 again, we get a covering family $\mathscr{F}$ of lines on F, and for a general point $x \in F$, we have a smooth hypersurface $S'_x \subset F$ of degree d and dimension m passing through x such that $F^1_x(S'_x) \subset \mathscr{F}_x$. Then

$$\dim \mathscr{F}_x \geq \dim F^1_x(S'_x) \geq m - d \geq [\frac{n}{2}] \geq \frac{f-1}{2}.$$

From Proposition 2.2.1, it follows that $\mathscr{F}_x \subset \mathbb{P}(T_xF^*)$ is smooth, irreducible, and non-degenerate.

We also know that

$$\dim F^1_x(S'_x) \geq [\frac{n}{2}] > [\frac{(f-1)-1}{2}].$$

So, by applying Proposition 2.2.11 to $F^1_x(S'_x) \subset \mathscr{F}_x$, we obtain

$$f - i(F) + 1 = \mathrm{codim}_{\mathbb{P}(T_xF^*)}(\mathscr{F}_x) \leq \mathrm{codim}_{\mathbb{P}(T_xS'^*_x)}(F^1_x(S'_x)) \leq d - 1.$$

Thus we conclude that $i(F) \geq f - d + 2$.

(a): Next, we assume that F is a linear space. Note that the inequality $f \geq m \geq [\frac{n}{2}] + d$ yields

$$\dim X - 2\dim Y = n - 2(n-f) > 0.$$

So φ is a scroll by [4, Theorem 1.7].

(b): Finally, we assume that F is a hypersurface. Then $i(F) \geq f - d + 2$ implies $\deg F \leq d$. On the other hand, for a hypersurface $S \subset F$ of degree d and dimension m, we see that $\langle S \rangle \not\subset F$. Indeed, if $\langle S \rangle \subset F$, then

$$\dim \langle S \rangle = m + 1 \geq [\frac{f}{2}] + d + 1 > [\frac{(f+1)-1}{2}].$$

It follows that $1 = \mathrm{codim}_{\langle F \rangle}(F) \leq \mathrm{codim}_{\langle S \rangle}(\langle S \rangle) = 0$ from Proposition 2.2.11, which is a contradiction. Hence $\langle S \rangle \cap F$ (which contains S) is an m-dimensional hypersurface whose degree is equal to $\deg F (\leq d)$. Thus we conclude that F is a hypersurface of degree d. $\square$

Now we prove Theorem 2.1.4.

Proof of Theorem 2.1.4. Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold of dimension $n \geq 3d-1$ which is swept out by smooth hypersurfaces of degree d and dimension $m \geq \frac{2n-1}{3} + d$. Furthermore, assume that Conjecture 2.1.3 is true.

When $d = 1$, the result follows from [28], so we assume that $d \geq 2$. Then Proposition 2.3.3 gives a contraction of an extremal ray $\varphi : X \to Y$ whose general fiber F satisfies (i), (ii), (iii), and (iv) in Proposition 2.3.3. Set $f := \dim F$ and $M := \dim \langle F \rangle$, and let x be a general point of F. We show that F is either a linear space or a hypersurface. So we assume that F is not a linear space. Then $\mathscr{F}_x \neq \mathbb{P}(T_xF^*)$, and we can define the second fundamental form $|II_{x,F}| \subset |\mathscr{O}_{\mathbb{P}(T_xF^*)}(2)|$.

Now we notice that the inequalities $m \geq \frac{2n-1}{3} + d$ and $n \geq m$ yield $f \geq m \geq 3d-1 (\geq 5)$. Thus $d \leq \frac{f+1}{3}$. It follows that

$$\dim \mathscr{F}_x = i(F) - 2 \geq f - d \geq \frac{2f-1}{3} > \frac{2(f-1)}{3} (> 2).$$

Hence we can apply Conjecture 2.1.3, so $\mathscr{F}_x \subsetneq \mathbb{P}(T_xF^*)$ is a complete intersection.

Moreover, the index $i(F)$ satisfies

$$i(F) \geq \frac{2f+5}{3} > \frac{2f}{3}.$$

According to Proposition 2.2.8, this implies that $\delta(F) > 0$. Therefore, the dimension of $|II_{x,F}|$ is equal to $M - f - 1$ by Proposition 2.2.10. Since $\mathscr{F}_x \subsetneq \mathbb{P}(T_xF^*)$ is contained in the base locus of $|II_{x,F}|$, we can apply the following lemma:

Lemma 2.3.4. *Let $X \subsetneq \mathbb{P}^N$ be a non-degenerate complete intersection. Assume that X is contained in a variety W which is an intersection of k linearly independent hyperquadrics in $\mathbb{P}^N$. Then $\operatorname{codim}_{\mathbb{P}^N}(X) \geq k$.*

Proof. Suppose that X is defined by polynomials $p_1, \ldots, p_c$, where $c := \operatorname{codim}_{\mathbb{P}^N}(X)$, and W is defined by quadratic polynomials $q_1, \ldots, q_k$. We may assume that $\deg p_1 = \cdots = \deg p_e = 2 < \deg p_{e+1} \leq \cdots \leq \deg p_c$. Now each q_i is contained in the ideal generated by $p_1, \ldots, p_c$, hence it is contained in the $\mathbb{C}$-vector space spanned by $p_1, \ldots, p_e$. Since $q_1, \ldots q_k$ are linearly independent, we conclude that $k \leq e \leq c$. $\square$

By Lemma 2.3.4, we have

$$(f-1) - \frac{2f-1}{3} \geq \operatorname{codim}_{\mathbb{P}(T_xF^*)}(\mathscr{F}_x) \geq M - f.$$

Hence $f \geq \frac{3M+2}{4}$, and this yields that

$$m \geq \frac{2f-1}{3} + d \geq \frac{M}{2} + d > [\frac{M-1}{2}].$$

Now we know that F contains a hypersurface S of degree d and dimension m. So, by applying Proposition 2.2.11 to $S \subset F$, we obtain that $\operatorname{codim}_{\langle F \rangle}(F) \leq \operatorname{codim}_{\langle S \rangle}(S) = 1$. Therefore, F is either a linear space or a hypersurface. By Proposition 2.3.3(a) and (b), we obtain the conclusion. $\square$

Next, we prove the following theorem, which is the case $d = 4$ in Conjecture 2.1.1.

Theorem 2.3.5. *Let $X \subset \mathbb{P}^N$ be a smooth complex projective manifold of dimension $n \geq 7$. Assume that there exists a smooth quartic hypersurface of dimension $m \geq [\frac{n}{2}]+4$ passing through a general point of X. Then X admits a morphism $\varphi : X \to Y$ such that*

- *φ is a contraction of an extremal ray;*
- *the relative dimension of φ is at least m;*
- *either φ is a scroll, or general fibers of φ are quartic hypersurfaces.*

Proof. By Proposition 2.3.3, we get a contraction of an extremal ray $\varphi : X \to Y$ whose general fiber F satisfies the conditions (i)-(iv). Then F is a Fano manifold with $i(F) \geq f - 2$ and $\mathrm{Pic}(F) \cong \mathbb{Z}\langle \mathscr{O}_F(1)\rangle$, where $f := \dim F$. We notice that the inequalities $m \geq [\frac{n}{2}] + 4$ and $n \geq m$ yield $f \geq m \geq 7$. According to Propositions 2.2.4, 2.2.5, and 2.2.6, F is isomorphic to one of the following:

If $i(F) = f + 1$,

(1) a linear space.

If $i(F) = f$,

(2) a quadric hypersurface.

If $i(F) = f - 1$,

(3) a cubic hypersurface,

(4) a complete intersection of two quadric hypersurfaces.

If $i(F) = f - 2$,

(5) a quartic hypersurface,

(6) a complete intersection of a quadric hypersurface and a cubic hypersurface,

(7) a complete intersection of three quadric hypersurfaces,

(8) a linear section of the spinor variety S_4 which is an irreducible component of the Fano variety of 4-planes in Q^8,

(9) a linear section of the Grassmann variety $G(2, \mathbb{C}^6) \subset \mathbb{P}^{14}$.

In any case, we know that $\mathrm{codim}_{\mathbb{P}^M}(F) \leq 6$, where $M := \dim \langle F\rangle$. This implies

$$m \geq [\frac{f}{2}] + 4 \geq [\frac{M-6}{2}] + 4 > [\frac{M-1}{2}].$$

Now F contains a quartic hypersurface S of dimension m, so we have $\mathrm{codim}_{\langle F\rangle}(F) \leq \mathrm{codim}_{\langle S\rangle}(S) = 1$ by Proposition 2.2.11 again. Therefore, F is either a linear space or a hypersurface. By Proposition 2.3.3(a) and (b), we obtain the conclusion. □

Chapter 3

On the Picard number of rationally quartic connected manifolds

3.1 Introduction

We consider a smooth complex projective manifold X of dimension $n \geq 2$ which is rationally connected by rational curves of degree d with respect to a fixed ample line bundle H, namely, whose two general points can be joined by a rational curve of H-degree d. We study structures of such a manifold. In particular, we investigate the Picard number ρ_X and lines on X, where a line means a curve of H-degree one. For small degree d, they have been studied well.

In case $d = 1$, X is called *line connected*, and it is known that $(\mathbb{P}^n, \mathscr{O}(1))$ is the unique manifold which is line connected.

