Projective geometry in arbitrary characteristic and Rational curves on algebraic varieties

任意標数の射影幾何と代数多様体上の有理曲線

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Introduction

This thesis is organized as three main research subjects. In Chapter I, we study geometric properties of the space of smooth rational curves lying in a hypersurface of projective space. In particular, we consider the smoothness, the dimension and the connectedness of the space. In Chapter II, as a research of geometry in positive characteristic, we investigate projective varieties which admit embeddings whose Gauss maps are of rank zero. This chapter contains the results of the joint work with S. Fukasawa and H. Kaji. In Chapter III, we study the Segre locus, which is the locus of points from which a closed subvariety in projective space is projected non-birationally. Here we give a method to compute polynomials generating the defining ideal of the Segre locus.

In the following, we state the details of each chapter.

I. Rational curves on hypersurfaces For a hypersurface $X \subset \mathbb{P}^n$ of degree d, we define $R_e(X)$ to be the open subscheme of the Hilbert scheme $\mathrm{Hilb}^{et+1}(X/k)$ which parametrizes smooth rational curves of degree e in $\mathbb{P}^n$ lying in X, where k is an algebraically closed field of arbitrary characteristic. We set

$$\mu := (n+1-d)e + n - 4, \tag{1}$$

which is called the *expected dimension* of $R_e(X)$. It is known that if X is smooth and if there exists $C \in R_e(X)$, then the dimension of $R_e(X)$ at C is greater than or equal to $\mu = \chi(N_{C/X})$ (see Ch. I, Remark 2.2 or [**45**], II, Theorem 1.2).

The starting point of our study is the following result for lines on hypersurfaces obtained by W. Barth and A. Van de Ven over $\mathbb{C}$, and by J. Kollár over an algebraically closed field of arbitrary characteristic.

Theorem A (W. Barth and A. Van de Ven ([**7**]), J. Kollár ([**45**], V, Theorem 4.3))**.** *Let X be as above. Then*

(a) $R_1(X) = \emptyset$ *for general X if $\mu < 0$,*
(b) $R_1(X)$ *is smooth of dimension μ for general X if $\mu \geqslant 0$,*
(c) $R_1(X)$ *is connected for any X if $\mu \geqslant 1$, except when $X \subset \mathbb{P}^3$ is a smooth quadric.*

What can we say about the family $R_e(X)$ for the degree $e \geqslant 2$? The following result has been obtained by J. Harris, M. Roth, and J. Starr.

Theorem B (J. Harris, M. Roth, and J. Starr ([**30**], Theorem 1.1))**.** *Assume that the ground field is $\mathbb{C}$, $d < (n+1)/2$, and $n \geqslant 3$. For general X and for any $e \geqslant 1$, the scheme $R_e(X)$ is an integral, local complete intersection scheme of dimension μ.*

We shall study $R_e(X)$ without the assumption on the characteristic of the ground field. Our main result is:

Theorem I. *Let $X \subset \mathbb{P}^n$ be as above with $n \geqslant 3$.*

(a) *Assume $d \geqslant \max\{ e-2, 1 \}$. Then $R_e(X) = \emptyset$ for general X if $\mu < 0$.*
(b) *Assume either $1 \leqslant e \leqslant 3$ and $d \geqslant 1$, or $e \geqslant 4$ and $d \geqslant 2e-3$. Then $R_e(X)$ is smooth of dimension μ for general X if $\mu \geqslant 0$.*
(c) *$R_2(X)$ is connected for general X if $\mu \geqslant 1$, except when $X \subset \mathbb{P}^3$ is a cubic.*

In characteristic zero, one can obtain the result of Theorem I(b) under a weaker assumption, which is a conclusion of Ch. I, §2 (Ch. I, Theorem 2.16). In the exceptional case of Theorem I(c), we certainly find that $R_2(X)$ is disconnected for any smooth cubic $X \subset \mathbb{P}^3$ (Ch. I, Proposition 4.4). Non-existence of rational curves on general hypersurfaces has been studied by several authors [**13**], [**17**], [**18**], [**52**], [**59**], [**60**]. Various properties of $R_e(X)$ (e.g., rational connectedness, singularity, e.t.c.), besides the ones stated in Theorem B, have been studied in [**31**], [**58**].

II. Gauss map of rank zero Let X be a projective variety of dimension n in $\mathbb{P}^N$ defined over an algebraically closed field K of characteristic $p \geq 0$. The *Gauss map* of $X \subseteq \mathbb{P}^N$, denoted by γ, is by definition the rational map from X to the Grassmann variety $\mathbb{G}(n, \mathbb{P}^N)$ which sends each smooth point x of X to the embedded tangent space T_xX to X at x in $\mathbb{P}^N$ ([**26**, §1, (e)], [**62**, I, §2]). To avoid trivial exceptions we treat γ only for a non-linear $X \subseteq \mathbb{P}^N$. According to a theorem of F. L. Zak [**62**, I, 2.8. Corollary], γ is finite for a smooth X, and it is well known that a general fibre of γ is linear if $p = 0$ ([**26**, (2.10)], [**62**, I, 2.3. Theorem]); hence γ is birational for a smooth X in $p = 0$.

Now we introduce an intrinsic property of a projective variety X as follows:

(GMRZ) *there exists an embedding ι of X into some $\mathbb{P}^M$ such that the Gauss map γ is of rank zero.*

Here the *rank* of a rational map is defined to be the rank of its differential at a general point, and the differential of a rational map is by definition the induced K-linear map between Zariski tangent spaces. Note that a variety X satisfies (GMRZ) only if $p > 0$, since the rank of a rational map is equal to the dimension of its image if $p = 0$.

In this paper, we consider the case where the variety X has a rational curve $f : \mathbb{P}^1 \to X$. We have the following basic theorem, where we find that the property (GMRZ) imposes Strong restrictions on rational curves on X:

Theorem II.1. *Let X be a projective variety, let $f : \mathbb{P}^1 \to X$ be an unramified morphism, and denote by N_f the dual of the kernel of the natural homomorphism $f^* : f^*\Omega^1_X \to \Omega^1_{\mathbb{P}^1}$. Assume that X is smooth along $f(\mathbb{P}^1)$ and $N_f^\vee \simeq \bigoplus_{i \geq -1} \mathcal{O}_{\mathbb{P}^1}(i)^{r_i}$ for some non-negative integers r_i ($i \geq -1$). Then we have:*

(a) *If X satisfies* (GMRZ), *then $r_{i-1}r_i = 0$ for any $i \geq 0$.*

(b) *Moreover if* $r_{-1} > 0$*, then* $p | \deg f^*\iota^*\mathcal{O}_{\mathbb{P}^M}(1) - 1$ *for any embedding* $\iota : X \hookrightarrow \mathbb{P}^M$ *with Gauss map of rank zero, and if* $r_i > 0$ *for some* $i \geq 0$*, then* $p = 2$ *or* $p | i + 1$.

Theorem II.1 is proved by investigating bundles of principal parts (Ch. II, §1). As a consequence of Theorem II.1, we have

Theorem II.2. (a) *Let* X *be a projective variety with a non-constant morphism* π *to a variety* Y*, and assume that there exists a smooth point* y *of* Y *such that the fibre* $X_y := \pi^{-1}(y)$ *is isomorphic to a projective space* $\mathbb{P}^l$ *and* π *is smooth along* X_y*. Then* X *satisfies* (GMRZ) *only if* $p = 2$ *and* $l = 1$*. Moreover, a product* $\prod_{1 \leq i \leq r} \mathbb{P}^{n_i}$ *of two or more projective spaces* $(r \geq 2, n_i \geq 1)$ *satisfies* (GMRZ) *if and only if* $p = 2$ *and* $n_i = 1$ *for any* i*.*

(b) *A Grassmann variety* $\mathbb{G}(l, l+m)$ *of* l*-dimensional subspaces of an* $(l+m)$*-dimensional vector space* $(l, m \geq 1)$ *satisfies* (GMRZ) *if and only if* $l = 1$ *or* $m = 1$*.*

(c) *A smooth quadric hypersurface* Q *in* $\mathbb{P}^N$ $(N \geq 3)$ *satisfies* (GMRZ) *if and only if* $p = 2$ *and* $N = 3$*.*

(d) *A smooth cubic hypersurface* X *in* $\mathbb{P}^N$ $(N \geq 3)$ *satisfies* (GMRZ) *only if* $p = 2$*.*

A rational curve (or a morphism) $f : \mathbb{P}^1 \to X$ is said to be *free* if the pull-back f^*T_X of the tangent bundle T_X on X is generated by its global sections ([**15**, p. 85], [**45**, II.3.1]), and a free f *minimal* if f^*T_X is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{d-2} \oplus \mathcal{O}_{\mathbb{P}^1}^{n-d+1}$ with $d = \deg(-f^*K_X)$ ([**15**, p. 93], [**45**, IV.2.8]). One of the most basic results in characteristic zero case to guarantee that existence is

Theorem C ([**45**, IV.2.10])**.** *Let* X *be a smooth projective variety in* $p = 0$*. If there exists a free rational curve on* X*, then there exists a minimal free rational curve on* X*.*

Note that for a smooth X in arbitrary characteristic $p \geq 0$, the existence of free rational curves is equivalent to the separable uniruledness ([**45**, IV.1.9]). In positive characteristic case, however, the conclusion of Theorem C turns out to fail, as we will see below. In fact, Theorem II.1 implies

Theorem II.3. *Let* X *be a projective variety, and assume that* X *satisfies* (GMRZ) *with an embedding* $\iota : X \hookrightarrow \mathbb{P}^M$*. Let* $f : \mathbb{P}^1 \to X$ *be a minimal free rational curve such that* X *is smooth along* $f(\mathbb{P}^1)$*, and set* $a := \deg f^*\iota^*\mathcal{O}_{\mathbb{P}^M}(1)$*. Then one of the following holds:*

(a) $\deg(-f^*K_X) = n + 1$*,* $a > p$ *and* $p \mid a - 1$*.*

(b) $\deg(-f^*K_X) = p = 2$ *and* $2 \mid a$*.*

In particular, we have $a > 1$*.*

Using Theorem II.3, one can give a counter-example for Theorem A, that is, a projective variety which admits a free rational curve, but no minimal free rational curve, in each characteristic $p > 0$ (Ch. II, Theorem 3.2; Cf. [**45**, IV.2.10.1]).

Next, we will consider a general hypersurface of low degree with (GMRZ):

Theorem II.4. *A general hypersurface* X *in* $\mathbb{P}^N$ *of degree* d *with* $3 \leqslant d \leqslant 2N - 3$ *satisfies* (GMRZ) *only if* $p = 2$ *and* $d = 2N - 3$*.*

For a higher dimensional cubic hypersurface, we have

Theorem II.5. *A smooth cubic hypersurface X in $\mathbb{P}^N$ with $N \geq 4$ satisfies* (GMRZ) *if and only if $p = 2$ and X is projectively equivalent to a Fermat hypersurface.*

To obtain Theorems II.4 and II.5 above, we need in addition detailed studies on the normal bundles of conics in a hypersurface (Ch. II, §4), and on projective geometry on cubic hypersurfaces with Gauss map of rank zero (Ch. II, §5, §6).

Next, in Ch. II, §7, we see that the statement of Theorem II.5 is no longer true in the case of $N = 3$ (Ch. II, Corollary 7.3). Moreover, we show that every smooth rational surface admits an embedding whose Gauss map is of rank zero if $p = 2$ (Ch. II, Theorem 7.5). This is deduced from the following result for blowing-ups:

Theorem II.6 (= Ch. II, Theorem 7.1)**.** *In the case of $p = 2$, the process of blowing-ups at points preserves the property* (GMRZ) *for projective varieties X.*

The results of Ch. II, §1–§5 is based on [**FFK**], the joint work with S. Fukasawa and H. Kaji. On the other hand, Ch. II, §6–§7 is the consequence of [**24**].

III. Defining ideal of the Segre locus B. Segre [**57**] studied the locus of points from which X is projected non-birationally, for a variety X embedded in $\mathbb{P}^N$. We rigorously define as follows:

$$\mathfrak{S}^{\mathrm{out}}(X) := \overline{\{\, z \in \mathbb{P}^N \setminus X \mid \pi_{z|_X} : X \to \pi_z(X) \text{ is not birational}\,\}},$$
$$\mathfrak{S}^{\mathrm{inn}}(X) := \overline{\{\, z \in X \mid \pi_{z|_X} : X \setminus \{\, z\,\} \to \pi_z(X \setminus \{\, z\,\}) \text{ is not birational}\,\}},$$

where $\pi_z : \mathbb{P}^N \setminus \{\, z\,\} \to \mathbb{P}^{N-1}$ is the projection from a point $z \in \mathbb{P}^N$. As an essential result, Segre proved that $\mathfrak{S}^{\mathrm{out}}(X)$ is a union of finitely many linear subspaces of $\mathbb{P}^N$ in characteristic zero case [**57**], [**10**, Thm. 1]. After him, $\mathfrak{S}^{\mathrm{out}}(X)$ is called the *Segre locus*. Recently, the study of $\mathfrak{S}^{\mathrm{out}}(X)$ and $\mathfrak{S}^{\mathrm{inn}}(X)$ has been developed by several authors (A. Calabri and C. Ciliberto [**10**], E. Ballico [**6**], A. Noma [**50**]). We denote by

$$\mathfrak{S}^{\mathrm{tot}}(X) := \mathfrak{S}^{\mathrm{out}}(X) \cup \mathfrak{S}^{\mathrm{inn}}(X),$$

and call this the *total Segre locus* of X. Our main result is:

Theorem III (= Ch. III, Theorem 2.8)**.** *Let $X \subset \mathbb{P}^N$ be a non-degenerate projective (reduced and irreducible) variety over an algebraically closed field k of characteristic p. If either $p \geqslant \deg(X)$ or $p = 0$, then the total Segre locus $\mathfrak{S}^{\mathrm{tot}}(X)$ is equal to a union of finitely many linear subspaces of $\mathbb{P}^N$.*

For the case of $p < \deg(X)$, we give an example of X such that $\mathfrak{S}^{\mathrm{tot}}(X)$ is *non-linear* (see Ch. III, Example 2.1). Note that the linearity of $\mathfrak{S}^{\mathrm{out}}(X)$ follows from Theorem III (Ch. III, Remark 2.11).