In case $d = 2$, X is called *conic connected.*

Fact 3.1.1 ([26, Theorem 7.4] and [16, Theorem 3.6])**.** *If X is conic connected, then*

(i) *$\rho_X \leq 2$;*

(ii) *X is covered by lines unless (X, H) is isomorphic to $(\mathbb{P}^n, \mathscr{O}(2))$.*

We remark that P. Ionescu and F. Russo classified the conic connected manifolds with $\rho_X = 2$ embedded in a projective space ([13, Theorem 2.2]), and V. Paterno generalized their classification for polarized manifolds ([26, Theorem 7.4]).

In case $d = 3$, X is called *rationally cubic connected.* Then there is no upper bound on the Picard number (see [24, Example 3.1]). However, G. Occhetta and V. Paterno obtained the following results:

Fact 3.1.2 ([24, Proposition 5.5 and Theorem 1.1])**.** *Suppose that X is rationally cubic connected with respect to a family $\mathscr{F}$.*

(i) *If $\mathscr{F}$ is not generically unsplit (see Definition 3.2.4), then $\rho_X \leq 3$ and X is covered by lines.*

(ii) *(Even when $\mathscr{F}$ is generically unsplit) If X is covered by lines, then $\rho_X \leq 3$.*

They also proved that rationally cubic connected manifolds which are not covered by lines are obtained by rationally cubic connected manifolds of Picard number one by blow-ups along smooth centers ([25, Theorem 1.1]).

In this chapter, we consider the case $d = 4$, namely, the case where X is *rationally quartic connected* with respect to a family $\mathscr{F}$, and one of our main problems is to find what conditions imply $\rho_X \leq 4$. In case $d = 4$, it turns out that there is no upper bound on the Picard number, even when $\mathscr{F}$ is not generically unsplit and X is covered by lines (see Example 3.3.2).

In general, if X is rationally connected with respect to $\mathscr{F}$, then $(-K_X.\mathscr{F}) \geq n+1$, and equality holds if and only if $\mathscr{F}$ is generically unsplit (see Remark 3.2.9). Recall that $\rho_X \leq 2$ holds unconditionally ($(-K_X.\mathscr{F}) \geq n+1$) in case $d = 2$, and that $\rho_X \leq 3$ holds if $(-K_X.\mathscr{F}) \geq n+2$ in case $d = 3$. So, when $d = 4$, it seems natural to consider the case $(-K_X.\mathscr{F}) \geq n+3$ for a first approach to our problem. In our main result, we will prove that, with two kinds of exceptions, this assumption implies $\rho_X \leq 4$ and X is covered by lines. The statement of our main theorem is as follows:

Theorem 3.1.3. *Let X be a smooth complex projective manifold of dimension $n \geq 2$ with a fixed ample line bundle H, and assume that X is rationally connected with respect to a family $\mathscr{F}$ which satisfies $(H.\mathscr{F}) = 4$ and $(-K_X.\mathscr{F}) \geq n+3$. Then we obtain at least one of the following:*

(a) *$\rho_X \leq 4$ and X is covered by lines;*

(b) *X is rationally cubic connected;*

(c) *X is 2-connected by conics which are numerically proportional to $\mathscr{F}$, namely, for two general points $x, y \in X$ there exist two conics C^1 and C^2 such that $[C^1] = [C^2] = \frac{1}{2}[\mathscr{F}]$, $x \in C^1$, $y \in C^2$, and $C^1 \cap C^2 \neq \emptyset$.*

This theorem fails under the assumption $(-K_X.\mathscr{F}) = n+2$ (see Example 3.3.2). In addition, as shown in Example 3.3.3, even when $(-K_X.\mathscr{F}) \geq n+3$, the Picard number may possibly be greater than four and X may not be covered by lines in cases (b) and (c).

Furthermore, we will provide a classification of rationally quartic connected surfaces (Theorem 3.5.1).

In our proof, Lemmas 3.4.6 and 3.4.7, which are generalizations of [24, Proposition 5.4], are key lemmas. In order to prove $\rho_X \leq 4$, we need to apply skillfully these lemmas and results of [1].

3.2 Families of rational curves

Throughout this section, we consider a smooth complex projective manifold X of dimension $n \geq 2$.

Definition 3.2.1. Let H be a fixed ample line bundle. Then a curve $C \subset X$ is called a *line*, *conic*, *cubic*, and *quartic*, if the intersection number $(H.C)$ is equal to one, two, three, and four, respectively.

Definition 3.2.2. We denote by $\mathrm{RatCurves}^n(X)$ the normalization of the scheme of rational curves on X (see [18, II.2]), and define a *family of rational curves* on X to be an irreducible component of $\mathrm{RatCurves}^n(X)$. Given a rational curve C on X, we define a *family of deformations* of C to be a family of rational curves containing C.

Definition 3.2.3. Let $\mathscr{V}$ be a family of rational curves. Let $\mathscr{U}$ be the universal family of $\mathscr{V}$, and let $p : \mathscr{U} \to \mathscr{V}$ and $q : \mathscr{U} \to X$ be the associated morphisms. $q(\mathscr{U})$ is denoted by $\mathrm{Locus}(\mathscr{V})$. We say that $\mathscr{V}$ is a *dominating* (resp. *covering*) family if $\overline{\mathrm{Locus}(\mathscr{V})} = X$ (resp. $\mathrm{Locus}(\mathscr{V}) = X$). For a subvariety $Y \subset X$, $p(q^{-1}(Y))$(the subscheme of $\mathscr{V}$ which parametrizes curves intersecting Y) is denoted by $\mathscr{V}_Y$, and $q(p^{-1}(\mathscr{V}_Y))$ is denoted by $\mathrm{Locus}(\mathscr{V}; Y)$. In particular, when Y is a point, $\mathscr{V}_{\{x\}}$ (resp. $\mathrm{Locus}(\mathscr{V}; \{x\})$) is also denoted by $\mathscr{V}_x$ (resp. $\mathrm{Locus}(\mathscr{V}; x)$). For families of rational curves $\mathscr{V}^1, \dots, \mathscr{V}^k$, we inductively define $\mathrm{Locus}(\mathscr{V}^k, \dots, \mathscr{V}^1; Y) := \mathrm{Locus}(\mathscr{V}^k; \mathrm{Locus}(\mathscr{V}^{k-1}, \dots, \mathscr{V}^1; Y))$.

Definition 3.2.4. For a family of rational curves $\mathscr{V}$,

(i) $\mathscr{V}$ is *unsplit* if it is proper;

(ii) $\mathscr{V}$ is *locally unsplit* if for a general point $x \in \mathrm{Locus}(\mathscr{V})$ $\mathscr{V}_x$ is proper;

(iii) $\mathscr{V}$ is *generically unsplit* if for a general point $x \in \mathrm{Locus}(\mathscr{V})$ and a general point $y \in \mathrm{Locus}(\mathscr{V}; x)$ there is at most a finite number of curves of $\mathscr{V}$ passing through both x and y.

Definition 3.2.5. Let $\mathscr{V}$ be a dominating family of rational curves. We say that X is *rationally connected* with respect to $\mathscr{V}$, if there exists a curve of $\mathscr{V}$ passing through two general points of X.

Definition 3.2.6. Let $\mathscr{V}^1, \dots, \mathscr{V}^k$ be unsplit families of rational curves. We say that two points $x, y \in X$ can be *connected by a $(\mathscr{V}^1, \dots, \mathscr{V}^k)$-chain of length m* if $x \in \mathrm{Locus}(\mathscr{V}^{i(1)}, \dots, \mathscr{V}^{i(m)}; y)$ (see Definition 3.2.3) for some $1 \leq i(j) \leq k$. We say that x and y are in *$rc(\mathscr{V}^1, \dots, \mathscr{V}^k)$ relation* if they can be connected by a $(\mathscr{V}^1, \dots, \mathscr{V}^k)$-chain of length m for some m. It is known (see [18, IV Theorem 4.16]) that there is an open subvariety $U \subset X$ and a proper morphism $\pi : U \to Z$ with connected fibers such that

(i) the $\mathrm{rc}(\mathscr{V}^1, \dots, \mathscr{V}^k)$ relation restricts to an equivalence relation on U;

(ii) $\pi^{-1}(z)$ coincides with an equivalence class for the $\mathrm{rc}(\mathscr{V}^1, \dots, \mathscr{V}^k)$ relation for every $z \in Z$;

(iii) any two points of $\pi^{-1}(z)$ can be connected by a $(\mathscr{V}^1, \dots, \mathscr{V}^k)$-chain of length at most $2^{\dim X - \dim Z} - 1$ for every $z \in Z$.

We call the morphism π *the* $rc(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ *fibration.* If Z is just a point, then we say that X is *rationally chain connected* with respect to $\mathscr{V}^1, \ldots, \mathscr{V}^k$.

Definition 3.2.7. For a family of rational curves $\mathscr{V}$, we denote by $\overline{\mathscr{V}}$ the closure of $\mathscr{V}$ in the Chow variety $\mathrm{Chow}(X)$. For a subvariety $Y \subset X$, we define $\overline{\mathscr{V}}_Y$ as in Definition 3.2.3.

Definition 3.2.8. $N_1(X)$ is the $\mathbb{R}$-vector space of 1-cycles with real coefficients modulo numerical equivalence. We denote by ρ_X the Picard number of X, which is the dimension of the $\mathbb{R}$-vector space $N_1(X)$. For a family of rational curves $\mathscr{V}$, we denote by $[\mathscr{V}]$ the numerical class of a curve of $\mathscr{V}$, and by $(D.\mathscr{V})$ the intersection number of a divisor D and a curve of $\mathscr{V}$.