In this paper, we propose a new approach to investigate the total Segre locus, working in arbitrary characteristic. In Ch. III, §1, we give a method to calculate polynomials generating the defining ideal of $\mathfrak{S}^{\mathrm{tot}}(X)$. In Ch. III, §2, by using this method, we determine the total Segre locus of Ch. III, Example 2.1, and next give the proof of Theorem III (Ch. III, Theorems 2.2 and 2.8).

CHAPTER I

Rational curves on hypersurfaces

1. Regularity of a power of an ideal sheaf

For later use, we investigate the Castelnuovo-Mumford regularity of a power of the defining ideal sheaf of a curve in projective space, by applying an argument which is similar to the proof of [**28**], Theorem 1.1.

Theorem 1.1. *Let $X \subset \mathbb{P}^r$ be a reduced irreducible non-degenerate curve of degree d, let $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^r}$ be the ideal sheaf of X, and let $\alpha \in \mathbb{N}$. Then $\mathcal{I}_X^\alpha$ is $\alpha(d+2-r)$-regular in the sense of Castelnuovo-Mumford.*

Proposition 1.2. *Let $X \subset \mathbb{P}^r$ be a reduced curve, with normalization $\tilde{X}$, and let $p : \tilde{X} \to \mathbb{P}^r$ denote the natural map. Set $M = p^*\Omega^1_{\mathbb{P}^r}(1)$, and suppose that A is a line bundle on $\tilde{X}$ such that*

$$H^1(\tilde{X}, \bigwedge^2 M \otimes A) = 0.$$

Then $\mathcal{I}_X^\alpha$ is $\alpha \cdot h^0(\tilde{X}, A)$-regular.

Proof. As in the proof of [**28**], Proposition 1.2, we have an exact sequence of sheaves on $\mathbb{P}^r$,

$$([\mathbf{28}], (1.3)) \qquad H^0(\tilde{X}, M \otimes A) \otimes_k \mathcal{O}_{\mathbb{P}^r}(-1) \xrightarrow{u} H^0(\tilde{X}, A) \otimes_k \mathcal{O}_{\mathbb{P}^r} \to p_*A \to 0.$$

As in [**28**], p.496, we set $n_0 = h^0(\tilde{X}, A)$ and set $\mathcal{J}$ to be the zeroth Fitting ideal sheaf of p_*A, i.e., $\mathcal{J} = \mathrm{im}(\wedge^{n_0} u) \subset \mathcal{O}_{\mathbb{P}^r}$. Here, since p_*A is supported on X, and since X is reduced, we have

$$\mathcal{J} \subset \mathcal{I}_X.$$

Moreover $\mathcal{I}_X/\mathcal{J}$ is supported in finitely many points of $\mathbb{P}^r$; hence so is $\mathcal{I}_X^\alpha/\mathcal{J}^\alpha$. Therefore we find that the sheaf $\mathcal{I}_X^\alpha$ is αn_0-regular if the sheaf $\mathcal{J}^\alpha$ is so.

On the other hand, the above sequence ([**28**], (1.3)) induces the following exact sequence of sheaves on $\mathbb{P}^r$,

$$H^0(\tilde{X}, M \otimes A)^{\oplus\alpha} \otimes_k \mathcal{O}_{\mathbb{P}^r}(-1) \xrightarrow{u^{\oplus\alpha}} H^0(\tilde{X}, A)^{\oplus\alpha} \otimes_k \mathcal{O}_{\mathbb{P}^r} \to p_*A^{\oplus\alpha} \to 0.$$

Then we have $\mathcal{J}^\alpha = \mathrm{im}(\wedge^{\alpha n_0} u^{\oplus\alpha}) \subset \mathcal{O}_{\mathbb{P}^r}$; hence the sheaf $\mathcal{J}^\alpha$ is the zeroth Fitting ideal sheaf of $p_*A^{\oplus\alpha}$. Now we have the Eagon-Northcott complex constructed from $u^{\oplus\alpha}$,

$$\cdots \to \mathcal{O}_{\mathbb{P}^r}^{M_{r-1}}(-\alpha n_0 + 1 - r) \to \cdots \to \mathcal{O}_{\mathbb{P}^r}^{M_1}(-\alpha n_0 - 1) \to \mathcal{O}_{\mathbb{P}^r}^{M_0}(-\alpha n_0) \xrightarrow{\varepsilon} \mathcal{J}^\alpha \to 0,$$

where $\varepsilon := \wedge^{\alpha n_0} u^{\oplus\alpha}$ is surjective. Since this complex is exact off X, it follows from [**28**], Lemma 1.6 that $H^i(\mathbb{P}^r, \mathcal{J}^\alpha(\alpha n_0 - m)) = 0$ for $1 \leqslant m \leqslant r$ and $i \geqslant m$. □

Proof of Theorem 1.1. By using Proposition 1.2 and [**28**], Lemma 1.7, we have the theorem. □

2. Projection of the incidence variety to the space of hypersurfaces

Let $\mathcal{R}$ be an open subset of $\mathrm{Hilb}^{et+1}(\mathbb{P}^n/k)$ whose general member corresponds to a smooth rational curve of degree e. Let $H := |\mathcal{O}_{\mathbb{P}^n}(d)|$, the space of hypersurfaces of degree d. We actually assume one of the following conditions (i-ii):

$$(2)\qquad \begin{cases} \text{(i)} & e \geqslant 2,\ d \geqslant \max\{\, e-2, 1\,\}, \text{ and } \mathcal{R} \subset \mathrm{Hilb}^{et+1}(\mathbb{P}^n/k) \text{ is} \\ & \text{the space of smooth rational curves of degree } e \text{ in } \mathbb{P}^n, \\ \text{(ii)} & e = 2,\ d \geqslant 1, \text{ and } \mathcal{R} = \mathrm{Hilb}^{2t+1}(\mathbb{P}^n/k), \end{cases}$$

and study the *incidence variety*

$$(3)\qquad I = \{\, (X, C) \in H \times \mathcal{R} \mid C \subset X \,\}$$

with the *projection* $p_H : I \to H$. Note that we additionally deal with the case where $\mathcal{R}$ is the whole space $\mathrm{Hilb}^{2t+1}(\mathbb{P}^n/k)$ as (2.ii) above, because that is necessary in the proof of Theorem I(c) (§4).

In this section, we first give an explicit construction of the incidence variety I, which is obtained as a projective bundle over $\mathcal{R}$ by assuming one of the conditions (i-ii) of (2). Then Theorem I(a) is proved by the calculation of the relative dimension of p_H. Next, as a preparation of the proof of Theorem I(b-c), we study properties of the projection p_H in terms of the normal sheaves $N_{C/\mathbb{P}^n}$ and $N_{X/\mathbb{P}^n}$.

In order to prove Theorem I(b), we need to establish the generic smoothness of the projection p_H in arbitrary characteristic (§3). However, just in the characteristic zero case, the statement of Theorem I(b) can be obtained by showing that $p_H(I)$ is dense in H (§2.3).

2.1. Basic construction. Since $\mathcal{R}$ is an open subscheme of $\mathrm{Hilb}^{et+1}(\mathbb{P}^n/k)$, there exist the universal family $u : \mathrm{Univ} \to \mathcal{R}$ and the projection $F : \mathrm{Univ} \to \mathbb{P}^n$. Then we have the following morphism of sheaves on $\mathcal{R}$:

$$\Phi : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \otimes_k \mathcal{O}_{\mathcal{R}} \to u_*(F^*(\mathcal{O}_{\mathbb{P}^n}(d))).$$

To construct the incidence variety, we show the following lemma. Here, we denote by $N_{A/B} := \mathcal{H}om_{\mathcal{O}_A}(\mathcal{I}_A/\mathcal{I}_A^2, \mathcal{O}_A)$ the normal sheaf of a subscheme A in a scheme B, where $\mathcal{I}_A \subset \mathcal{O}_B$ is the ideal sheaf of A in B. We denote by $\mathbb{P}_*E := \mathrm{Proj}(\mathrm{Sym}(E^\vee))$ the covariant projectivization of a k-linear space or a locally free sheaf E, where the points of $\mathbb{P}_*E$ correspond to lines in fibers of E.

Lemma 2.1. *Let $n \geqslant 3$, and assume one of the conditions* (i-ii) *of* (2)*. Then the following holds.*

(a) *The scheme $\mathcal{R}$ is a smooth irreducible subvariety of* $\mathrm{Hilb}^{et+1}(\mathbb{P}^n/k)$ *of dimension $(n+1)e+n-3$, where $(n+1)e+n-3 = h^0(N_{C/\mathbb{P}^n})$ for any $C \in \mathcal{R}$.*

(b) *We have $h^0(C, \mathcal{O}_C(d)) = de+1$ and have that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(C, \mathcal{O}_C(d))$ is surjective for any $C \in \mathcal{R}$.*

(c) *We have that* $\ker(\Phi)\otimes k_C$ *is isomorphic to* $H^0(\mathbb{P}^n, \mathcal{I}_C(d))$ *for any* $C \in \mathcal{R}$*, where* k_C *is the residue field of the Hilbert point* C *on* $\mathcal{R}$*. Hence,* $\ker\Phi$ *is a locally free sheaf on* $\mathcal{R}$ *of rank* $h^0(\mathbb{P}^n, \mathcal{O}(d)) - (de+1)$*.*

Thus the projective bundle $I := \mathbb{P}_*(\ker\Phi)$ *over* $\mathcal{R}$ *is a smooth irreducible variety with* $\dim I = \dim H + \mu$*, where we set* $H = |\mathcal{O}_{\mathbb{P}^n}(d)|$ *and* $\mu = (n+1-d)e+n-4$ *as in* §*, (1).*

Remark 2.2. We have $\mu = \chi(N_{C/X})$, in the case where $X \subset \mathbb{P}^n$ is a hypersurface of degree d, and $C \subset X$ is a smooth rational curve of degree e, such that X is smooth along C. The reason is the following: Let $f : \mathbb{P}^1 \to \mathbb{P}^n$ be a morphism parametrizing C. From the exact sequence $0 \to f^*T_X \to f^*T_{\mathbb{P}^n} \to f^*N_{X/\mathbb{P}^n} \to 0$, we have $\chi(f^*T_X) = (n+1-d)e+n-1$. Thus, from the exact sequence $0 \to T_{\mathbb{P}^1} \to f^*T_X \to f^*N_{C/X} \to 0$, we have $\chi(N_{C/X}) = (n+1-d)e+n-4$.

By applying [**45**], I, Theorem 2.8 and Proposition 2.14, we find that the scheme $R_e(X)$ is of dimension $\geqslant \chi(N_{C/X}) = \mu$ at C.

Remark 2.3. Let $C \subset \mathbb{P}^n$ be a smooth rational curve of degree $e \geqslant 2$, let $L \subset \mathbb{P}^n$ be the linear subspace spanned by C, and let $r = \dim(L)$. Here we have the following information about the dimension of L and regularity of C.

(a) We have $r \leqslant e$. In addition we have $r \geqslant 3$ in the case $e \geqslant 3$, because every plane rational curve of degree $\geqslant 3$ must be singular.

(b) We have $r = e$ in the case $e \leqslant 3$, as follows: If $e = 2$, then C is a conic; hence we have $r = 2$. If $e = 3$, then it follows from (a) that we have $r = 3$.

(c) We have $\max\{\,(e-2), 1\,\} \geqslant e+1-r$. This is because it follows from (b) that we have $r = e$ if $e \leqslant 3$, and it follows from (a) that we have $r \geqslant 3$ if $e \geqslant 4$.

(d) The ideal sheaf $\mathcal{I}_{C/L}$ of C in L is $(e+2-r)$-regular, and hence $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(C, \mathcal{O}_C(d))$ is surjective for any $d \geqslant e+1-r$. The reason is as follows: Since C is non-degenerate in L, it follows from [**28**], Theorem 1.1 that $\mathcal{I}_{C/L}$ is $(e+2-r)$-regular. By regularity, we have $H^1(\mathcal{I}_{C/L}(d)) = 0$ for $d+1 \geqslant e+2-r$, which implies that

$$H^0(L, \mathcal{O}(d)) \to H^0(C, \mathcal{O}(d))$$

is surjective. Since $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(L, \mathcal{O}(d))$ is also surjective, so is the composite map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(C, \mathcal{O}_C(d))$.