Remark 3.2.9. Assume that X is rationally connected with respect to a family $\mathscr{V}$. Then by [18, II Theorem 3.11], the dimension of the subscheme of $\mathscr{V}$ which parametrizes curves passing through two general points of X, is equal to $(-K_X.\mathscr{V})-n-1$. Thus we have $(-K_X.\mathscr{V}) \geq n+1$, and equality holds if and only if $\mathscr{V}$ is generically unsplit. If $\mathscr{V}$ is not generically unsplit, then Mori's Bend-and-Break ([21, Theorem 4]) gives a reducible connected 1-cycle parametrized by $\overline{\mathscr{V}}$ (see Definition 3.2.7) passing through two general points of X.

Proposition 3.2.10 ([1, Lemma 4.1])**.** *Let $Y \subset X$ be a closed subvariety, $\mathscr{V}$ a family of rational curves on X. Then every curve contained in $\overline{\mathrm{Locus}(\mathscr{V};Y)}$ (see Definition 3.2.3) is numerically equivalent to a linear combination with rational coefficients $aC_Y + \sum_{i=1}^k b_i C_i$, where C_Y is a curve contained in Y, and each C_i is an irreducible component of a cycle parametrized by $\overline{\mathscr{V}}_Y$ (see Definition 3.2.7).*

Proposition 3.2.11 ([1, Corollary 4.4])**.** *Suppose that X is rationally chain connected with respect to some unsplit families $\mathscr{V}^1, \ldots, \mathscr{V}^k$. Then every curve in X is numerically equivalent to a linear combination of curves in $\mathscr{V}^1, \ldots, \mathscr{V}^k$. In particular, $\rho_X \leq k$.*

Proposition 3.2.12 (proof of [1, Lemma 5.4])**.** *Let x be a point of X, and let $\mathscr{V}^1, \ldots, \mathscr{V}^k$ be numerically independent families of rational curves on X. Set $Y^j := \mathrm{Locus}(\mathscr{V}^j, \ldots, \mathscr{V}^1; Y)$ and $Y^0 := \{x\}$. Assume that*

- *$\mathscr{V}^j_y$ is proper for every $1 \leq j \leq k$ and every point $y \in Y^{j-1}$;*
- *Y^k is nonempty.*

Then we have

$$\dim Y^k \geq \sum_{j=1}^{k} (-K_X.\mathscr{V}^j) - k.$$

3.3 Examples

Example 3.3.1. Products of four projective spaces $(\mathbb{P}^r \times \mathbb{P}^s \times \mathbb{P}^t \times \mathbb{P}^u, \mathscr{O}(1,1,1,1))$ are trivial examples of rationally quartic connected manifolds with Picard number four.

Example 3.3.2. Let $\varphi : Y \to \mathbb{P}^r$ be the blow-up of $\mathbb{P}^r$ at general k points $P_1, \dots, P_k$ with $k \le \binom{r+3}{3} - 2r - 2$, and $E_i := \varphi^{-1}(P_i)$ the exceptional divisor. Let X be $Y \times \mathbb{P}^s$, and set

$$H := p_1^*\{\varphi^*\mathscr{O}_{\mathbb{P}^r}(3) - \sum_{i=1}^{k} E_i\} + p_2^*\mathscr{O}_{\mathbb{P}^s}(1),$$

where p_1 and p_2 are the projections. Then H is ample by [3]. Notice that $(Y, \varphi^*\mathscr{O}_{\mathbb{P}^r}(3) - \sum E_i)$ is a rationally cubic connected manifold which was given by [24, Example 3.1]

Now for a general line l on $\mathbb{P}^r$ and for any line m on $\mathbb{P}^s$, we get a rational curve in $|\mathscr{O}(1,1)|$ on $\tilde{l} \times m$, where $\tilde{l} \subset Y$ is the strict transform of l. We define $\mathscr{F}$ to be the family of deformations of such a rational curve. Then we know that $(H.\mathscr{F}) = 4$ and X is rationally connected with respect to $\mathscr{F}$. Remark that

$$-K_X = p_1^*\{\varphi^*\mathscr{O}_{\mathbb{P}^r}(r+1) - (r-1)\sum_{i=1}^{k} E_i\} + p_2^*\mathscr{O}_{\mathbb{P}^s}(s+1),$$

so we have $(-K_X.\mathscr{F}) = r + s + 2$, thus $\mathscr{F}$ is not generically unsplit. In addition, X is covered by lines. On the other hand, X has large Picard number $\rho_X = k + 2$. Moreover, we also know that this manifold satisfies none of the three conclusions of Theorem 3.1.3.

Example 3.3.3. Let $\varphi : X \to Q_n$ be the blow-up of a smooth quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$ of dimension $n \ge 3$ at general k points $P_1, \dots, P_k$ with $k \le 2n + 1$, and $E_i := \varphi^{-1}(P_i)$ the exceptional divisor. Set

$$H := \varphi^*\mathscr{O}_{Q_n}(2) - \sum_{i=1}^{k} E_i.$$

Then it follows from the next lemma (Lemma 3.3.4) that for any curve $C \subset X$,

$$(H.C) \ge \frac{m(C)}{n+1},$$

where $m(C)$ means the maximum of the multiplicities at the points of C, so H is ample by Seshadri's criterion ([8, Theorem 7.1]).

Let $\mathscr{F}$ be the family of deformations of the strict transform of a general conic on Q_n. Note that

$$-K_X = \varphi^*\mathscr{O}_{Q_n}(n) - (n-1)\sum_{i=1}^{k} E_i.$$

Then we know that $(H.\mathscr{F}) = 4$, $(-K_X.\mathscr{F}) = 2n \ge n + 3$, and X is rationally connected with respect to $\mathscr{F}$. However, $\rho_X = k + 1$. We also see that every line with $(-K_X)$-degree at least two is contracted by φ, so X is not covered by lines.

On the other hand, we show that X satisfies both (b) and (c) in Theorem 3.1.3. Let $\mathscr{E}$ be the family of deformations of the strict transform of a general conic on Q_n passing through P_1. Then $(H, \mathscr{E}) = 3$, and X is rationally connected with respect to $\mathscr{E}$ because there is a conic on Q_n passing through three general points, so (b) holds. Next, let $\mathscr{C}$ be the family of deformations of the strict transform of a general line on Q_n. Then $[\mathscr{C}] = \frac{1}{2}[\mathscr{F}]$, and since Q_n is 2-connected by lines, X has the same property with respect to $\mathscr{C}$. Thus (c) also holds.

Lemma 3.3.4. *Let $P_1, \ldots, P_k$ be general points of a smooth quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$ with $k \leq 2n+1$. Let $C \subset Q_n$ be an irreducible curve of degree d, and let m_i be the multiplicity of C at P_i (in case $P_i \notin C$, set $m_i := 0$). Then we have*

$$\sum_{i=1}^{k} m_i \leq \frac{2n+1}{n+1} d.$$

Proof. We may assume $m_1 \geq \cdots \geq m_k$. We only have to prove the case $k = 2n+1$. Let $L \subset \mathbb{P}^{n+1}$ be the hyperplane passing through $P_1, \ldots, P_{n+1}$. Note that none of $P_{n+2}, \ldots, P_k$ is contained in L.

If C is not contained in L, then Bézout's theorem ([9, I Theorem 7.7]) yields $\sum_{i=1}^{n+1} m_i \leq d$. This implies the conclusion.

If C is contained in L, then $C \subset L \cap Q_n = Q_{n-1}$. In case $n = 2$, C must be a smooth conic, so

$$\sum_{i=1}^{5} m_i = \sum_{i=1}^{3} m_i \leq 3 < \frac{10}{3}.$$

In case $n \geq 3$, by induction on n, we obtain

$$\sum_{i=1}^{k} m_i = \sum_{i=1}^{n+1} m_i \leq \frac{2n-1}{n} d < \frac{2n+1}{n+1} d.$$

□

3.4 Preliminaries

In this section, let X be a smooth complex projective manifold of dimension $n \geq 2$.

Definition 3.4.1. Let $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ be a k-tuple of families of rational curves on X. We define a *1-cycle parametrized by* $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ as a 1-cycle $C^1 + \cdots + C^k$ such that each C^i is a curve parametrized by $\mathscr{V}^i$. We say that X is *dominated by connected 1-cycles of* $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ (resp. $(\mathscr{V}^1, \ldots, \mathscr{V}^k; i)$), if for a general point $x \in X$ there exists a connected 1-cycle $C^1 + \cdots + C^k$ parametrized by $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ such that $x \in C^1 \cup \cdots \cup C^k$ (resp. $x \in C^i$). We use the word "*covered*" instead of "dominated" if the same conditions hold for every point $x \in X$. We also say that X is *connected by 1-cycles of* $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ (resp. $(\mathscr{V}^1, \ldots, \mathscr{V}^k; i, j)$), if for two general points $x, y \in X$ there exists a connected 1-cycle $C^1 + \cdots + C^k$ parametrized by $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ such that $x, y \in C^1 \cup \cdots \cup C^k$ (resp. $x \in C^i$ and $y \in C^j$).

Definition 3.4.2. In graph theory, a *tree* is an undirected connected graph without simple cycles. Let t be a tree with k vertices, and $V(t) = \{v^1, \ldots, v^k\}$ the set of vertices of t. Then we define a *1-cycle parametrized by* $(\mathscr{V}^1, \ldots, \mathscr{V}^k; t)$ as a connected 1-cycle $C^1 + \cdots + C^k$ parametrized by $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ satisfying the following condition: if t has an edge connecting v^i and v^j, then C^i intersects C^j (see Figures 1 and 2).