Remark 2.4. Each element $C \in \mathrm{Hilb}^{2t+1}(\mathbb{P}^n/k)$ is given by a complete intersection of a linear plane $L \subset \mathbb{P}^n$ and a quadric hypersurface of $\mathbb{P}^n$, which is a smooth conic, a union of two lines intersecting in one point, or a double line. Here we have an exact sequence

$$0 \to \mathcal{O}_L(-2) \to \mathcal{O}_L \to \mathcal{O}_C \to 0. \tag{4}$$

In addition, we have $N_{C/L} \simeq \mathcal{O}_C(2)$ and

$$N_{C/\mathbb{P}^n} \simeq N_{C/L} \oplus N_{L/\mathbb{P}^n}|_C \simeq \mathcal{O}_C(2) \oplus \mathcal{O}_C(1)^{\oplus n-2}. \tag{5}$$

On the other hand, we have a morphism

$$\pi : \mathrm{Hilb}^{2t+1}(\mathbb{P}^n/k) \to G(2, \mathbb{P}^n)$$

by sending each C to the linear plane L spanned by C, where each fiber $\pi^{-1}(L)$ at $L \in G(2, \mathbb{P}^n)$ is isomorphic to $\mathrm{Hilb}^{2t+1}(L/k) \simeq |\mathcal{O}_L(2)|$. Indeed, $\mathrm{Hilb}^{2t+1}(\mathbb{P}^n/k)$ is obtained as the projective bundle $\mathbb{P}_*(S^2(U))$ on $G(2, \mathbb{P}^n)$, where U is the universal bundle on $G(2, \mathbb{P}^n)$ of rank 3.

Proof of Lemma 2.1. (a) Assume the condition (2.i). Then $\mathcal{R}$ is the space of smooth rational curves of degree e in $\mathbb{P}^n$. Let $C \in \mathcal{R}$ and let $f : \mathbb{P}^1 \to \mathbb{P}^n$ be a morphism parametrizing C. Since $T_{\mathbb{P}^n}$ is ample, so is $f^* N_{C/\mathbb{P}^n}$ on $\mathbb{P}^1$. Hence $h^1(N_{C/\mathbb{P}^n}) = 0$. Thus it follows from [**45**], I, Theorem 2.8 and Proposition 2.14 that $\mathcal{R}$ is smooth of dimension $h^0(N_{C/\mathbb{P}^n}) = (n+1)e + n - 3$ at every $C \in \mathcal{R}$.

Next, for any elements $C_1, C_2 \in \mathcal{R}$, we give an irreducible curve in $\mathrm{Hilb}^{et+1}(\mathbb{P}^n/k)$ connecting C_1 and C_2, as follows: We have morphisms $f_i = (f_{i,j})_{j=0}^n : \mathbb{P}^1 \to \mathbb{P}^n$ parametrizing C_i with $i = 1, 2$. Here we set $\Lambda \subset \mathbb{P}^1 \times \mathbb{P}^n$ to be the closure of the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the rational map,

$$\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^n : ((a,b), P) \mapsto ((a,b), (af_{1,j}(P) + bf_{2,j}(P))_{j=0}^n).$$

Then the first projection $\Lambda \to \mathbb{P}^1$ gives a flat family whose fibers at $(1.0), (0.1) \in \mathbb{P}^1$ are isomorphic to $C_1, C_2 \subset \mathbb{P}^n$. Thus we have a morphism $\mathbb{P}^1 \to \mathrm{Hilb}^{et+1}(\mathbb{P}^n/k)$ whose image contains the elements C_1 and C_2. This implies that $\mathcal{R}$ is irreducible, since we already showed that $\mathcal{R}$ is smooth.

Assume the condition (2.ii). Then $\mathcal{R} = \mathrm{Hilb}^{2t+1}(\mathbb{P}^n/k)$, which is a projective bundle on $G(2, \mathbb{P}^n)$ whose fibers are of dimension 5, as we saw in Remark 2.4. Thus $\mathcal{R}$ is a smooth irreducible variety of dimension $\dim(G(2, \mathbb{P}^n)) + 5 = (n+1)2 + n - 3$. By the formula (5) in Remark 2.4, one can calculate $h^0(N_{C/\mathbb{P}^n}) = (n+1)2 + n - 3$.

(b) Assume the condition (2.i). Let $C \in \mathcal{R}$ and let $f : \mathbb{P}^1 \to \mathbb{P}^n$ be a morphism parametrizing C. Since $f^*(\mathcal{O}_C(d)) = \mathcal{O}_{\mathbb{P}^1}(de)$, the k-linear space $H^0(C, \mathcal{O}_C(d))$ is isomorphic to $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(de))$, which is of dimension $de+1$. Since $d \geqslant \max\{(e-2), 1\}$, it follows from Remark 2.3(c-d) that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(C, \mathcal{O}_C(d))$ is surjective.

Assume the condition (2.ii). Let $C \in \mathcal{R}$, and let $L \subset \mathbb{P}^n$ be the linear plane spanned by C. From the exact sequence (4) in Remark 2.4, we have $h^0(C, \mathcal{O}_C(d)) = 2d + 1$. Since $H^1(\mathcal{O}_L(d-2)) = 0$, the k-linear map $H^0(L, \mathcal{O}_L(d)) \to H^0(C, \mathcal{O}_C(d))$ is surjective, and hence so is $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(C, \mathcal{O}_C(d))$.

(c) From (b), the function $h^0(C, \mathcal{O}_C(d))$ is constant for any $C \in \mathcal{R}$. Thus, from [**32**], III, Corollary 12.9, we have $u_*(F^*(\mathcal{O}_{\mathbb{P}^n}(d))) \otimes k_C \simeq H^0(C, \mathcal{O}_C(d))$. For any $C \in \mathcal{R}$, since $\ker(\Phi) \otimes k_C$ is isomorphic to the kernel of the morphism

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \otimes k_C \to u_*(F^*(\mathcal{O}_{\mathbb{P}^n}(d))) \otimes k_C \simeq H^0(C, \mathcal{O}_C(d)), \tag{6}$$

we have $\ker(\Phi) \otimes k_C \simeq H^0(\mathbb{P}^n, \mathcal{I}_C(d))$. From (b) again, the k-linear map (6) is surjective. Hence $\dim_k \ker(\Phi) \otimes k_C = h^0(\mathbb{P}^n, \mathcal{O}(d)) - (de + 1)$.

Note that, from (a) and (c), it follows that $\dim I = \dim \mathcal{R} + \mathrm{rk}(\ker \Phi) - 1$ is equal to $\dim H + \mu$. In addition, since $\mathcal{R}$ is smooth and irreducible, so is I. □

Definition 2.5. Assume one of the conditions (i-ii) of (2). Then we define the *incidence variety* I as the projective bundle $\mathbb{P}_*(\ker \Phi)$ over $\mathcal{R}$, as in Lemma 2.1. Since $\ker \Phi$ is a subbundle of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \otimes_k \mathcal{O}_{\mathcal{R}}$, it follows $I \subset H \times \mathcal{R}$, where $H = |\mathcal{O}_{\mathbb{P}^n}(d)|$. Hence we have *projections*,

$$\begin{array}{ccc} I & \xrightarrow{p_H} & H. \\ \downarrow{\scriptstyle p_{\mathcal{R}}} & & \\ \mathcal{R} & & \end{array}$$

For each $C \in \mathcal{R}$, we set I_C to be the fiber $p_{\mathcal{R}}^{-1}(C)$, which is isomorphic to the set of hypersurfaces $X \in H$ containing C, via the projection p_H. This is because $\ker(\Phi) \otimes k_C$ is isomorphic to $H^0(\mathbb{P}^n, \mathcal{I}_C(d))$ due to Lemma 2.1(c). Therefore the variety I is described as the formula (3).

Under the condition (2.i), since $\mathcal{R}$ is the space of smooth rational curves of degree e in $\mathbb{P}^n$, and since the formula (3) holds, the fiber $p_H^{-1}(X)$ is isomorphic to $R_e(X)$ for each hypersurface $X \in H$. Similarly, under the condition (2.ii), since $\mathcal{R} = \mathrm{Hilb}^{2t+1}(\mathbb{P}^n/k)$, the fiber $p_H^{-1}(X)$ is isomorphic to $\mathrm{Hilb}^{2t+1}(X/k)$.

Here we prove the emptiness of $R_e(X)$ for $\mu < 0$, stated in Theorem I(a).

Proof of Theorem I(a). The case $e = 1$ is nothing but Theorem A(a). Thus we consider the case (2.i) with $\mu < 0$. From Lemma 2.1, we have $\dim I = \dim H + \mu < \dim H$; hence the subset $p_H(I)$ is not dense in H. Since $R_e(X) \simeq p_H^{-1}(X) = \emptyset$ for all $X \in H \setminus p_H(I)$, the statement follows. □

Remark 2.6. Under the condition (2.i), the incidence variety I is smooth as in Lemma 2.1. In characteristic zero, applying the generic smoothness theorem to the morphism $p_H : I \to p_H(I)$, we find that the scheme $R_e(X) \simeq p_H^{-1}(X)$ is smooth of dimension $\mu + \dim H - \dim(p_H(I))$ for general $X \in p_H(I)$, and moreover for general $X \in H$ if $p_H(I)$ is dense in H.

From now on, we investigate the projection p_H in more detail.

Lemma 2.7. *Assume one of the conditions* (i-ii) *of* (2), *and let* $(X, C) \in I$. *Then the following are equivalent:*

(a) *The k-linear map $d_{(X,C)}p_H : T_{(X,C)}I \to T_X H$ of Zariski tangent spaces is surjective.*
(b) *The natural morphism $H^0(C, N_{C/\mathbb{P}^n}) \to H^0(C, N_{X/\mathbb{P}^n}|_C)$ is surjective.*

Proof. We have a morphism $d_{(X,C)}p_{\mathcal{R}} : T_{(X,C)}I \to T_C\mathcal{R}$, which is surjective since I is a projective bundle over $\mathcal{R}$. Here $T_C\mathcal{R}$, the Zariski tangent space of the Hilbert scheme, is isomorphic to $H^0(N_{C/\mathbb{P}^n})$. Similarly $T_X H$ is isomorphic to $H^0(X, N_{X/\mathbb{P}^n})$.

On the other hand, we have an exact sequence $0 \to N_{X/\mathbb{P}^n} \otimes \mathcal{I}_C \to N_{X/\mathbb{P}^n} \to N_{X/\mathbb{P}^n}|_C \to 0$. It follows from Lemma 2.1(b) that $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(C, \mathcal{O}(d))$ is surjective, hence so is $H^0(X, N_{X/\mathbb{P}^n}) \to H^0(C, N_{X/\mathbb{P}^n}|_C)$.

For the sheaf $\Omega^1_{I/\mathcal{R}}$ of relative differentials, considering the base change, we obtain $\Omega^1_{I/\mathcal{R}} \otimes k_{(X,C)} \simeq \Omega^1_{I_C} \otimes k_{(X,C)}$; hence $\ker(d_{(X,C)}p_{\mathcal{R}}) \simeq T_{(X,C)}I_C$, where $T_{(X,C)}I_C$ is isomorphic to $H^0(X, N_{X/\mathbb{P}^n} \otimes \mathcal{I}_C)$.

As a consequence, we have the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{(X,C)}I_C & \longrightarrow & T_{(X,C)}I & \xrightarrow{d_{(X,C)}p_{\mathcal{R}}} & T_C\mathcal{R} & \longrightarrow & 0 \\
 & & \Big\downarrow \simeq & & \Big\downarrow d_{(X,C)}p_H & & \Big\downarrow \simeq & & \\
 & & & & T_XH & & H^0(C, N_{C/\mathbb{P}^n}) & & \\
 & & & & \Big\downarrow \simeq & & \Big\downarrow & & \\
0 & \longrightarrow & H^0(X, N_{X/\mathbb{P}^n} \otimes \mathcal{I}_C) & \longrightarrow & H^0(X, N_{X/\mathbb{P}^n}) & \longrightarrow & H^0(C, N_{X/\mathbb{P}^n}|_C) & \longrightarrow & 0,
\end{array}$$

which implies the equivalence between (a) and (b). □

Definition 2.8. For a local complete intersection curve $C \subset \mathbb{P}^n$, we define

$$\delta_C = \delta_{C/\mathbb{P}^n} : H^0(\mathbb{P}^n, \mathcal{I}_C(d)) \to H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d))$$

as the k-linear map induced from the surjective morphism $\mathcal{I}_C \to N^\vee_{C/\mathbb{P}^n} = \mathcal{I}_C/\mathcal{I}_C^2$ of sheaves on $\mathbb{P}^n$, where we note that $N_{C/\mathbb{P}^n}$ is defined as $\mathcal{H}om_{\mathcal{O}_C}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ and is locally free on C.

Under the identification of $H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d))$ with $\mathrm{Hom}_{\mathcal{O}_C}(N_{C/\mathbb{P}^n}, \mathcal{O}_C(d))$, each polynomial $h \in H^0(\mathbb{P}^n, \mathcal{I}_C(d))$ gives a morphism of sheaves on C,

$$\delta_C(h) : N_{C/\mathbb{P}^n} \to \mathcal{O}_C(d).$$

Remark 2.9. Let $X \subset \mathbb{P}^n$ be the hypersurface defined by h. Then the morphism $\delta_C(h)$ factors through the natural morphism $N_{C/\mathbb{P}^n} \to N_{X/\mathbb{P}^n}|_C$ of normal bundles. The reason is as follows: Multiplication with h yields an isomorphism $\mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{I}_X$ on $\mathbb{P}^n$. Restricting this to X, we have an isomorphism of sheaves on X,

$$\mathcal{O}_X(-d) \to \mathcal{I}_X/\mathcal{I}_X^2. \tag{7}$$

On the other hand, the inclusion $\mathcal{I}_X \hookrightarrow \mathcal{I}_C$ induces the following morphism,

$$\mathcal{I}_X/\mathcal{I}_X^2 \otimes \mathcal{O}_C \to \mathcal{I}_C/\mathcal{I}_C^2. \tag{8}$$

The dual morphism $\delta_C(h)^\vee : \mathcal{O}_C(-d) \to \mathcal{I}_C/\mathcal{I}_C^2$ is given by

$$(f \mod \mathcal{I}_C) \mapsto (h \cdot f \mod \mathcal{I}_C^2)$$

for a section $f \in H^0(U, \mathcal{O}_{\mathbb{P}^n}(-d))$ with an open subset $U \subset \mathbb{P}^n$. Hence $\delta_C(h)^\vee$ factors into the restriction of (7) to C, followed by the morphism (8):

$$\begin{array}{ccc}
\mathcal{O}_C(-d) & \xrightarrow{\delta_C(h)^\vee} & \mathcal{I}_C/\mathcal{I}_C^2. \\
\Big\downarrow \simeq & \nearrow & \\
\mathcal{I}_X/\mathcal{I}_X^2 \otimes \mathcal{O}_C & &
\end{array}$$

By considering the dual of this diagram, we have the assertion.

From Lemma 2.7 and Remark 2.9, we have a criterion for $d_{(X,C)}p_H$ to be surjective.

Proposition 2.10. *Assume one of the conditions* (i-ii) *of* (2). *Let* $(X, C) \in I$, *and let* $h \in H^0(\mathbb{P}^n, \mathcal{O}(d))$ *be a defining equation of* X. *Then* $d_{(X,C)}p_H : T_{(X,C)}I \to T_X H$ *is surjective if and only if*

$$H^0(\delta_C(h)) : H^0(C, N_{C/\mathbb{P}^n}) \to H^0(C, \mathcal{O}_C(d))$$

is surjective.

We additionally have the following lemma:

Lemma 2.11. *Let* $C \subset \mathbb{P}^n$ *be a local complete intersection curve, and let* $X \in H$ *be a hypersurface containing* C *and defined by* $h \in H^0(\mathbb{P}^n, \mathcal{O}(d))$. *Then* X *is singular at a point* $P \in C$ *if the* k*-linear map* $\delta_C(h)(P) : N_{C/\mathbb{P}^n} \otimes k(P) \to \mathcal{O}_C(d) \otimes k(P)$ *is identically zero.*

Proof. The assumption implies that h is equal to zero as an element of $N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d) \otimes k(P)$. By taking an element $h_0 \in H^0(\mathbb{P}^n, \mathcal{O}(d))$ with $h_0(P) \neq 0$, we have that h/h_0 is equal to zero in $N^\vee_{C/\mathbb{P}^n} \otimes k(P)$. For the maximal ideal $m_P \subset \mathcal{O}_{\mathbb{P}^n,P}$, since $\mathcal{I}_{C,P} \subset m_P$ and since $N^\vee_{C/\mathbb{P}^n} \otimes k(P) \simeq \mathcal{I}_{C,P}/\mathcal{I}_{C,P} \cdot m_P$, it follows that $(h/h_0 \mod m_P^2) = 0$ in m_P/m_P^2, which means that X is singular at P. □

In the following, we study the surjectivity of the k-linear map δ_C. For a linear subspace $L \subset \mathbb{P}^n$ containing C, we set $\delta_{C/L} : H^0(L, \mathcal{I}_{C/L}(d)) \to H^0(C, N^\vee_{C/L} \otimes \mathcal{O}_C(d))$ to be the k-linear map induced from $\mathcal{I}_{C/L} \to N^\vee_{C/L} = \mathcal{I}_{C/L}/\mathcal{I}^2_{C/L}$, where $\mathcal{I}_{C/L} \subset \mathcal{O}_L$ is the ideal sheaf of C in L. Then the surjectivity of δ_C is reduced to that of $\delta_{C/L}$, as follows:

Lemma 2.12. *Let* $C \subset \mathbb{P}^n$ *be a local complete intersection curve, let* $L \subset \mathbb{P}^n$ *be a linear subspace containing* C, *and let* $d \geqslant 1$. *Suppose that the restriction map* $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)) \to H^0(C, \mathcal{O}_C(d-1))$ *is surjective. Then* δ_C *is surjective if* $\delta_{C/L}$ *is so.*

Proof. From the exact sequence $0 \to \mathcal{I}_L \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_L \to 0$, we have that $H^1(\mathbb{P}^n, \mathcal{I}_L(d)) = 0$. From the exact sequence $0 \to N_{C/L} \to N_{C/\mathbb{P}^n} \to N_{L/\mathbb{P}^n}|_C \to 0$ on C, we have the following commutative diagram of k-linear spaces with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{I}_L(d)) & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{I}_C(d)) & \longrightarrow & H^0(L, \mathcal{I}_{C/L}(d)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \delta_C} & & \downarrow{\scriptstyle \delta_{C/L}} & & \\
0 & \longrightarrow & H^0(C, N^\vee_{L/\mathbb{P}^n} \otimes \mathcal{O}_C(d)) & \longrightarrow & H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d)) & \longrightarrow & H^0(C, N^\vee_{C/L} \otimes \mathcal{O}_C(d)) & &
\end{array}$$

By assumption, the k-linear map $\delta_{C/L}$ of the third column is surjective. Thus, in order to prove that δ_C is surjective, it is sufficient to show that ε is surjective. By choosing coordinates $(z_0, \dots, z_n)$ on $\mathbb{P}^n$, we may assume that L is the zero set of polynomials $z_{r+1}, \dots, z_n$, where we set $r := \dim L$. Then $N^\vee_{L/\mathbb{P}^n} = \mathcal{I}_L/\mathcal{I}^2_L = \bigoplus_{i=r+1}^n \mathcal{O}_L(-1) \cdot \bar{z}_i$.

Let $g \in H^0(C, N^\vee_{L/\mathbb{P}^n} \otimes \mathcal{O}_C(d))$. Then we have $g = g_{r+1} \cdot \bar{z}_{r+1} + \dots + g_n \cdot \bar{z}_n$ with $g_i \in H^0(C, \mathcal{O}_C(d-1))$. By assumption, there exist sections $f_{r+1}, \dots, f_n \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1))$ such that $g_i = f_i|_C$. We set

$$f := f_{r+1} \cdot z_{r+1} + \dots + f_n \cdot z_n \in H^0(\mathbb{P}^n, \mathcal{I}_L(d)).$$

Since $\varepsilon(f) = (\partial f/\partial z_{r+1})|_C \cdot \bar{z}_{r+1} + \cdots + (\partial f/\partial z_n)|_C \cdot \bar{z}_n$ and since $(\partial f/\partial z_i)|_C = g_i$ for $r+1 \leqslant i \leqslant n$, we obtain $\varepsilon(f) = g$. □

Proposition 2.13. *Suppose that $C \subset \mathbb{P}^n$ is a smooth rational curve of degree $e \geqslant 2$. Then δ_C is surjective if $d \geqslant 2(e-r)+3$, where the integer r with $r \leqslant e$ is the dimension of the linear subspace of $\mathbb{P}^n$ spanned by C.*

Proof. Let $L \subset \mathbb{P}^n$ be the r-dimensional linear subspace spanned by C. Then, since $d-1 \geqslant 2(e-r)+2 \geqslant e+1-r$, it follows from Remark 2.3(d) that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)) \to H^0(C, \mathcal{O}_C(d-1))$ is surjective.

In addition, Theorem 1.1 implies that the second power $\mathcal{I}^2_{C/L}$ is $2(e+2-r)$-regular. Since $d+1 \geqslant 2(e+2-r)$, we have $H^1(L, \mathcal{I}^2_{C/L}(d)) = 0$. From the exact sequence $0 \to \mathcal{I}^2_{C/L} \to \mathcal{I}_{C/L} \to N^\vee_{C/L} \to 0$ on L, the k-linear map $\delta_{C/L}$ is surjective. Hence Lemma 2.12 implies that δ_C is surjective. □

2.2. Bounds of the splitting type of the normal bundle of a rational curve. Let $C \subset \mathbb{P}^n$ be a rational curve of degree $e \geqslant 2$ parametrized by a morphism,

$$f : \mathbb{P}^1 \to \mathbb{P}^n : (s,t) \mapsto (f_0(s,t), f_1(s,t), \ldots, f_n(s,t)). \tag{9}$$

It is known that every vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles on $\mathbb{P}^1$. We study such a splitting type of the pullback $f^*N_{C/\mathbb{P}^n}$ on $\mathbb{P}^1$.

For a smooth variety Y and for an invertible sheaf $\mathcal{L}$ on Y, we denote by $\mathcal{P}^1_Y(\mathcal{L})$ the bundle of principal parts of $\mathcal{L}$ of first order, which gives an exact sequence

$$0 \to \Omega^1_Y \otimes \mathcal{L} \to \mathcal{P}^1_Y(\mathcal{L}) \to \mathcal{L} \to 0. \tag{$\xi_{Y,\mathcal{L}}$}$$

Note that in the case $Y = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$, it follows $\mathcal{P}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1)) \simeq H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{P}^n}$ and the sequence $(\xi_{\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)})$ gives the Euler sequence.

As in [**38**], for the morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$, we set $\mathcal{P}^1 := \mathcal{P}^1_{\mathbb{P}^1}(f^*(\mathcal{O}_{\mathbb{P}^n}(1)))$. Then the following commutative diagram with exact rows is induced functorially:

([**38**], (1.1))

$$\begin{array}{cccccccc}
(f^*\xi_{\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)}) & 0 \to & f^*(\Omega^1_{\mathbb{P}^n}) \otimes f^*(\mathcal{O}_{\mathbb{P}^n}(1)) & \to & H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes_k \mathcal{O}_{\mathbb{P}^1} & \to & f^*(\mathcal{O}_{\mathbb{P}^n}(1)) & \to 0 \\
 & & \downarrow & & \downarrow a^1_f & & \| & \\
(\xi_{\mathbb{P}^1, f^*(\mathcal{O}_{\mathbb{P}^n}(1))}) & 0 \to & \Omega^1_{\mathbb{P}^1} \otimes f^*(\mathcal{O}_{\mathbb{P}^n}(1)) & \to & \mathcal{P}^1 & \to & f^*(\mathcal{O}_{\mathbb{P}^n}(1)) & \to 0,
\end{array}$$

where we denote by a^1_f the morphism of the second column. In the case where f is unramified, since $f^*(\Omega^1_{\mathbb{P}^n}) \to \Omega^1_{\mathbb{P}^1}$ is surjective and its kernel is isomorphic to $f^*N^\vee_{C/\mathbb{P}^n}$, the above diagram ([**38**], (1.1)) induces the following exact sequence of sheaves on $\mathbb{P}^1$:

$$0 \to f^*N^\vee_{C/\mathbb{P}^n} \otimes f^*(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes_k \mathcal{O}_{\mathbb{P}^1} \xrightarrow{a^1_f} \mathcal{P}^1 \to 0. \tag{10}$$

Proposition 2.14. *Let $C \subset \mathbb{P}^n$ be a non-degenerate smooth rational curve of degree $e \geqslant 2$, let f parametrize C as in (9), and let $f^*N_{C/\mathbb{P}^n} \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}(a_i)$ be the splitting on $\mathbb{P}^1$ with $a_i \in \mathbb{Z}$. Then we have inequality $e+2 \leqslant a_i \leqslant 3e-2n+2$ for each $1 \leqslant i \leqslant n-1$.*

PROOF. Let $(z_0, z_1, \dots, z_n)$ be homogeneous coordinates on $\mathbb{P}^n$, and let $p \geqslant 0$ be the characteristic of the ground field k. From [**38**], Lemma (1.2), we have the following description of the bundle $\mathcal{P}^1$,

$$\begin{cases} \mathcal{P}^1 \simeq \mathcal{O}(e-1) \oplus \mathcal{O}(e-1) \ \text{ and } \ a^1_f(z_i) = (\partial f_i/\partial s, \partial f_i/\partial t) & \text{if } p = 0 \text{ or } p \nmid e, \\ \mathcal{P}^1 \simeq \mathcal{O}(e) \oplus \mathcal{O}(e-2) \ \text{ and } \ a^1_f(z_i) = (f_i, t^{-1}\partial f_i/\partial s) & \text{if } p \mid e. \end{cases}$$

Here, note that the description of $a^1_f(z_i)$ depends on the choice of an isomorphism from $\mathcal{P}^1$ to its direct sum decomposition, and note that equality $t^{-1}\partial f_i/\partial s = -s^{-1}\partial f_i/\partial t$ holds in the case $p \mid e$ ([**38**], Remark 1.4).