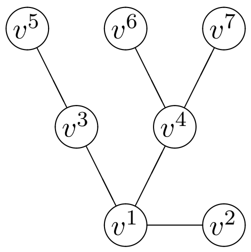


Figure 1. A tree t.

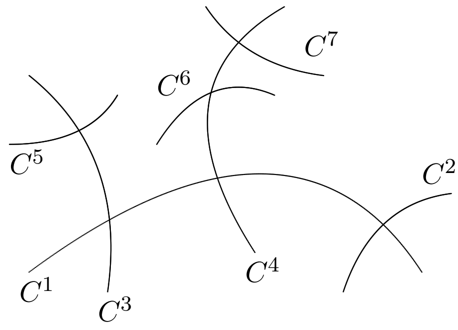


Figure 2. A 1-cycle parametrized by $(\mathscr{V}^1, \ldots, \mathscr{V}^7; t)$.

When the same conditions as in Definition 3.4.1 hold for a 1-cycle parametrized by $(\mathscr{V}^1, \ldots, \mathscr{V}^k; t)$, X is said to be, respectively, $\{$*dominated* or *covered*$\}$ *by connected 1-cycles of* $\{(\mathscr{V}^1, \ldots, \mathscr{V}^k; t)$ or $(\mathscr{V}^1, \ldots, \mathscr{V}^k; i; t)\}$, and *connected by 1-cycles of* $\{(\mathscr{V}^1, \ldots, \mathscr{V}^k; t)$ or $(\mathscr{V}^1, \ldots, \mathscr{V}^k; i, j; t)\}$.

Lemma 3.4.3. *Suppose that X is dominated by a family of rational curves $\mathscr{V}$ which is not locally unsplit. Then we obtain an integer $k \geq 2$, a k-tuple of families of rational curves $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$, a number $1 \leq i \leq k$, and a tree t with k vertices such that*

- $[\mathscr{V}^1] + \cdots + [\mathscr{V}^k] = [\mathscr{V}]$;
- *X is dominated by connected 1-cycles of $(\mathscr{V}^1, \ldots, \mathscr{V}^k; i; t)$ (see Definition 3.4.2).*

Proof. Let H be an ample line bundle. For an integer $2 \leq k \leq (H.\mathscr{V})$, let Λ_k be the set of k-tuples of families of rational curves $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ such that $[\mathscr{V}^1]+\cdots+[\mathscr{V}^k] = [\mathscr{V}]$, and let T_k be the set of trees with k vertices. Remark that both Λ_k and T_k are finite sets. For each $\lambda = (\mathscr{V}^1, \ldots, \mathscr{V}^k) \in \Lambda_k$, each $1 \leq i \leq k$, and each $t \in T_k$, let $A(k; \lambda; i; t)$ be the set of $x \in X$ such that there exists a connected 1-cycle $C^1+\cdots+C^k$ parametrized by $(\mathscr{V}^1, \ldots, \mathscr{V}^k; t)$ which satisfies $x \in C^i$. By assumption, we get a reducible connected 1-cycle parametrized by $\overline{\mathscr{V}}$ (see Definition 3.2.7) passing through a general point of X. It follows that $\bigcup_{2\leq k\leq (H.\mathscr{V})} \bigcup_{\lambda\in\Lambda_k} \bigcup_{1\leq i\leq k} \bigcup_{t\in T_k} A(k; \lambda; i; t)$ dominates X. Since X is irreducible, we conclude that $\overline{A(k; \lambda; i; t)} = X$ for some k, λ, i, and t, as desired. $\square$

We can also prove the following lemma in the same way as the proof of Lemma 3.4.3, by considering subsets of $X \times X$.

Lemma 3.4.4. *Suppose that X is rationally connected with respect to a family $\mathscr{V}$ which is not generically unsplit. Then we obtain an integer $k \geq 2$, a k-tuple of families of rational curves $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$, two numbers $1 \leq i, j \leq k$, and a tree t with k vertices such that*

- $[\mathscr{V}^1] + \cdots + [\mathscr{V}^k] = [\mathscr{V}]$;
- *X is connected by 1-cycles of $(\mathscr{V}^1, \ldots, \mathscr{V}^k; i, j; t)$ (see Definition 3.4.2).*

Lemma 3.4.5. *Let $\pi : U \to Z$ be a dominant morphism mapping from an open subvariety $U \subset X$ to some variety Z, and $\mathscr{V}$ a dominating family of rational curves on X. Assume that $\mathscr{V}$ is numerically equivalent to a linear combination of some curves $C^1, \ldots, C^k$ which are contained in U and contracted by π. Then general curves of $\mathscr{V}$ are also contracted by π.*

Proof. Set $d := \dim Z$. Let $W_1, \ldots, W_d$ be general effective divisors on Z passing through none of the k points $\pi(C^1), \ldots, \pi(C^k)$, and let Y_j be a prime divisor on X contained in the closure of $\pi^{-1}(W_j)$. Then by construction, $(Y_j.C^i) = 0$ for every i and j. Let $x \in \bigcap Y_j$ be a general point. We may assume that $x \in U$, and that $\mathscr{V}_x$ is nonempty because $\mathscr{V}$ is dominating. Then for any curve $C \in \mathscr{V}_x$, $(Y_j.C) = 0$ by assumption, hence $C \subset Y_j$. Thus $\pi(C) \subset \bigcap W_j$. Since $\bigcap W_j$ is a finite set, we conclude that $\pi(C)$ is a point. $\square$

Lemma 3.4.6. *Suppose that X is rationally connected with respect to a family $\mathscr{V}$. Let $\mathscr{V}^1, \ldots, \mathscr{V}^k$ be unsplit families of rational curves. We assume that*

- *$[\mathscr{V}]$ is a linear combination of $[\mathscr{V}^1], \ldots, [\mathscr{V}^k]$;*
- *X is covered by connected 1-cycles of $(\mathscr{V}^1, \ldots, \mathscr{V}^k)$ (see Definition 3.4.1).*

Then $N_1(X)$ is spanned by $[\mathscr{V}^1], \ldots, [\mathscr{V}^k]$. In particular, $\rho_X \leq k$.

We show the following lemma, which is a strong form of Lemma 3.4.6.

Lemma 3.4.7. *Suppose that X is rationally connected with respect to a family of $\mathscr{V}$. Let $\mathscr{W}$ be a dominating family of rational curves, and let $\mathscr{V}^1, \ldots, \mathscr{V}^k$, $\mathscr{W}^1, \ldots, \mathscr{W}^m$ be unsplit families of rational curves. We assume that*

- *$[\mathscr{V}]$ is a linear combination of $[\mathscr{V}^1], \ldots, [\mathscr{V}^k]$;*
- *$[\mathscr{W}]$ is a linear combination of $[\mathscr{W}^1], \ldots, [\mathscr{W}^m]$;*
- *X is dominated by connected 1-cycles of $(\mathscr{V}^1, \ldots, \mathscr{V}^k, \mathscr{W}; k+1)$ (see Definition 3.4.1);*
- *X is covered by connected 1-cycles of $(\mathscr{W}^1, \ldots, \mathscr{W}^m)$.*

Then $N_1(X)$ is spanned by $[\mathscr{V}^1], \ldots, [\mathscr{V}^k], [\mathscr{W}^1], \ldots, [\mathscr{W}^m]$.

Proof. Let $\pi : U \to Z$ be the $\mathrm{rc}(\mathscr{V}^1, \ldots, \mathscr{V}^k, \mathscr{W}^1, \ldots, \mathscr{W}^m)$ fibration (see Definition 3.2.6), and $x \in U$ a general point. By assumption, we get a connected 1-cycle $\sum_{j=1}^m D^j$ passing through x such that $D^j \in \mathscr{W}^j$. Then since each D^j is contained in U and contracted by π, Lemma 3.4.5 yields that general curves of $\mathscr{W}$ are also contracted by π.

Now let $y \in U$ be another general point, and let $\sum_{i=1}^k C^i + D$ be a connected 1-cycle such that $C^i \in \mathscr{V}^i$ and $D \in \mathscr{W}_y$. Then D is contracted by π. Observe that D is contained in U because $\pi : U \to Z$ is proper, and that each C^i is also contained in U and contracted by π. By Lemma 3.4.5 again, we obtain that general curves of $\mathscr{V}$ are also contracted by π. Since two general points of X can be joined by a curve of $\mathscr{V}$, Y must be a point. Therefore, Proposition 3.2.11 implies the conclusion. $\square$

3.5 Rationally quartic connected surfaces

In this section, we give a classification of rationally quartic connected surfaces.

Theorem 3.5.1. *Let X be a smooth complex projective surface with a fixed ample line bundle H, and assume that X is rationally connected with respect to a family $\mathscr{F}$ with $(H.\mathscr{F}) = 4$.*

(1) *If X is covered by lines, then (X, H) is isomorphic to one of the following:*

(i) $(\mathbb{P}^2, \mathscr{O}(1))$,

(ii) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(1,3))$, (iii) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(1,2))$, (iv) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(1,1))$,

(v) $(\mathbb{F}_1, C_0 + 4f)$, (vi) $(\mathbb{F}_1, C_0 + 3f)$, (vii) $(\mathbb{F}_1, C_0 + 2f)$,

(viii) $(\mathbb{F}_2, C_0 + 4f)$, (ix) $(\mathbb{F}_2, C_0 + 3f)$,

(x) $(\mathbb{F}_3, C_0 + 4f)$,

where we denote by C_0 a minimal section and by f a fiber on $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(-e) \oplus \mathscr{O})$.