From the exact sequence (10), it follows $H^0(\mathbb{P}^1, f^*N^\vee_{C/\mathbb{P}^n} \otimes f^*(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, which implies $e - a_i - 1 \leqslant -1$ for each i; hence $\min\{ a_i \}_i \geqslant e$.

Suppose $\min\{ a_i \}_i = e$. Then there exists a nonzero element $\xi = \sum_{i=0}^n \alpha_i z_i \in H^0(\mathbb{P}^n, \mathcal{O}(1))$ with $\alpha_i \in k$ such that $a^1_f(\xi) = 0$ in $H^0(\mathbb{P}^1, \mathcal{P}^1)$. In the case $p \nmid e$, we have

$$\sum_{i=0}^n \alpha_i \partial f_i/\partial s = \sum_{i=0}^n \alpha_i \partial f_i/\partial t = 0.$$

From Euler's formula, we get $\sum_{i=0}^n \alpha_i f_i = 0$, which contradicts that C is non-degenerate. In the case $p \mid e$, we straightforwardly have $\sum_{i=0}^n \alpha_i f_i = 0$, a contradiction.

Suppose $\min\{ a_i \}_i = e + 1$. Then there exists a nonzero element

$$\xi = \sum_{i=0}^n (\alpha_i s + \beta_i t) z_i \in H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(1))$$

with $\alpha_i, \beta_i \in k$ such that $a^1_f(\xi) = 0$ in $H^0(\mathbb{P}^1, \mathcal{P}^1(1))$. Here, since ξ is nonzero, at least one of the elements $\{ \alpha_i, \beta_i \}_{i=0}^n$ must be nonzero. Without loss of generality, we can assume $\alpha_0 \neq 0$. In the case $p \nmid e$, it follows that

$$\sum_{i=0}^n (\alpha_i s + \beta_i t) \partial f_i/\partial s = \sum_{i=0}^n (\alpha_i s + \beta_i t) \partial f_i/\partial t = 0.$$

From Euler's formula, we get $F = \sum_{i=0}^n (\alpha_i s + \beta_i t) f_i = 0$. This implies $\sum_{i=0}^n \alpha_i f_i = \sum_{i=0}^n \alpha_i f_i + \sum_{i=0}^n (\alpha_i s + \beta_i t) \partial f_i/\partial s = \partial F/\partial s = 0$. Since $\alpha_0 \neq 0$, it follows that C is degenerate, a contradiction. In the case $p \mid e$, it follows that $F = \sum_{i=0}^n (\alpha_i s + \beta_i t) f_i = 0$ and $\sum_{i=0}^n (\alpha_i s + \beta_i t) t^{-1} \partial f_i/\partial s = 0$. This implies $\sum_{i=0}^n \alpha_i f_i = \partial F/\partial s = 0$. Thus we have a contradiction again.

In consequence, we get the inequality $\min\{ a_i \}_i \geqslant e + 2$. Thus, putting $a_1 = \max\{ a_i \}_i$, we have $a_1 = \sum_{i=1}^{n-1} a_i - \sum_{i=2}^{n-1} a_i \leqslant ((n+1)e - 2) - (n-2)(e+2) = 3e - 2n + 2$ □

2.3. Projection dominating the space of hypersurfaces.

Proposition 2.15. *Assume the condition* (2.i) *with $n \geqslant 3$, and assume either $e \leqslant n$ and $d \geqslant 3$, or $e > n$ and $d \geqslant 2(e-n)+3$. If $\mu \geqslant 0$, then the projection p_H is dominant on H, and is smooth at a general point of I.*

Proof. We can take a smooth rational curve $C \subset \mathbb{P}^n$ of degree e such that the linear subspace L of $\mathbb{P}^n$ spanned by C is of dimension $r := \min\{e, n\}$. The reason is as follows: Suppose $e \leqslant n$. Then we set $C \subset \mathbb{P}^e \subset \mathbb{P}^n$ to be a rational normal curve of degree e (i.e., the rational curve defined by the morphism $\mathbb{P}^1 \to \mathbb{P}^e : (s,t) \mapsto (s^e, \ldots, s^{e-i}t^i, \ldots, t^e)$). Suppose $e > n$. Then we first take $C' \subset \mathbb{P}^e$ to be a rational normal curve of degree e. Since $n \geqslant 3$ and since the secant variety of C is of dimension $\leqslant 3$, there exists a linear projection $\pi : \mathbb{P}^e \dashrightarrow \mathbb{P}^n$ which gives an isomorphism from C' to its image. Then we set $C = \pi(C') \subset \mathbb{P}^n$, which is a non-degenerate smooth rational curve of degree e.

Now, we give a morphism $\alpha : N_{C/\mathbb{P}^n} \to \mathcal{O}_C(d)$ such that $H^0(\alpha)$ is surjective, as follows. Let $f : \mathbb{P}^1 \to \mathbb{P}^n$ be a morphism parametrizing C. Since $N_{C/\mathbb{P}^n} \simeq N_{C/L} \oplus N_{L/\mathbb{P}^n}|_C \simeq N_{C/L} \oplus \mathcal{O}_C(1)^{\oplus n-r}$, we have an isomorphism

$$f^* N_{C/\mathbb{P}^n} \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}(a_i) \oplus \mathcal{O}(e)^{\oplus n-r}$$

on $\mathbb{P}^1$ with $a_i \in \mathbb{Z}$. From Proposition 2.14, it follows $e+2 \leqslant a_i \leqslant 3e-2r+2$ for each $1 \leqslant i \leqslant r-1$; hence we particularly have $a_i \leqslant de$. We set $a_i = e$ for $r \leqslant i \leqslant n-1$, and set $m_0 := 0$, $m_i := m_{i-1} + (a_i+1) = \sum_{j=1}^{i}(a_j+1)$ for $1 \leqslant i \leqslant n-1$.

Since $m_{n-1} - (de+1) = \chi(N_{C/\mathbb{P}^n}) - (de+1) = \mu \geqslant 0$, there exists an integer $1 \leqslant i_0 \leqslant n-2$ such that $m_{i_0} < de+1$ and $m_{i_0+1} \geqslant de+1$. Let (s,t) be homogeneous coordinates on $\mathbb{P}^1$. We set

$$\xi_i := s^{de+1-m_i} t^{m_{i-1}} \in H^0(\mathbb{P}^1, \mathcal{O}(de - a_i)),$$

and set $\alpha \in H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d)) \simeq \bigoplus_{i=1}^{n-1} H^0(\mathbb{P}^1, \mathcal{O}(de-a_i))$ as

$$\alpha := (\xi_1, \xi_2, \cdots, \xi_{i_0}, t^{de-a_{i_0+1}}, *, \cdots, *).$$

For each $1 \leqslant i \leqslant i_0$, the k-linear subspace $\xi_i \cdot H^0(\mathbb{P}^1, \mathcal{O}(a_i)) \subset H^0(\mathbb{P}^1, \mathcal{O}(de))$ is spanned by the following a_i+1 monomials,

$$s^{de-m_{i-1}} t^{m_{i-1}},\ s^{de-m_{i-1}-1} t^{m_{i-1}+1},\ \cdots,\ s^{de+1-m_i} t^{m_i-1}. \tag{11}$$

By identification of $H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d))$ with $\mathrm{Hom}_{\mathcal{O}_C}(N_{C/\mathbb{P}^n}, \mathcal{O}_C(d))$, we regard α as a morphism $N_{C/\mathbb{P}^n} \to \mathcal{O}_C(d)$, which induces the following k-linear map,

$$H^0(\alpha) : H^0(C, N_{C/\mathbb{P}^n}) \simeq \bigoplus_{i=1}^{n-1} H^0(\mathbb{P}^1, \mathcal{O}(a_i)) \to H^0(C, \mathcal{O}_C(d)) \simeq H^0(\mathbb{P}^1, \mathcal{O}(de)).$$

Here all monomials of $H^0(\mathbb{P}^1, \mathcal{O}(de))$ are given by the monomials of (11) with $1 \leqslant i \leqslant i_0$ and $t^{de-a_{i_0+1}} \cdot H^0(\mathbb{P}^1, \mathcal{O}(a_{i_0+1}))$. Hence the k-linear map $H^0(\alpha)$ is surjective.

From Proposition 2.13, the k-linear map δ_C is surjective; hence we have $h \in H^0(\mathbb{P}^n, \mathcal{I}_C(d))$ such that $\alpha = \delta_C(h)$. Let $X \subset \mathbb{P}^n$ be a hypersurface defined by h. Then Proposition 2.10 implies that $d_{(X,C)} p_H$ is surjective; thus the subset $p_H(I) \subset H$ is dense, and the projection p_H is smooth at (X, C). In particular, p_H is smooth on an open neighborhood of (X, C) in I. □

Now the statement of Theorem I(b) is proved in the characteristic zero case:

Theorem 2.16. *Let the characteristic be equal to zero, and assume $n \geqslant 3$, $e \geqslant 2$, and $d \geqslant \max\{\, e-2, 2(e-n)+3, 3\,\}$. Then $R_e(X)$ is smooth and of dimension μ, for a general hypersurface $X \subset \mathbb{P}^n$ of degree d with $\mu \geqslant 0$.*

Remark 2.17. The conclusion of Theorem 2.16 also holds if $e = 1, 2$ or 3 and $d \geqslant 1$. Indeed, the case $e = 1$ follows from Theorem A(b). The case $e = 2$ or 3 with $d \geqslant 3$ follows from Theorem 2.16. The case $d = 1$ or 2 follows from the homogeneity of X.

Proof. Let $\mathcal{R} \subset \mathrm{Hilb}^{et+1}(\mathbb{P}^n/k)$ be the space of smooth rational curves of degree e in $\mathbb{P}^n$. Since the condition (2.i) holds, Proposition 2.15 implies that $p_H(I) \subset H$ is dense. Thus, by the argument of Remark 2.6, the result follows. □

3. Generic smoothness of the projection

Recall that $\mu = (n+1-d)e + n - 4$, the expected dimension defined as in Introduction, Eq. (1). To establish the generic smoothness of the projection $p_H : I \to H$ in the case $\mu \geqslant 0$ in arbitrary characteristic, we consider the *non-smooth locus Z of the projection p_H* as follows.

Definition 3.1. We define Z to be the set of $(X, C) \in I$ such that the projection p_H is *not* smooth at (X, C), i.e, the k-linear map $d_{(X,C)}p_H : T_{(X,C)}I \to T_X H$ is *not* surjective. In addition, we define I^0 to be the set of $(X, C) \in I$ such that the hypersurface X is smooth along C.

In this section, we set $\mathcal{R}$ to be the space of smooth rational curves of degree e, as in the condition (2.i). In §3.1, we will show that the subset $Z^0 := Z \cap I^0$ is sufficiently small:

Proposition 3.2. *Let $n \geqslant 3$, assume the condition* (2.i)*, and assume $d \geqslant \max\{\, 2e - 3, 4\,\}$. Then we have* $\mathrm{codim}(Z^0, I) \geqslant \mu + 1$.

Here, note that I^0 is a dense subset in I (see Corollary 3.5), and note that we consider the subset Z^0 instead of Z because we need to shrink Z in the process of proving that the codimension is "$\geqslant \mu + 1$" (Lemma 3.13).

The statement of Proposition 3.2 above does not cover the case $e \leqslant 3$ and $d \leqslant 3$. For this case, the following will be shown in §3.2:

Proposition 3.3. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d = 2$ or 3, and assume one of the following:*

(i) *X is a quadric hypersurface, and C is a smooth rational curve of degree $e = 2$ or 3 such that X is smooth along C.*

(ii) *X is a cubic hypersurface, and C is a smooth rational curve of degree $e = 2$ or 3 such that X is smooth along C and that the linear subspace of $\mathbb{P}^n$ spanned by C is not contained in X.*

(iii) *X is a general cubic hypersurface, and C is a smooth rational curve of degree $e = 2$ or 3 such that X is smooth along C.*

Then we have $H^1(N_{C/X}) = 0$.

At the end of the section, the generic smoothness of p_H and Theorem I(b) will be proved by Propositions 3.2 and 3.3.

3.1. Codimension of the non-smooth locus of the projection. For $C \in \mathcal{R}$, we set $I_C := p_{\mathcal{R}}^{-1}(C)$ as in Definition 2.5. Then the subset $I_C \setminus I^0$ is isomorphic to the set of hypersurfaces $X \in H$ containing C and being singular at some point of C, via the projection p_H. Here, under a general setting, we have:

Proposition 3.4. *Let $C \subset \mathbb{P}^n$ be a smooth curve, let $d \geqslant 2$ satisfy that $\mathcal{I}_C(d)$ is generated by its global sections, and denote by H_C the set of hypersurfaces $X \subset \mathbb{P}^n$ of degree d containing C. Then the set of $X \in H_C$ being singular at some point of C is of codimension $\geqslant n-2$ in H_C.*

PROOF. We denote by $S_C^P := \{\, X \in H_C \mid X \text{ is singular at } P \,\}$ for a point $P \in C$. Then the set of $X \in H_C$ being singular at some point of C is given by the union of S_C^P with $P \in C$. Thus it suffices to show that $\operatorname{codim}(S_C^P, H_C) = n-1$ for each $P \in C$.

As in the proof of Bertini's theorem [**32**], II, Theorem 8.18, we have a k-linear map

$$\varphi_P : H^0(\mathbb{P}^n, \mathcal{O}(d)) \to \mathcal{O}_{C,P}/m_P^2 : h \mapsto (h/h_0 \mod m_P^2),$$

where $m_P \subset \mathcal{O}_{\mathbb{P}^n,P}$ is the maximal ideal, and h_0 is a polynomial of degree d satisfying $h_0(P) \neq 0$. Here, for a hypersurface $X \subset \mathbb{P}^n$ containing P, and for a defining polynomial h of X, it follows that $\varphi_P(h) = 0$ in m_P/m_P^2 if and only if X is singular at P. In particular, we have

$$h \in \ker(\varphi_P) \cap H^0(\mathbb{P}^n, \mathcal{I}_C(d)) \text{ if and only if } X \in S_C^P. \tag{12}$$

By assumption, the $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{I}_{C,P}$ is generated by global sections $g_1, g_2, \ldots, g_m \in H^0(\mathbb{P}^n, \mathcal{I}_C(d))$. Hence the k-linear space $\mathcal{I}_{C,P}/\mathcal{I}_{C,P} \cap m_P^2$ is generated by the elements $\varphi_P(g_1), \varphi_P(g_2), \ldots, \varphi_P(g_m)$, which implies that

$$\varphi_P(H^0(\mathbb{P}^n, \mathcal{I}_C(d))) = \mathcal{I}_{C,P}/\mathcal{I}_{C,P} \cap m_P^2.$$

Thus the dimension of the k-linear space $\mathcal{I}_{C,P}/\mathcal{I}_{C,P} \cap m_P^2$ and the codimension of the k-linear space $\ker(\varphi_P) \cap H^0(\mathbb{P}^n, \mathcal{I}_C(d))$ in $H^0(\mathbb{P}^n, \mathcal{I}_C(d))$ are the same, and are equal to $\operatorname{codim}(S_C^P, H_C)$ due to the equivalence (12).

We consider the following exact sequence,

$$0 \to \mathcal{I}_{C,P}/\mathcal{I}_{C,P} \cap m_P^2 \to m_P/m_P^2 \to \bar{m}_P/\bar{m}_P^2 \to 0,$$

where $\bar{m}_P$ is the maximal ideal of $\mathcal{O}_{C,P}$. Since P is a smooth point of $\mathbb{P}^n$, we have $\dim_k m_P/m_P^2 = n$. Since P is a smooth point of C, we have $\dim_k \bar{m}_P/\bar{m}_P^2 = 1$. Hence $\dim_k(\mathcal{I}_{C,P}/\mathcal{I}_{C,P} \cap m_P^2)$ is equal to $n-1$, and so is $\operatorname{codim}(S_C^P, H_C)$. □

Corollary 3.5. *Assume the condition* (2.i) *and assume $d \geqslant \max\{\,(e-1), 2\,\}$. Then we have* $\operatorname{codim}(I \setminus I^0, I) \geqslant n-2$. *In particular, the subset $I^0 \subset I$ is dense if $n \geqslant 3$.*

PROOF. For any $C \in \mathcal{R}$, since $\mathcal{I}_{C/L}$ is $(\max\{\,(e-1), 2\,\})$-regular as in Remark 2.3(c-d), it follows that $\mathcal{I}_{C/L}(d)$ is generated by its global sections, and hence so is $\mathcal{I}_C(d)$. Thus Proposition 3.4 implies that $\operatorname{codim}(I_C \setminus I^0, I_C) \geqslant n-2$. Since C is arbitrary, we obtain $\operatorname{codim}(I \setminus I^0, I) \geqslant n-2$. □

Next, we fix a smooth rational curve $C \subset \mathbb{P}^n$ of degree e. Let $\delta_C(h) : N_{C/\mathbb{P}^n} \to \mathcal{O}_C(d)$ be the morphism defined in Definition 2.8.

For $(X, C) \in I$ and for a defining equation h of X, it follows from Proposition 2.10 that $(X, C) \in Z$ if and only if $H^0(\delta_C(h))$ is not surjective. Here we remark that the k-linear map $H^0(\delta_C(h))$ is not surjective if and only if $\psi(\mathrm{im}(H^0(\delta_C(h)))) = 0$ for some nonzero linear functional $\psi : H^0(C, \mathcal{O}_C(d)) \to k$. We denote by σ_ψ the composite k-linear map,

$$\begin{aligned} H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d)) &\simeq \mathrm{Hom}_{\mathcal{O}_C}(N_{C/\mathbb{P}^n}, \mathcal{O}_C(d)) \\ &\xrightarrow{H^0} \mathrm{Hom}_k(H^0(C, N_{C/\mathbb{P}^n}), H^0(C, \mathcal{O}_C(d))) \\ &\xrightarrow{\psi\circ -} \mathrm{Hom}_k(H^0(C, N_{C/\mathbb{P}^n}), k), \end{aligned}$$

By the above argument, the following lemma holds:

Lemma 3.6. *Assume the condition* (2.i)*. Let $(X, C) \in I$, and let $h \in H^0(\mathbb{P}^n, \mathcal{I}_C(d))$ be a defining equation of X. Then $(X, C) \in Z$ if and only if $\sigma_\psi(\delta_C(h)) = 0$ for some nonzero linear functional $\psi : H^0(C, \mathcal{O}_C(d)) \to k$.*

Let $Z_C := Z \cap I_C$, which is isomorphic to the set of hypersurfaces $X \in H$ containing C such that p_H is not smooth at (X, C). If $\delta_C : H^0(\mathbb{P}^n, \mathcal{I}_C(d)) \to H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d))$ is surjective (as in the conclusion of Proposition 2.13), then Lemma 3.6 implies that the codimension of Z_C in I_C is equal to the codimension of the union,

$$\bigcup_{\psi \in \mathrm{Hom}_k(H^0(C, \mathcal{O}_C(d)), k)} \ker(\sigma_\psi) \quad \text{in} \quad H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d)). \tag{13}$$

In order to study these codimensions, we investigate σ_ψ and its kernel.

We take a morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ which parametrizes the smooth rational curve C of degree e. Then we have a splitting $f^* N_{C/\mathbb{P}^n} \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(a_i)$ on $\mathbb{P}^1$ with $a_i \in \mathbb{Z}$. For $\psi : H^0(\mathbb{P}^1, f^*(\mathcal{O}(d))) \to k$, the k-linear map σ_ψ is identified with the direct sum of k-linear maps,

$$\oplus_{i=1}^{n-1} \sigma_\psi^{a_i} : \bigoplus_{i=1}^{n-1} H^0(\mathbb{P}^1, \mathcal{O}(de - a_i)) \to \bigoplus_{i=1}^{n-1} \mathrm{Hom}_k(H^0(\mathbb{P}^1, \mathcal{O}(a_i)), k), \tag{14}$$

where $\sigma_\psi^{a_i}$ is defined as follows.

Definition 3.7. Let $\varepsilon \geqslant 0$ be an integer, and let $\psi : H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon)) \to k$ be a linear functional. For an integer α with $0 \leqslant \alpha \leqslant \varepsilon$, we denote by σ_ψ^α the composite k-linear map

$$\begin{aligned} H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon - \alpha)) &\to \mathrm{Hom}_k(H^0(\mathbb{P}^1, \mathcal{O}(\alpha)), H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon))) \\ &\xrightarrow{\psi\circ -} \mathrm{Hom}_k(H^0(\mathbb{P}^1, \mathcal{O}(\alpha)), k), \end{aligned}$$

where the first transformation is the adjoint of the usual multiplication.

Now, we investigate σ_ψ^α in detail. Let (s, t) be homogeneous coordinates on $\mathbb{P}^1$. Then $H^0(\mathbb{P}^1, \mathcal{O}(\alpha))$ has a standard ordered basis

$$(s^\alpha, \dots, s^{\alpha - i} t^i, \dots, t^\alpha)$$

for every integer α. Let $\psi : H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon)) \to k$ be a linear functional. Then the matrix of ψ with respect to the standard ordered basis for $H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon))$ is denoted by

$$M^0_\psi = \begin{bmatrix} c_0 & \cdots & c_i & \cdots & c_\varepsilon \end{bmatrix}.$$

Lemma 3.8. *Let ε, α be integers with $1 \leqslant \alpha \leqslant \varepsilon - 1$, and let $\psi : H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon)) \to k$. Then the matrix of σ^α_ψ with respect to the standard ordered basis for $H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon - \alpha))$ and the dual standard ordered basis $((s^\alpha)^\vee, \dots, (s^{\alpha-i}t^i)^\vee, \dots, (t^\alpha)^\vee)$ for $H^0(\mathbb{P}^1, \mathcal{O}(\alpha))^\vee$ is equal to the $(\alpha+1) \times (\varepsilon - \alpha + 1)$ catalecticant matrix,*

$$M^\alpha_\psi = \begin{bmatrix} c_0 & c_1 & \cdots & c_{\varepsilon-\alpha} \\ c_1 & c_2 & \cdots & c_{\varepsilon-\alpha+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_\alpha & c_{\alpha+1} & \cdots & c_\varepsilon \end{bmatrix}.$$

Proof. The k-linear map $H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon-\alpha)) \to H^0(\mathbb{P}^1, \mathcal{O}(\alpha))^\vee \otimes H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon))$ sends each $s^{\varepsilon-\alpha-j}t^j$ to $\sum_{i=0}^{\alpha}(s^{\alpha-i}t^i)^\vee \otimes (s^{\varepsilon-\alpha-j}t^j)(s^{\alpha-i}t^i) = \sum_{i=0}^{\alpha}(s^{\alpha-i}t^i)^\vee \otimes s^{\varepsilon-(i+j)}t^{i+j}$. Thus we have

$$\sigma^\alpha_\psi(s^{\varepsilon-\alpha-j}t^j) = \sum_{i=0}^{\alpha}(s^{\alpha-i}t^i)^\vee \otimes \psi(s^{\varepsilon-(i+j)}t^{i+j}) = \sum_{i=0}^{\alpha} c_{i+j}((s^{\alpha-i}t^i)^\vee \otimes 1)$$

for $0 \leqslant j \leqslant \varepsilon - \alpha$. Hence σ^α_ψ is represented by the matrix M^α_ψ. □

Let $G := \mathbb{P}_* \operatorname{Hom}_k(H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon)), k) \simeq \mathbb{P}^\varepsilon$, where we regard $M^0_\psi = [c_0 \ \cdots \ c_i \ \cdots \ c_\varepsilon]$ as homogeneous coordinates of $\psi = \sum c_i (s^{\varepsilon-i}t^i)^\vee$ on G. We set $G^\alpha_\rho := \{ \bar\psi \in G \mid \operatorname{rk} \sigma^\alpha_\psi \leqslant \rho \}$ for integers ρ, α with $\rho \leqslant \min\{ \alpha, \varepsilon - \alpha \} + 1$.

Lemma 3.9. *Let ε, α be integers with $1 \leqslant \alpha \leqslant \varepsilon - 1$, Then the following holds.*

(a) *We have that G^α_1 coincides with the rational normal curve in G parametrized by the morphism,*

$$\mathbb{P}^1 \to G : (a, b) \mapsto \begin{bmatrix} a^\varepsilon & \cdots & a^{\varepsilon-i}b^i & \cdots & b^\varepsilon \end{bmatrix} = \sum_{i=0}^{\varepsilon} a^{\varepsilon-i}b^i (s^{\varepsilon-i}t^i)^\vee.$$

Moreover G^α_1 coincides with the locus

$$\{ \bar\psi \in G \mid \ker(\psi) = H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon)(-P)) \text{ for some } P \in \mathbb{P}^1 \},$$

where we remark that the definition of this locus does not depend on the integer α and the choice of coordinates on $\mathbb{P}^1$.

(b) *Let ρ be an integer with $\rho \leqslant \min\{ \alpha, \varepsilon - \alpha \}$. Then G^α_ρ coincides with the ρ-secant variety $S_{\rho-1}(G^\alpha_1)$ of the rational normal curve G^α_1 in G. Hence we have $\dim G^\alpha_\rho = 2\rho - 1$.*

(c) *Let ρ be an integer with $\rho \leqslant \min\{ \alpha, \varepsilon - \alpha \} + 1$, and let $\psi = r_1\psi_1 + \cdots + r_\rho\psi_\rho$ be a linear functional with $r_1, \dots, r_\rho \in k$ and $\bar\psi_1, \dots, \bar\psi_\rho \in G^\alpha_1$. If $\operatorname{rk} M^\alpha_\psi = \rho$, then we have $\ker \sigma^\alpha_\psi \subset \ker \sigma^\alpha_{\psi_1} \cap \cdots \cap \ker \sigma^\alpha_{\psi_\rho}$.*

PROOF. (a) We show that G_1^α is contained in the rational normal curve in G, as follows: We take $\bar{\psi} \in G_1^\alpha$ with $M_\psi^0 = \begin{bmatrix} c_0 & \cdots & c_i & \cdots & c_\varepsilon \end{bmatrix}$. Then we set

$$v_i := \begin{bmatrix} c_i & c_{i+1} & \cdots & c_{\varepsilon-\alpha+i} \end{bmatrix},$$

the i-th row vector of M_ψ^α for $0 \leqslant i \leqslant \alpha$. Since $\operatorname{rk} M_\psi^\alpha = 1$, any two vectors v_i and v_j are linearly dependent. First, we consider the case $c_0 \neq 0$. Then we have $v_1 = \lambda v_0$ with $\lambda \in k$, which implies

$$c_1 = \lambda c_0,\ c_2 = \lambda c_1 = \lambda^2 c_0,\ \ldots,\ c_{\varepsilon-\alpha} = \lambda c_{\varepsilon-\alpha-1} = \lambda^{\varepsilon-\alpha} c_0.$$

Since $c_2 = \lambda c_1$, we also have $v_2 = \lambda v_1$; hence $c_{\varepsilon-\alpha+1} = \lambda c_{\varepsilon-\alpha} = \lambda^{\varepsilon-\alpha+1} c_0$. Similarly, we have $c_i = \lambda^i c_0$ for all $0 \leqslant i \leqslant \varepsilon$. Thus $M_\psi^0 = c_0 \cdot \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^\varepsilon \end{bmatrix}$. Next, we consider the case $c_0 = 0$. Then we have $c_1 = 0$, because if $c_1 \neq 0$, then v_0 and v_1 must be linearly independent. Similarly, we find $c_1 = c_2 = \cdots = c_{\varepsilon-1} = 0$. Thus we have $M_\psi^0 = \begin{bmatrix} 0 & \cdots & 0 & c_\varepsilon \end{bmatrix}$ with $c_\varepsilon \neq 0$, and the assertion follows.