(2) *If X is not covered by lines and $\mathscr{F}$ is not generically unsplit (see Definition 3.2.4), then (X, H) is isomorphic to one of the following:*

(xi) $(\mathbb{P}^2, \mathscr{O}(2))$,

(xii) $(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}(2,2))$,

(xiii) $(S_k, -K_{S_k})$ *for some* $2 \leq k \leq 8$,

where S_k means a blow-up of $\mathbb{P}^2$ at k general points.

(3) *If X is not covered by lines and $\mathscr{F}$ is generically unsplit, then (X, H) is isomorphic to one of the following:*

(xiv) $(\mathbb{P}^2, \mathscr{O}(4))$,

(xv) $(T_k, 4L - E_1 - \cdots - E_k)$ *for some* $1 \leq k \leq 15$,

(xvi) $(T_k, 4L - 2E_1 - E_2 - \cdots - E_k)$ *for some* $1 \leq k \leq 12$,

(xvii) $(\tilde{T}_k, 4\tilde{L} - 3\tilde{E} - 2\tilde{E}_1 - \tilde{E}_2 - \cdots - \tilde{E}_k)$ *for some* $1 \leq k \leq 11$,

where T_k is a blow-up of $\mathbb{P}^2$ at k (possibly not general) points, and we denote by L the pullback of $\mathscr{O}(1)$ and by E_i the exceptional curve, and furthermore, $\tilde{T}_k$ is a blow-up of T_k at a point in E_1, and we denote by $\tilde{L}$ the pullback of L, by $\tilde{E}$ the exceptional curve, and by $\tilde{E}_i$ the strict transform of E_i.

Remark 3.5.2. In Theorem 3.5.1, (1) and (2) are complete classifications. On the other hand, (3) is not. In some cases in (xv), (xvi), and (xvii), H is not ample. It seems difficult to classify completely the possible values of k and the possible positions of centers of the blow-up (for instance, we need to consider the case where three points among them are colinear).

Proof of Theorem 3.5.1. Since X is a rational surface, if $\rho_X = 1$, then X is isomorphic to $\mathbb{P}^2$, and we obtain (i), (xi), and (xiv), respectively. So, from now on, we suppose that $\rho_X \geq 2$.

First, we consider the case (1). Then X is a ruled surface $\mathbb{F}_e$, and f is a line with respect to H. So $H \equiv C_0 + hf$ for some integer h. Since H is ample, we have

$$h - e \geq 1. \tag{3.1}$$

Suppose that $\mathscr{F} \equiv aC_0 + bf$, then

$$4 = (H.\mathscr{F}) = a(h - e) + b. \tag{3.2}$$

Notice that $\mathscr{F}$ is numerically equivalent to neither C_0 nor f because X is rationally connected with respect to $\mathscr{F}$. So we know (see [9, V Corollary 2.18]) that

$$a \geq 1,\ b \geq 1, \text{and } b \geq ae. \tag{3.3}$$

Furthermore, the genus formula yields

$$0 = 1 + \frac{1}{2}\{(\mathscr{F})^2 + (K_X.\mathscr{F})\} = (a-1)(b - \frac{1}{2}ae - 1). \tag{3.4}$$

Then we can easily check that every (a, b, e, h) satisfying all of (3.1), (3.2), (3.3), and (3.4) is listed in the following table (Table 3.1). Thus we get one of (ii)-(x).

Table 3.1: Possible (a, b, e, h).

	$a=1$ $b=1$	$a=1$ $b=2$	$a=1$ $b=3$	$a=2$ $b=2$	$a=3$ $b=1$
$e=0$	$(1,1,0,3)$	$(1,2,0,2)$	$(1,3,0,1)$		$(3,1,0,1)$
$e=1$	$(1,1,1,4)$	$(1,2,1,3)$	$(1,3,1,2)$	$(2,2,1,2)$	
$e=2$		$(1,2,2,4)$	$(1,3,2,3)$		
$e=3$			$(1,3,3,4)$		

Next, we consider the case (2). Then $(-K_X.\mathscr{F}) \geq 4$ by Remark 3.2.9, so it follows from the next proposition (Proposition 3.5.3) that X is a del Pezzo surface with $-K_X = H$. We finally show that X cannot be isomorphic to S_1. Assume by contradiction that $X = S_1 = \mathbb{F}_1$. Suppose that $\mathscr{F} \equiv aC_0 + bf$. Since $H = -K_X = 2C_0 + 3f$, we have

$$4 = (H.\mathscr{F}) = a + 2b. \tag{3.5}$$

On the other hand, the genus formula implies

$$2 = (-K_X.\mathscr{F}) - 2 = (\mathscr{F})^2 = -a^2 + 2ab. \tag{3.6}$$

However, there is no pair of integers (a, b) satisfying both (3.5) and (3.6). Therefore, we obtain either (xii) or (xiii).

Finally, we consider the case (3). In this case, we have $(-K_X.\mathscr{F}) = 3$, which is equivalent to $(\mathscr{F})^2 = 1$ by the genus formula. Moreover, X is neither conic connected nor rationally cubic connected. Indeed, since X is not covered by lines, conic connectedness implies $(X,H)\cong(\mathbb{P}^2,\mathscr{O}(2))$ (see Fact 3.1.1), and cubic connectedness implies $H=-K_X$ by Proposition 3.5.3 (see also [25, Proposition 3.3]). Therefore, for a rational curve $C\subset X$, if $(H.C)=2,3$ then we have $(-K_X.C)\le 2$ (i.e., $(C)^2\le 0$), and if $(H.C)=1$ then we have $(-K_X.C)\le 1$ (i.e., $(C)^2<0$).

Now Proposition 3.2.10 yields that $\mathscr{F}$ is not locally unsplit because $\rho_X\ge 2$. By Lemma 3.4.3, at least one of the following holds:

(a) There is a family of cubics $\mathscr{E}$ and a line l such that

- $[\mathscr{E}]+[l]=[\mathscr{F}]$;
- X is dominated by connected 1-cycles of $(\mathscr{E},l;1)$ (see Definition 3.4.1).

(b) There are two families of conics $\mathscr{C}$ and $\mathscr{D}$ such that

- $[\mathscr{C}]+[\mathscr{D}]=[\mathscr{F}]$;
- X is dominated by connected 1-cycles of $(\mathscr{C},\mathscr{D};1)$.

(c) There is a family of conics $\mathscr{C}$ and two lines l and m such that

- $[\mathscr{C}]+[l]+[m]=[\mathscr{F}]$;
- X is dominated by connected 1-cycles of $(\mathscr{C},l,m;1)$.

In case (a), $\mathscr{E}$ is dominating, so we get $(-K_X.\mathscr{E})=2$ (i.e., $(\mathscr{E})^2=0$), and $(-K_X.l)=(-K_X.\mathscr{F}-\mathscr{E})=1$ (i.e., $(l)^2=-1$). Thus we obtain $\varphi:X\to X'$ which is a blow-up of a smooth surface at a point with exceptional curve l. Then $H+l$ is a supporting divisor for l, so $H=\varphi^*H'-l$ for some ample divisor H' on X'. Let $\mathscr{F}'$ be the family of deformations of the image of a general curve parametrized by $\mathscr{F}$. Now

$$0=(\mathscr{F}-l)^2=1-2(\mathscr{F}.l)-1$$

implies $(\mathscr{F}.l)=0$, therefore we find that (X',H') is also rationally quartic connected with respect to the generically unsplit family $\mathscr{F}'$, and it is not covered by lines.

In case (b), since $\mathscr{C}$ is dominating, we have $(-K_X.\mathscr{C})=2$ (i.e., $(\mathscr{C})^2=0$), and $(-K_X.\mathscr{D})=1$ (i.e., $(\mathscr{D})^2=-1$). In particular, $\mathscr{D}$ consists of a curve D. Thus, in a similar way, we can contract D by a morphism φ and get another rationally quartic connected surface (X',H') which satisfies the same conditions and $H=\varphi^*H'-2D$.

In case (c), we know $(-K_X.\mathscr{C})=2$ (i.e., $(\mathscr{C})^2=0$), and

$$(-K_X.l)+(-K_X.m)=(-K_X.\mathscr{F}-\mathscr{C})=1,$$

so we may assume that $(-K_X.l)=1$ (i.e., $(l)^2=-1$) and $(-K_X.m)=0$ (i.e., $(m)^2=-2$). Then

$$-1\le 2(\mathscr{C}.l)-1=(\mathscr{C}+l)^2=(\mathscr{F}-m)^2=-2(\mathscr{F}.m)-1\le -1$$

yields $(\mathscr{C}.l)=(\mathscr{F}.m)=0$. Hence $(\mathscr{C}.m)>0$ because there exists a connected 1-cycle parametrized by $(\mathscr{C},l,m)$, and

$$0\le 2(\mathscr{C}.m)-2=(\mathscr{C}+m)^2=(\mathscr{F}-l)^2=-2(\mathscr{F}.l)\le 0$$

implies $(\mathscr{C}.m) = 1$ and $(\mathscr{F}.l) = 0$. Thus we can contract l by a morphism φ and get another surface (X', H') which satisfies the same conditions and $H = \varphi^* H' - l$. Note that $-K_X = \varphi^*(-K_{X'}) - l$. Let m' be the image of m in X', and let $\mathscr{F}'$ be as in the case (a). Since $(l.m) = (l.\mathscr{F} - \mathscr{C} - l) = 1$, we have $(H'.m') = 2$, $(-K_{X'}.m') = 1$ (i.e., $(m')^2 = -1$), and $(\mathscr{F}'.m') = 0$. Therefore, we can contract m' by a morphism φ' and get another surface (X'', H'') which satisfies the same conditions and $H' = \varphi'^* H'' - 2m'$. Then $H = \varphi^* \varphi'^* H'' - 3l - 2m$.