Conversely, one can similarly show that G_1^α contains the normal rational curve in G.

On the other hand, we find that if $\bar{\psi} \in G$ is an element of the rational normal curve with $M_\psi^0 = \begin{bmatrix} a^\varepsilon & \cdots & a^{\varepsilon-i} b^i & \cdots & b^\varepsilon \end{bmatrix}$, then $\ker(\psi)$ is equal to $H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon)(-P))$ with $P := (a, b) \in \mathbb{P}^1$. This is because each polynomial $f = \sum f_i s^{\varepsilon-i} t^i \in H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon))$ with $f_i \in k$ satisfies that $\psi(f) = \sum f_i a^{\varepsilon-i} b^i$, which is equal to $f(P)$. Hence $\psi(f) = 0$ if and only if $f(P) = 0$.

Finally, we show that if $\bar{\psi} \in G$ satisfies $\ker(\psi) = H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon)(-P))$ for some $P \in \mathbb{P}^1$, then $\bar{\psi}$ is contained in the rational normal curve in G, as follows: Without loss of generality, we can assume $P = (1, \lambda) \in \mathbb{P}^1$ with $\lambda \in k$. Let $M_\psi^0 = \begin{bmatrix} c_0 & \cdots & c_i & \cdots & c_\varepsilon \end{bmatrix}$. Since each polynomial $f = \lambda^i s^\varepsilon - s^{\varepsilon-i} t^i$ with $0 \leqslant i \leqslant \varepsilon$ satisfies $f(P) = 0$, we have $\lambda^i c_0 - c_i = \psi(f) = 0$. Therefore $M_\psi^0 = c_0 \cdot \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^\varepsilon \end{bmatrix}$. Thus the assertion follows.

(b) This is straightforward from [**29**], Propositions 9.7 and 11.32.

(c) From (a), there exists $(a_i, b_i) \in \mathbb{P}^1$ such that $M_{\psi_i}^0 = [a_i^\varepsilon \ \cdots \ a_i^{\varepsilon-i} b_i^i \ \cdots \ b_i^\varepsilon]$ for each $1 \leqslant i \leqslant \rho$. Then we have $\operatorname{im}(\sigma_{\psi_i}^\alpha) = k \cdot e_i$, where we set $e_i \in H^0(\mathbb{P}^1, \mathcal{O}(\alpha))^\vee$ to be the element given by $[a_i^\alpha \ \cdots \ a_i^{\alpha-i} b_i^i \ \cdots \ b_i^\alpha]$ with respect to the dual standard ordered basis for $H^0(\mathbb{P}^1, \mathcal{O}(\alpha))^\vee$. Note that e_i corresponds to a point of the rational normal curve of degree α.

Here, we show that the elements $\psi_1, \ldots, \psi_\rho$ are linearly independent, as follows: Suppose that $\psi_1, \ldots, \psi_\rho$ are linearly dependent. Then $\bar{\psi}$ is contained in the $(\rho-1)$-secant variety $S_{\rho-2}(G_1^\alpha)$. Since $\rho - 1 \leqslant \min\{\alpha, \varepsilon - \alpha\}$, it follows from (b) that we have $\bar{\psi} \in G_{\rho-1}^\alpha$, which contradicts that the assumption $\operatorname{rk} M_\psi^\alpha = \rho$.

Since $\psi_1, \ldots, \psi_\rho$ are linearly independent, it follows that $(a_1, b_1), \ldots, (a_\rho, b_\rho) \in \mathbb{P}^1$ are distinct points. This implies that $e_1, \ldots, e_\rho$ are linearly independent ([**29**], Example 1.14).

For any $h \in \ker(\sigma_\psi^\alpha)$, we take $\beta_1, \ldots, \beta_\rho \in k$ such that $\sigma_{\psi_i}^\alpha(h) = \beta_i e_i$. Since

$$r_1 \beta_1 e_1 + \cdots + r_\rho \beta_\rho e_\rho = r_1 \sigma_{\psi_1}^\alpha(h) + \cdots + r_\rho \sigma_{\psi_\rho}^\alpha(h) = \sigma_\psi^\alpha(h) = 0,$$

and since $r_i \neq 0$, we find that $\beta_1 = \cdots = \beta_\rho = 0$. □

Recall that we identify σ_ψ with $\oplus_{i=1}^{n-1}\sigma_\psi^{a_i}$, as in (14). Applying Lemma 3.8 in the case $\varepsilon = de$ and $\alpha = a_i$, we find that the k-linear map $\sigma_\psi^{a_i}$ is represented by a $(a_i{+}1)\times(de{-}a_i{+}1)$ catalecticant matrix. We consider the case where $2\max\{\, a_i \,\}_i \leqslant de$, that is to say, $a_i + 1 \leqslant de - a_i + 1$ for all i.

Definition 3.10. Let $G = \mathbb{P}_* \operatorname{Hom}_k(H^0(\mathbb{P}^1, \mathcal{O}(de)), k) \simeq \mathbb{P}^{de}$. We set $G_\rho := G_\rho^{\max\{\, a_i \,\}_i}$ and $U_\rho := G_\rho \setminus G_{\rho-1}$ for each integer $1 \leqslant \rho \leqslant \max\{\, a_i \,\}_i + 1$.

We regard $\mathcal{A} := H^0(\mathbb{P}^1, f^* N_{C/\mathbb{P}^n}^\vee \otimes f^*(\mathcal{O}_C(d)))$ as affine space. Then we set

$$\mathcal{K}(S) := \overline{\bigcup_{\bar\psi \in S} \ker(\sigma_\psi)},$$

which is a closed affine subvariety of $\mathcal{A}$ for a subset $S \subset G$.

In this notation, the union of kernels given in (13) is expressed as the variety $\mathcal{K}(G)$, and is equal to the union of $\mathcal{K}(U_\rho)$ with $1 \leqslant \rho \leqslant \max\{\, a_i \,\}_i + 1$. For a linear functional $\psi : H^0(\mathbb{P}^1, \mathcal{O}(\varepsilon)) \to k$, we have the following equality

$$\operatorname{codim}(\ker(\sigma_\psi), \mathcal{A}) = \sum_{i=1}^{n-1} \operatorname{rk} \sigma_\psi^{a_i}, \tag{15}$$

because of $\operatorname{codim}_k(\ker(\sigma_\psi^{a_i}), H^0(\mathbb{P}^1, \mathcal{O}(de - a_i))) = \dim_k(\operatorname{im}(\sigma_\psi^{a_i})) = \operatorname{rk} \sigma_\psi^{a_i}$ for each i.

Lemma 3.11. *Let C be a smooth rational curve of degree e, and assume $2\max\{\, a_i \,\}_i \leqslant de$. For integers $1 \leqslant \rho \leqslant \max\{\, a_i \,\}_i + 1$ and $1 \leqslant i \leqslant n - 1$, the following holds.*

(a) *If $\rho \leqslant a_i$, then we have $G_\rho = G_\rho^{a_i}$ in G.*
(b) *If $\bar\psi \in U_\rho$, then $\operatorname{rk} \sigma_\psi^{a_i} = \min\{\, a_i + 1, \rho \,\}$.*

Proof. (a) Lemma 3.9(a) implies that the varieties G_1 and $G_1^{a_i}$ coincide and are equal to the rational normal curve in G. From Lemma 3.9(b), we have $G_\rho = S_{\rho-1}(G_1) = G_\rho^{a_i}$ in G.

(b) Let $\bar\psi \in U_\rho$. From (a), we have $G_{a_i} = G_{a_i}^{a_i}$. Thus if $\rho > a_i$, then we have $\bar\psi \notin G_{a_i}^{a_i}$, and hence $\operatorname{rk} \sigma_\psi^{a_i} = a_i + 1$. If $\rho \leqslant a_i$, then it follows from (a) that $\bar\psi \in G_\rho^{a_i} \setminus G_{\rho-1}^{a_i}$. Hence $\operatorname{rk} \sigma_\psi^{a_i} = \rho$. □

Lemma 3.12. *Under the assumption of* Lemma 3.11*, the following holds.*

(a) $\operatorname{codim}(\mathcal{K}(U_{\max\{\, a_i \,\}_i+1}), \mathcal{A}) \geqslant \mu + 1$.
(b) $\mathcal{K}(U_\rho) \subset \mathcal{K}(G_1)$ *if $\rho \leqslant \min\{\, a_i \,\}_i + 1$ and $\rho \leqslant \max\{\, a_i \,\}_i$.*

Proof. (a) Let $\bar\psi \in U_{\max\{\, a_i \,\}_i+1}$. For each $1 \leqslant i \leqslant n - 1$, it follows from Lemma 3.11(b) that $\operatorname{rk} \sigma_\psi^{a_i} = a_i + 1$. Hence the equality (15) implies that

$$\operatorname{codim}(\ker(\sigma_\psi), \mathcal{A}) = \sum_{i=1}^{n-1} (a_i + 1) = \chi(N_{C/\mathbb{P}^n})$$

. Since $\dim G = de$, we have $\operatorname{codim}(\mathcal{K}(U_{\max\{\, a_i \,\}_i+1}), \mathcal{A}) \geqslant \chi(N_{C/\mathbb{P}^n}) - de = \mu + 1$.

(b) Let $\bar\psi \in U_\rho$. Since $\rho \leqslant \max\{\, a_i \,\}_i$, it follows from Lemma 3.9(b) that $G_\rho = S_{\rho-1}(G_1)$. Thus we have $\bar\psi_1, \ldots, \bar\psi_\rho \in G_1$ and $r_1, \ldots, r_\rho \in k$ such that $\psi = r_1\psi_1 +$

$\cdots + r_\rho\psi_\rho$. Since $\rho \leqslant \min\{\, a_i \,\}_i + 1$, it follows from Lemma 3.11(b) that $\operatorname{rk}\sigma_{\psi}^{a_i} = \rho$. Hence Lemma 3.9(c) implies that $\ker(\sigma_{\psi}^{a_i}) \subset \ker(\sigma_{\psi_1}^{a_i}) \cap \cdots \cap \ker(\sigma_{\psi_\rho}^{a_i})$ for each i. Thus $\ker(\sigma_\psi) \subset \ker(\sigma_{\psi_1}) \cap \cdots \cap \ker(\sigma_{\psi_\rho}) \subset \mathcal{K}(G_1)$. Since $\bar{\psi} \in U_\rho$ is arbitrary, we have $\mathcal{K}(U_\rho) \subset \mathcal{K}(G_1)$. □

For the fiber $I_C = p_{\mathcal{R}}^{-1}(C)$, let us calculate the codimension of $Z_C^0 := Z^0 \cap I_C$ in I_C, where Z_C^0 is isomorphic to the set of hypersurfaces $X \in H$ containing C and being smooth along C such that the projection p_H is not smooth at (X, C). We denote by $\hat{Z}_C^0 \subset H^0(\mathbb{P}^n, \mathcal{I}_C(d))$ the affine subset isomorphic to the affine cone of $Z_C^0 \subset I_C$.

Lemma 3.13. *Let $n \geqslant 3$ and assume the condition* (2.i)*. Let $C \subset \mathbb{P}^n$ be a smooth rational curve of degree $e \geqslant 2$ parametrized by a morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$, and assume $d \geqslant \max\{\, 2(e-r)+3, 4 \,\}$, where r is the dimension of the linear subspace spanned by C. Then the following holds.*

(a) $2\max\{\, a_i \,\}_i \leqslant de$,
(b) $\delta_C(\hat{Z}_C^0) \subset \mathcal{K}(G) \setminus \mathcal{K}(G_1)$,
(c) $\operatorname{codim}(\mathcal{K}(G) \setminus \mathcal{K}(G_1), \mathcal{A}) \geqslant \mu + 1$.

As a result, we have $\operatorname{codim}(Z_C^0, I_C) \geqslant \mu + 1$.

Proof. Let $f^*N_{C/\mathbb{P}^n} \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(a_i) \oplus \mathcal{O}_{\mathbb{P}^1}(e)^{\oplus n-r}$ be the splitting on $\mathbb{P}^1$. From Proposition 2.14, we obtain the following inequality,

$$e + 2 \leqslant a_i \leqslant 3e - 2r + 2 \tag{16}$$

for $1 \leqslant i \leqslant r-1$. In addition we set $a_i = e$ for $r \leqslant i \leqslant n-1$.