The procedure stops when the Picard number becomes one, namely, eventually we get $(\mathbb{P}^2, \mathscr{O}(4))$. Now we know that (X, H) contains at most one conic with self-intersection -1. Indeed, if it contains two such conics C and D, then $H = 4L - 2C - 2D - \cdots$, where L is the pull back of $\mathscr{O}(1)$, so H cannot be positive on the strict transform of the line on $\mathbb{P}^2$ passing through the two centers. Therefore, (X, H) is isomorphic to one of (xv), (xvi), and (xvii). Since $(H)^2 > 0$, we get the upper bound for k in each case. $\square$

Proposition 3.5.3. *Let X be a smooth complex projective surface with a fixed ample line bundle H, and assume that*

- $\rho_X \geq 2$;
- *X is not covered by lines;*
- *X is rationally connected with respect to a family $\mathscr{F}$ with $(H.\mathscr{F}) = d$ and $(-K_X.\mathscr{F}) \geq d$.*

Then $-K_X$ is linearly equivalent to H. In particular, X is a del Pezzo surface.

Proof. Now $K_X + H$ is nef. Indeed, if it is not nef, then there exists an extremal ray on which it is negative, so this ray has length at least two, and hence X must be either $\mathbb{P}^2$ or a ruled surface.

Let $\sum_{i=1}^k C^i$ be any 1-cycle parametrized by $\overline{\mathscr{F}}$. Then

$$d \leq \sum_{i=1}^{k} (-K_X.C^i) \leq \sum_{i=1}^{k} (H.C^i) = d.$$

Thus we obtain that $(-K_X.C^i) = (H.C^i)$ for each i.

Let $x \in X$ be a general point, then by assumption, we have $\overline{\mathrm{Locus}(\mathscr{F}; x)} = X$. It follows from Proposition 3.2.10 that any curve $C \subset X$ is numerically equivalent to a linear combination of irreducible components of cycles parametrized by $\overline{\mathscr{F}}_x$, so we have $(-K_X.C) = (H.C)$. This means that $-K_X$ and H are numerically equivalent. Recalling that X is a rational surface, we conclude that they are also linearly equivalent. $\square$

3.6 Proof of Theorem 3.1.3

Proof. By our assumption, $\mathscr{F}$ is not generically unsplit. So Lemma 3.4.4 implies at least one of the following:

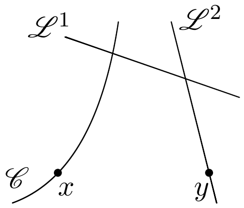
(1) There are four families of lines $\mathscr{L}^1$, $\mathscr{L}^2$, $\mathscr{L}^3$, and $\mathscr{L}^4$ such that

- $[\mathscr{L}^1] + [\mathscr{L}^2] + [\mathscr{L}^3] + [\mathscr{L}^4] = [\mathscr{F}]$;
- X is connected by 1-cycles of $(\mathscr{L}^1, \mathscr{L}^2, \mathscr{L}^3, \mathscr{L}^4)$ (see Definition 3.4.1).

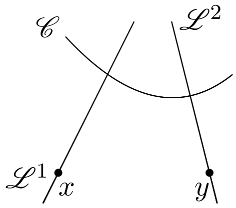
(2) There is a family of conics $\mathscr{C}$ and two families of lines $\mathscr{L}^1$ and $\mathscr{L}^2$ such that

- $[\mathscr{C}] + [\mathscr{L}^1] + [\mathscr{L}^2] = [\mathscr{F}]$;
- for two general points $x, y \in X$ there exists one of the following 1-cycles:

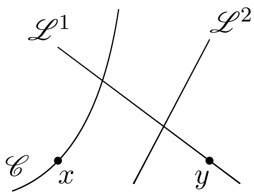
(2.1)



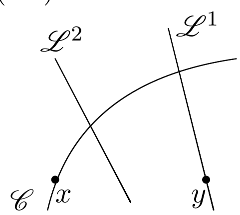
(2.2)



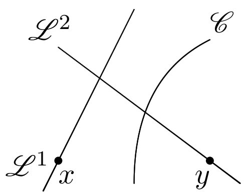
(2.3)



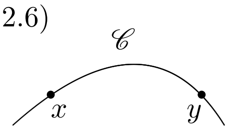
(2.4)



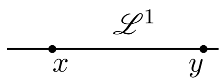
(2.5)



(2.6)



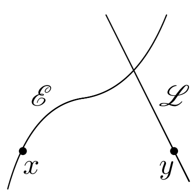
(2.7)



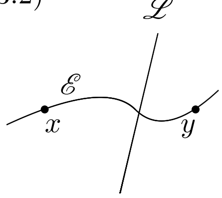
(3) There is a family of cubics $\mathscr{E}$ and a family of lines $\mathscr{L}$ such that

- $[\mathscr{E}] + [\mathscr{L}] = [\mathscr{F}]$;
- for two general points $x, y \in X$ there exists one of the following 1-cycles:

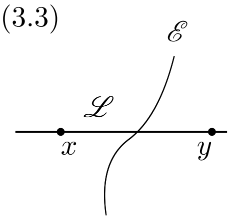
(3.1)



(3.2)



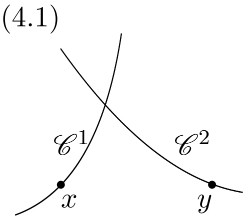
(3.3)



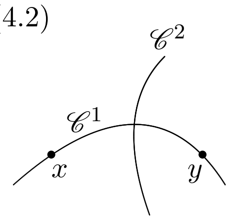
(4) There are two families of conics $\mathscr{C}^1$ and $\mathscr{C}^2$ such that

- $[\mathscr{C}^1] + [\mathscr{C}^2] = [\mathscr{F}]$;
- for two general points $x, y \in X$ there exists either of the following 1-cycles:

(4.1)



(4.2)



Remark that X is covered by lines except the cases (2.6), (3.2), (4.1), and (4.2).

Case (1):

Since X is rationally chain connected with respect to $(\mathscr{L}^1, \mathscr{L}^2, \mathscr{L}^3, \mathscr{L}^4)$, we obtain $\rho_X \leq 4$ by Proposition 3.2.11.

Case (2.1):

In this case, $\overline{\mathrm{Locus}(\mathscr{L}^2, \mathscr{L}^1, \mathscr{C}; x)} = X$ for a general point $x \in X$. If $\mathscr{C}$ is locally unsplit, then after employing Proposition 3.2.10 three times, we know that $N_1(X)$ is spanned by $[\mathscr{C}]$, $[\mathscr{L}^1]$, and $[\mathscr{L}^2]$, hence $\rho_X \leq 3$.

If $\mathscr{C}$ is not locally unsplit, then Lemma 3.4.3 gives two families of lines $\mathscr{M}^1$ and $\mathscr{M}^2$ which satisfy the following:

- $[\mathscr{M}^1] + [\mathscr{M}^2] = [\mathscr{C}]$;
- for a general point $x \in X$, there are two lines $m^1 \in \mathscr{M}^1_x$ and $m^2 \in \mathscr{M}^2$ such that $m^1 \cap m^2 \neq \emptyset$.

Since $\mathrm{Locus}(\mathscr{L}^2; \mathrm{Locus}(\mathscr{L}^1)) = X$, we get two lines $l^2 \in \mathscr{L}^2$ and $l^1 \in \mathscr{L}^1$ such that $m^2 \cap l^2 \neq \emptyset$ and $l^2 \cap l^1 \neq \emptyset$. Thus X is dominated by connected 1-cycles of $(\mathscr{M}^1, \mathscr{M}^2, \mathscr{L}^2, \mathscr{L}^1)$. Since $[\mathscr{M}^1] + [\mathscr{M}^2] + [\mathscr{L}^2] + [\mathscr{L}^1] = [\mathscr{F}]$, Lemma 3.4.6 yields $\rho_X \leq 4$.

Case (2.2):

In this case, $\overline{\mathrm{Locus}(\mathscr{L}^2, \mathscr{C}, \mathscr{L}^1; x)} = X$ for a general point $x \in X$. If $\mathscr{C}$ is proper at every point in $\mathrm{Locus}(\mathscr{L}^1; x)$, then by Proposition 3.2.10, we see that $N_1(X)$ is spanned by $[\mathscr{L}^1]$, $[\mathscr{C}]$, and $[\mathscr{L}^2]$, hence $\rho_X \leq 3$.

If $\mathscr{C}$ is not proper at a point in $\mathrm{Locus}(\mathscr{L}^1; x)$ for a general point $x \in X$, then as in the proof of Lemma 3.4.3, we get two families of lines $\mathscr{M}^1$ and $\mathscr{M}^2$ such that

- $[\mathscr{M}^1] + [\mathscr{M}^2] = [\mathscr{C}]$;
- X is covered by connected 1-cycles of $(\mathscr{L}^1, \mathscr{M}^1, \mathscr{M}^2; 1)$ (see Definition 3.4.1).

Since $\mathscr{L}^2$ is covering, X is also covered by connected 1-cycles of $(\mathscr{L}^1, \mathscr{M}^1, \mathscr{M}^2, \mathscr{L}^2)$. Therefore, we obtain $\rho_X \leq 4$ by Lemma 3.4.6.

Case (2.3):

In this case, $\overline{\mathrm{Locus}(\mathscr{L}^1, \mathscr{C}; x)} = X$ for a general point $x \in X$. If $\mathscr{C}$ is locally unsplit, then by applying Proposition 3.2.10 two times, we see that $N_1(X)$ is spanned by $[\mathscr{C}]$ and $[\mathscr{L}^1]$, so $\rho_X \leq 2$.

If $\mathscr{C}$ is not locally unsplit, then by Lemma 3.4.3 we have two families of lines $\mathscr{M}^1$ and $\mathscr{M}^2$ such that

- $[\mathscr{M}^1] + [\mathscr{M}^2] = [\mathscr{C}]$;
- X is covered by connected 1-cycles of $(\mathscr{M}^1, \mathscr{M}^2)$.

Since $\mathrm{Locus}(\mathscr{L}^1; \mathrm{Locus}(\mathscr{L}^2)) = X$, X is also covered by connected 1-cycles of $(\mathscr{M}^1, \mathscr{M}^2, \mathscr{L}^1, \mathscr{L}^2)$, so $\rho_X \leq 4$ by Lemma 3.4.6.

Case (2.4):

If $\mathscr{C}$ is locally unsplit, $\rho_X \leq 2$ holds for the same reason as in Case (2.3).

If $\mathscr{C}$ is not locally unsplit, then Lemma 3.4.3 gives two families of lines $\mathscr{M}^1$ and $\mathscr{M}^2$ such that

- $[\mathscr{M}^1] + [\mathscr{M}^2] = [\mathscr{C}]$;
- X is covered by connected 1-cycles of $(\mathscr{M}^1, \mathscr{M}^2)$.

So we also know that X is dominated by connected 1-cycles of $(\mathscr{C}, \mathscr{L}^1, \mathscr{L}^2, \mathscr{M}^1, \mathscr{M}^2; 1)$. By employing Lemma 3.4.7 ($\mathscr{V} = \mathscr{F}$, $\mathscr{W} = \mathscr{C}$, $\mathscr{V}^1 = \mathscr{L}^1$, $\mathscr{V}^2 = \mathscr{L}^2$, $\mathscr{V}^3 = \mathscr{W}^1 = \mathscr{M}^1$, $\mathscr{V}^4 = \mathscr{W}^2 = \mathscr{M}^2$), we obtain $\rho_X \leq 4$.

Case (2.5):

Since X is rationally chain connected with respect to $(\mathscr{L}^1, \mathscr{L}^2)$, $\rho_X \leq 2$ follows from Proposition 3.2.11.

Case (2.6):

In this case, X is conic connected, so $\rho_X \leq 2$ according to Fact 3.1.1. Moreover, since (X, H) cannot be isomorphic to $(\mathbb{P}^n, \mathscr{O}(2))$ (which has no lines), X is covered by lines.

Case (2.7):

Since X is line connected, (X, H) is isomorphic to $(\mathbb{P}^n, \mathscr{O}(1))$, which satisfies all the three conclusions.

Case (3.1):

Let Λ be the set of triples of families of lines $(\mathscr{M}^1, \mathscr{M}^2, \mathscr{M}^3)$ such that

- $[\mathscr{M}^1] + [\mathscr{M}^2] + [\mathscr{M}^3] = [\mathscr{E}]$;
- X is covered by connected 1-cycles of $(\mathscr{M}^1, \mathscr{M}^2, \mathscr{M}^3)$;

and let Γ_1 (resp. Γ_2) be the set of pairs of families of conics and of lines $(\mathscr{D}, \mathscr{M})$ such that

- $[\mathscr{D}] + [\mathscr{M}] = [\mathscr{E}]$;
- X is dominated by connected 1-cycles of $(\mathscr{D}, \mathscr{M}; 1)$ (resp. $(\mathscr{D}, \mathscr{M}; 2)$).

Now let $x \in X$ be a general point. Since $\overline{\mathrm{Locus}(\mathscr{L}, \mathscr{E}; x)} = X$, by applying Proposition 3.2.10 two times, we find that $N_1(X)$ is spanned by the numerical classes of $\mathscr{L}$, $\mathscr{E}$, and families contained in one of Λ, Γ_1, and Γ_2.

If there exists a triple $(\mathscr{M}^1, \mathscr{M}^2, \mathscr{M}^3) \in \Lambda$, then X is also covered by connected 1-cycles of $(\mathscr{M}^1, \mathscr{M}^2, \mathscr{M}^3, \mathscr{L})$ because $\mathscr{L}$ is covering. This yields $\rho_X \leq 4$ by Lemma 3.4.6. So we may assume $\Lambda = \emptyset$. Then we have

$$N_1(X) = \langle [\mathscr{L}], [\mathscr{E}], \{[\mathscr{M}]\}_{(\mathscr{D},\mathscr{M}) \in \Gamma_1 \cup \Gamma_2} \rangle.$$

If $[\mathscr{E}] = 3[\mathscr{L}]$, then $[\mathscr{F}] = 4[\mathscr{L}]$. Since $\mathscr{L}$ is covering, this implies that $\rho_X = 1$ by Lemma 3.4.6. Thus we may assume that $\mathscr{E}$ and $\mathscr{L}$ are numerically independent.

If $[\mathscr{M}]$ is a linear combination of $[\mathscr{L}]$ and $[\mathscr{E}]$ for every pair $(\mathscr{D}, \mathscr{M}) \in \Gamma_1 \cup \Gamma_2$, then $N_1(X)$ is spanned by just $[\mathscr{L}]$ and $[\mathscr{E}]$, hence $\rho_X = 2$. Therefore, we may assume that there exists a pair $(\mathscr{D}, \mathscr{M}) \in \Gamma_1 \cup \Gamma_2$ satisfying $[\mathscr{M}] \notin \langle [\mathscr{L}], [\mathscr{E}] \rangle$, namely, $\mathscr{D}, \mathscr{M}$, and $\mathscr{L}$ are numerically independent.

Case (i) $(\mathscr{D}, \mathscr{M}) \in \Gamma_1$.

Now X is dominated by connected 1-cycles of $(\mathscr{D}, \mathscr{M}; 1)$. Since $\mathscr{L}$ is covering, $\mathrm{Locus}(\mathscr{L}, \mathscr{M}, \mathscr{D}; x)$ is nonempty for a general point $x \in X$. If $\mathscr{D}$ is locally unsplit, then by applying Proposition 3.2.12, we see that

$$\dim \mathrm{Locus}(\mathscr{L}, \mathscr{M}, \mathscr{D}; x) \geq (-K_X.\mathscr{F}) - 3 \geq n.$$

Thus $\mathrm{Locus}(\mathscr{L}, \mathscr{M}, \mathscr{D}; x) = X$. By using Proposition 3.2.10 three times, we conclude that $\rho_X \leq 3$.

Next, if $\mathscr{D}$ is not locally unsplit, then we get two families of lines $\mathscr{M}^1$ and $\mathscr{M}^2$ such that

- $[\mathscr{M}^1] + [\mathscr{M}^2] = [\mathscr{C}]$;
- X is covered by connected 1-cycles of $(\mathscr{M}^1, \mathscr{M}^2)$ by Lemma 3.4.3.

Then X is dominated by connected 1-cycles of $(\mathscr{D}, \mathscr{M}, \mathscr{L}, \mathscr{M}^1, \mathscr{M}^2; 1)$. Therefore, Lemma 3.4.7 implies $\rho_X \leq 4$.

Case (ii) $(\mathscr{D}, \mathscr{M}) \in \Gamma_2$.

In this case, $\mathrm{Locus}(\mathscr{L}, \mathscr{D}, \mathscr{M}; x)$ is nonempty for a general point $x \in X$. If $\mathscr{D}$ is proper at every point of $\mathrm{Locus}(\mathscr{M}; x)$, then as in the first half of Case (i), we can prove $\mathrm{Locus}(\mathscr{L}, \mathscr{D}, \mathscr{M}; x) = X$, and hence $\rho_X \leq 3$.

If $\mathscr{D}$ is not proper at some point of $\mathrm{Locus}(\mathscr{M}; x)$ for a general point $x \in X$, then as in the proof of Lemma 3.4.3, we get two families of lines $\mathscr{M}^1$ and $\mathscr{M}^2$ such that

- $[\mathscr{M}^1] + [\mathscr{M}^2] = [\mathscr{D}]$;
- X is covered by connected 1-cycles of $(\mathscr{M}, \mathscr{M}^1, \mathscr{M}^2; 1)$.

Then X is also covered by connected 1-cycles of $(\mathscr{M}, \mathscr{M}^1, \mathscr{M}^2, \mathscr{L})$, hence Lemma 3.4.6 allows us to conclude that $\rho_X \leq 4$.

Case (3.2):

X is rationally cubic connected, so the second conclusion holds.

Case (3.3):

(X, H) is isomorphic to $(\mathbb{P}^n, \mathscr{O}(1))$.

Case (4.1):

If $[\mathscr{C}^1] = [\mathscr{C}^2]$, then just the third conclusion holds. So we suppose that $\mathscr{C}^1$ and $\mathscr{C}^2$ are numerically independent. Then we show that at least one among $\mathscr{C}^1$ and $\mathscr{C}^2$ is not locally unsplit. By contradiction, assume that both of them are locally unsplit. Then Proposition 3.2.12 implies

$$\dim \mathrm{Locus}(\mathscr{C}^1_x) + \dim \mathrm{Locus}(\mathscr{C}^2_y) \geq (-K_X.\mathscr{F}) - 2 \geq n + 1$$

for general points $x, y \in X$. It follows that $\mathrm{Locus}(\mathscr{C}^1_x) \cap \mathrm{Locus}(\mathscr{C}^2_y)$ has positive dimension, so we get a curve contained in this intersection. Then according to Proposition 3.2.10, this curve must be numerically proportional to both $\mathscr{C}^1$ and $\mathscr{C}^2$, a contradiction. So, from now on, we suppose that $\mathscr{C}^1$ is not locally unsplit, in particular, X is covered by lines. Then we only have to show that $\rho_X \leq 4$.

If $\mathscr{C}^2$ is not locally unsplit either, then Lemma 3.4.3 gives four families of lines $\mathscr{M}^1$, $\mathscr{M}^2$, $\mathscr{M}^3$, and $\mathscr{M}^4$ such that

- $[\mathscr{M}^1] + [\mathscr{M}^2] = [\mathscr{C}^1]$;
- $[\mathscr{M}^3] + [\mathscr{M}^4] = [\mathscr{C}^2]$;
- X is covered by connected 1-cycles of $(\mathscr{M}^1, \mathscr{M}^2, \mathscr{M}^3, \mathscr{M}^4)$.

Thus $\rho_X \leq 4$ by Lemma 3.4.6.

We finally consider the case where $\mathscr{C}^2$ is locally unsplit. Let Λ be the set of pairs of families of lines $(\mathscr{N}^1, \mathscr{N}^2)$ such that

- $[\mathscr{N}^1] + [\mathscr{N}^2] = [\mathscr{C}^1]$;
- X is dominated by connected 1-cycles of $(\mathscr{C}^2, \mathscr{N}^1, \mathscr{N}^2; 1; t)$ (see Definition 3.4.2), where t is a tree as Figure 3 (see also Figure 4).

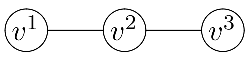


Figure 3.

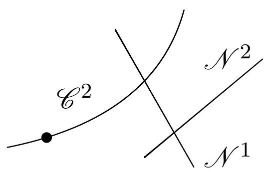


Figure 4.

Since $\overline{\mathrm{Locus}(\mathscr{C}^1, \mathscr{C}^2; x)} = X$ for a general point $x \in X$, Proposition 3.2.10 yields that $N_1(X)$ is spanned by the numerical classes of $\mathscr{C}^1$, $\mathscr{C}^2$, and families contained in Λ, hence

$$N_1(X) = \langle \mathscr{C}^1, \mathscr{C}^2, \{[\mathscr{N}^1]\}_{(\mathscr{N}^1, \mathscr{N}^2) \in \Lambda} \rangle.$$

If $[\mathscr{N}^1]$ is a linear combination of $[\mathscr{C}^1]$ and $[\mathscr{C}^2]$ for every pair $(\mathscr{N}^1, \mathscr{N}^2) \in \Lambda$, then $\rho_X = 2$. So we may assume that there is a pair $(\mathscr{N}^1, \mathscr{N}^2) \in \Lambda$ such that $\mathscr{C}^2$, $\mathscr{N}^1$, and $\mathscr{N}^2$ are numerically independent.

Now $\mathrm{Locus}(\mathscr{N}^2, \mathscr{N}^1, \mathscr{C}^2; x)$ is nonempty for a general point $x \in X$. Then as is Case (3.1)(i), Proposition 3.2.12 implies $\mathrm{Locus}(\mathscr{N}^2, \mathscr{N}^1, \mathscr{C}^2; x) = X$, and hence

$\rho_X \leq 3$ by Proposition 3.2.10.

Case (4.2):

It follows from Fact 3.1.1 that $\rho_X \leq 2$ and either X is covered by lines or (X, H) is isomorphic to $(\mathbb{P}^n, \mathscr{O}(2))$. Note that the third conclusion holds in the latter case. $\square$

Remark 3.6.1. In the proof of Theorem 3.1.3, we use the assumption $(-K_X.\mathscr{F}) \geq n+3$ only in Cases (3.1) and (4.1), and use $(-K_X.\mathscr{F}) \geq n+2$ in order to show that at least one among (1)-(4) holds.

Bibliography

[1] M. Andreatta, E. Chierici, and G. Occhetta, *Generalized Mukai conjecture for special Fano varieties*, Cent. Eur. J. Math. **2**(2) (2004), 272-293.

[2] M. C. Beltrametti, P. Ionescu, *On manifolds swept out by high-dimensional quadrics*, Math. Z. **260**(1) (2008) 229-234.

[3] M. Coppens, *Very ample linear systems on blowings-up at general points of projective spaces*, Canad. Math. Bull. **45**(3) (2002), 349-354.

[4] L. Ein, *Varieties with small dual variety. II*, Duke Math. J. **52** (1985) 895-907.

[5] B. Fu, *Inductive characterizations of hyperquadrics*, Math. Ann. **340**(1) (2008) 185-194.

[6] T. Fujita, *On the structure of polarized manifolds with total deficiency one, I*, J. Math. Soc. Japan **32** (1980) 709-725.

[7] T. Fujita, *On the structure of polarized manifolds with total deficiency one, II*, J. Math. Soc. Japan **33** (1981) 415-434.

[8] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Springer-Verlag, Lecture Notes in Mathematics, 156 (1970).

[9] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York (1977).

[10] J. M. Hwang, *Geometry of minimal rational curves on Fano manifolds*, in School on Vanishing Theorems and Effective Results in Algebraic Geometry(Trieste, 2000), ICTP Lect. Notes, vol. 6, pp335-393. Abdus Salam Int. Cent. Theoret. Phys. (2001).

[11] J. M. Hwang, S. Kebekus, *Geometry of chains of minimal rational curves*, J. Reine Angew. Math. **584** (2005) 173-194.

[12] P. Ionescu and F. Russo, *Conic connected manifolds*, J. Reine Angew. Math. **644** (2010), 145-157.

[13] P. Ionescu, F. Russo, *Manifolds covered by lines, defective manifolds and a restricted Hartshorne Conjecture*, arXiv:0909.2763.

[14] T. A. Ivey, J. M. Landsberg, *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*, Grad. Stud. Math., vol. 61, Amer. Math. Soc., Providence, RI, 2003.

[15] Y. Kachi, E. Sato, *Segre's reflexivity and an inductive characterization of hyperquadrics* , Mem. Amer. Math. Soc. **160**(763) (2002).

[16] S. Kebekus, *Families of singular rational curves*, J. Alg. Geom. **11**(2) (2002), 245-256.

[17] S. Kobayashi, T. Ochiai, *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. **13** (1973) 31-47.

[18] J. Kollár, *Rational Curves on Algebraic Varieties*, Ergeb. Math. Grenzgeb., vol. 32. Springer, Berlin (1996).

[19] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties, Cambridge Tracts in Mathematics*, vol. 134. Cambridge University Press, Cambridge (1998).

[20] J. Kollár, Y. Miyaoka, S. Mori, *Rational curves on Fano varieties*, in Proc. Alg. Geom. Conf., Trento 1990, Springer LNM 1515 (1992) 100-105.

[21] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. Math.(2) **110**(3) (1979), 593-606.

[22] S. Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Natl. Acad. Sci. USA **86** (1989) 3000-3002.

[23] C. Novelli, G. Occhetta, *Projective manifolds containing a large linear subspace with nef normal bundle*, *Michigan Math. J.* **60**(2) (2011) 441-462.

[24] G. Occhetta and V. Paterno, *Rationally cubic connected manifolds I: manifolds covered by lines*, J. Math. Soc. Japan **64**(3) (2012), 941-967.

[25] G. Occhetta and V. Paterno, *Rationally cubic connected manifolds II*, Rev. Mat. Iberoam. **28**(3) (2012), 815-838.

[26] V. Paterno, *Special rationally connected manifolds*, Ph. D. Thesis, University of Trento, 2009.

[27] F. Russo, *Varieties with quadratic entry locus. I*, Math. Ann. **344** (2009) 597-617.

[28] E. Sato, *Projective manifolds swept out by large-dimensional linear spaces*, Tohoku Math. J. **49**(2) (1997) 299-321.

[29] E. Sernesi, *Deformations of Algebraic Schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334. Springer-Verlag, Berlin, (2006).

[30] K. Watanabe, *On projective manifolds swept out by cubic varieties*, Int. J. Math. **23**, 1250058 (2012).

[31] J. A. Wiśniewski, *On a conjecture of Mukai*, Manuscripta Math. **68**(2) (1990) 135-141.

[32] F. L. Zak, *Tangents and Secants of Algebraic Varieties*, Translations of Mathematical Monographs, Vol. 127 (American Mathematical Society, Providence, RI, 1993).

List of papers by Taku SUZUKI

(1) *On manifolds swept out by high dimensional hypersurfaces*, to appear in Journal of Pure and Applied Algebra.

(2) *On the Picard number of rationally quartic connected manifolds*, to appear in International Journal of Mathematics.