(a) Suppose $e = 2$ or 3. Then we have $r = e$ as in Remark 2.3(b). Thus the right hand side of inequality (16) is equal to $e+2$, that is to say, $a_i = e+2$ for $1 \leqslant i \leqslant r-1$. Hence, by the assumption $d \geqslant 4$, we have $2\max\{\, a_i \,\}_i \leqslant de$.

Suppose $e \geqslant 4$. Then, by the assumption $d \geqslant 2(e-r)+3$, the right hand side of inequality (16) is less than or equal to $e+d-1$. Thus we have $2\max\{\, a_i \,\}_i \leqslant 2(e+d-1) \leqslant 4\max\{\, d, e \,\} - 2$. By the assumption $d \geqslant 4$, we have $\min\{\, d, e \,\} \geqslant 4$, and hence $2\max\{\, a_i \,\}_i \leqslant \min\{\, d, e \,\}\max\{\, d, e \,\} - 2 = de - 2$.

(b) Let $h \in \hat{Z}_C^0$ and let $X \in H$ be the hypersurface defined by h. Since $(X, C) \in Z^0$, it follows from Lemma 3.6 that we have $\sigma_\psi(\delta_C(h)) = 0$ for some ψ; hence $\delta_C(h) \in \mathcal{K}(G)$.

Suppose that $\delta_C(h) \in \ker(\sigma_\psi)$ with some $\bar{\psi} \in G_1$. Then $H^0(\delta_C(h))$ is contained in $\ker(\psi)$, where we have $\ker(\psi) = H^0(\mathbb{P}^1, \mathcal{O}(de)(-P))$ for some $P \in \mathbb{P}^1$ due to Lemma 3.9(a). Hence $\delta_C(h)(f(P)) = 0$. From Lemma 2.11, the hypersurface X is singular at $f(P) \in C$, which contradicts that X is smooth along C.

(c) From Lemma 3.12(b), it follows that

$$\mathcal{K}(G) \setminus \mathcal{K}(G_1) \subset \mathcal{K}(U_{\max\{\, a_i \,\}_i+1}) \cup \bigcup_{\min\{\, a_i \,\}_i+2 \leqslant \rho \leqslant \max\{\, a_i \,\}_i} \mathcal{K}(U_\rho).$$

From Lemma 3.12(a), we have already seen $\operatorname{codim}(\mathcal{K}(U_{\max\{\, a_i \,\}_i+1}), \mathcal{A}) \geqslant \mu + 1$. Thus it is sufficient to show the integer $\nu := \min\{\, \operatorname{codim}(\mathcal{K}(U_\rho), \mathcal{A}) \mid \min\{\, a_i \,\}_i + 2 \leqslant \rho \leqslant \max\{\, a_i \,\}_i \,\}$ is greater than or equal to $\mu + 1$.

Now, for each integer ρ with $\min\{ a_i \}_i + 2 \leqslant \rho \leqslant \max\{ a_i \}_i$, we will calculate the codimension of $\mathcal{K}(U_\rho)$ in $\mathcal{A}$. Here we set α to be the smallest integer a_i with $1 \leqslant i \leqslant r-1$. Since inequality (16) holds for $1 \leqslant i \leqslant r-1$ and since $a_i = e$ for $r \leqslant i \leqslant n-1$, we have

$$e+2 \leqslant \rho \leqslant 3e-2r+2 \quad \text{and} \quad \alpha \geqslant e+2.$$

By definition, $\mathcal{K}(U_\rho)$ is the closure of the union of $\ker(\sigma_\psi)$ with $\bar{\psi} \in U_\rho$. Here it follows from the equality (15) that we have $\operatorname{codim}(\ker(\sigma_\psi), \mathcal{A}) \geqslant \sum_{i=1}^{r-1} \operatorname{rk} \sigma_\psi^{a_i} + (n-r) \operatorname{rk} \sigma_\psi^{e}$, where Lemma 3.11(b) implies that the right hand side is greater than or equal to $(r-1)\min\{ \alpha+1, \rho \} + (n-r)(e+1)$. In addition, it follows from Lemma 3.9(b) that $\dim U_\rho = 2\rho - 1$. Thus we have inequality,

$$\operatorname{codim}(\mathcal{K}(U_\rho), \mathcal{A}) \geqslant (r-1)\min\{ \alpha+1, \rho \} + (n-r)(e+1) - (2\rho-1). \tag{17}$$

Assume $\rho \geqslant e+3$. Then we have $\min\{ \alpha+1, \rho \} \geqslant e+3$. By using this inequality and $\rho \leqslant 3e-2r+2$, we can see that the right hand side of inequality (17) is greater than or equal to

$$(r-1)(e+3) + (n-r)(e+1) - (2(3e-2r+2)-1) = ne+n+6r-7e-6. \tag{18}$$

From the assumption $d \geqslant 2(e-r)+3$, by calculating (18), we have

$$\operatorname{codim}(\mathcal{K}(U_\rho), \mathcal{A}) \geqslant ne+n-e-3d+3. \tag{19}$$

Assume $\rho = e+2$. Then we have $\min\{ \alpha+1, \rho \} \geqslant e+2$. By using this inequality and by substituting $\rho = e+2$, we can see that the right hand side of inequality (17) is greater than or equal to

$$(r-1)(e+2) + (n-r)(e+1) - (2(e+2)-1) = ne+n+r-3e-5. \tag{20}$$

Thus, from the assumption $d \geqslant 2(e-r)+3$, by calculating (20), we have

$$\operatorname{codim}(\mathcal{K}(U_{e+2}), \mathcal{A}) \geqslant ne+n-2e-d/2-7/2. \tag{21}$$

Here if equality $e = r$ holds, then by applying this equality to (20), we have

$$\operatorname{codim}(\mathcal{K}(U_{e+2}), \mathcal{A}) \geqslant (n-2)e+n-5. \tag{22}$$

Next, let us show $\nu \geqslant \mu+1$ by calculating $\nu - (\mu+1)$, where we recall that $\mu = (n+1-d)e+n-4$. Suppose $e = 2$. Then we have $\max\{ a_i \}_i = 4$ and $\nu = \operatorname{codim}(\mathcal{K}(U_4), \mathcal{A})$. Hence inequality (22) implies that $\nu - (\mu+1) \geqslant 2d-8$.

Suppose $e \geqslant 3$. Then inequality (19) implies that

$$\operatorname{codim}(\mathcal{K}(U_\rho), \mathcal{A}) - (\mu+1) \geqslant (e-3)(d-2) \quad \text{for} \quad \rho \geqslant e+3.$$

In addition, for the case $\rho = e+2$, the inequality (21) implies that

$$\operatorname{codim}(\mathcal{K}(U_{e+2}), \mathcal{A}) - (\mu+1) \geqslant (e-1/2)(d-3)-2.$$

Thus $\nu - (\mu+1) \geqslant \min\{ (e-3)(d-2), (e-1/2)(d-3)-2 \}$.

In consequence, for $e \geqslant 2$ and $d \geqslant 4$, it follows that the integer ν is greater than or equal to $\mu+1$, hence so is $\operatorname{codim}(\mathcal{K}(G) \setminus \mathcal{K}(G_1), \mathcal{A})$. Therefore the assertion (c) follows.

Here Proposition 2.13 implies that the k-linear map δ_C is surjective. Since we can regard $\delta_C : H^0(\mathbb{P}^n, \mathcal{I}_C(d)) \to \mathcal{A}$ as a smooth morphism of affine spaces, it follows from (b) that we have the following inequality,

$$\operatorname{codim}(Z^0_C, I_C) = \operatorname{codim}(\hat{Z}^0_C, H^0(\mathbb{P}^n, \mathcal{I}_C(d))) \geqslant \operatorname{codim}(\mathcal{K}(G) \setminus \mathcal{K}(G_1), \mathcal{A}).$$

From (c), we have $\operatorname{codim}(Z^0_C, I_C) \geqslant \mu + 1$. □

Proof of Proposition 3.2. Let $C \in \mathcal{R}$, and let r be the dimension of the linear subspace spanned by C. By the assumption $d \geqslant \max\{\,2e-3, 4\,\}$, we have inequality $d \geqslant \max\{\,2(e-r)+3, 4\,\}$, as follows: If $e = 2$ or 3, then we have $e = r$ as in Remark 2.3(b); hence $4 \geqslant 2(e-r)+3$. If $e \geqslant 4$, then we have $r \geqslant 3$ as in Remark 2.3(a); hence $2e-3 \geqslant 2(e-r)+3$.

Thus Lemma 3.13 implies $\operatorname{codim}(Z^0_C, I_C) \geqslant \mu + 1$. Since $C \in \mathcal{R}$ is arbitrary, we have $\operatorname{codim}(Z^0, I) \geqslant \mu + 1$. □

3.2. Quadric and cubic hypersurfaces. Let $X \subset \mathbb{P}^n$ be a hypersurface, and let $(\mathbb{P}^n)^\vee = G(n-1, \mathbb{P}^n)$ be the space of hyperplanes in $\mathbb{P}^n$. We consider the Gauss map,

$$\gamma_X : X \dashrightarrow (\mathbb{P}^n)^\vee$$

which sends each smooth point $P \in X$ to the embedded tangent space of X at P in $\mathbb{P}^n$.

Let $(z_0, z_1, \dots, z_n)$ be homogeneous coordinates on $\mathbb{P}^n$, and let h be the defining equation of X. We denote by $(z_0^\vee, z_1^\vee, \dots, z_n^\vee)$ the dual basis of $H^0((\mathbb{P}^n)^\vee, \mathcal{O}(1)) = H^0(\mathbb{P}^n, \mathcal{O}(1))^\vee$. Since the embedded tangent space of X at P is defined as the zero set of $(\partial h/\partial z_0)|_P \cdot z_0 + \cdots + (\partial h/\partial z_n)|_P \cdot z_n$, the k-linear map $\gamma_X^* : H^0((\mathbb{P}^n)^\vee, \mathcal{O}(1)) \to H^0(X, \mathcal{O}(d-1))$ is given by

$$\gamma_X^*(z_i^\vee) = (\partial h/\partial z_i)|_X \tag{23}$$

with $0 \leqslant i \leqslant n$. To prove Proposition 3.3, we show the following two lemmas.

Lemma 3.14. *Let X be a cubic hypersurface, and let $C \subset X$ be a smooth rational curve of degree $e = 2$ or 3. Let $L \subset \mathbb{P}^n$ be the linear subspace spanned by C, and let $L^* = \{\, M \in (\mathbb{P}^n)^\vee \mid L \subset M \,\}$. Suppose that X is smooth along C, and suppose $\gamma_X(C) \subset L^*$. Then we have $L \subset X$.*

Proof. By changing coordinates on $\mathbb{P}^n$, we may assume that L is the zero set of $z_{e+1}, \dots, z_n \in H^0(\mathbb{P}^n, \mathcal{O}(1))$, and assume that the rational curve $C \subset \mathbb{P}^n$ of degree $e = 2$ or 3 is parametrized by a morphism

$$f : \mathbb{P}^1 \to \mathbb{P}^n : (s,t) \mapsto (s^e, s^{e-1}t, \dots, t^e, 0, \dots, 0).$$

We denote by φ^i the k-linear map $H^0(\mathbb{P}^n, \mathcal{O}(i)) \to H^0(\mathbb{P}^1, \mathcal{O}(i \cdot e))$ induced from f. Then the composite morphism $\gamma_X \circ f : \mathbb{P}^1 \to (\mathbb{P}^n)^\vee$ induces the composite k-linear map,

$$\varphi^2 \circ \gamma_X^* : H^0(\mathbb{P}^n, \mathcal{O}(1))^\vee \to H^0(\mathbb{P}^n, \mathcal{O}(2)) \to H^0(\mathbb{P}^1, \mathcal{O}(2e)).$$

Here we find $(\varphi^2 \circ \gamma_X^*)(z_i^\vee) = 0$ for $0 \leqslant i \leqslant e$, as follows: Since $\gamma_X(C) \subset L^*$, it follows $(\varphi^2 \circ \gamma_X^*)(H^0((\mathbb{P}^n)^\vee, \mathcal{I}_{L^*/(\mathbb{P}^n)^\vee}(1))) = 0$. We note that $\mathbb{P}^n \simeq (\mathbb{P}^n)^{\vee\vee}$ by sending $x \in \mathbb{P}^n$ to $x^* = \{\, M \in (\mathbb{P}^n)^\vee \mid x \in M \,\}$, a hyperplane of $(\mathbb{P}^n)^\vee$. In particular, L is isomorphic to the space of hyperplanes of $(\mathbb{P}^n)^\vee$ containing L^*. This means that $H^0(L, \mathcal{O}(1))^\vee \simeq H^0((\mathbb{P}^n)^\vee, \mathcal{I}_{L^*/(\mathbb{P}^n)^\vee}(1))$. Since $H^0(L, \mathcal{O}(1))^\vee \subset H^0(\mathbb{P}^n, \mathcal{O}(1))^\vee$ is spanned by $z_0^\vee, \dots, z_e^\vee$, we get $(\varphi^2 \circ \gamma_X^*)(z_i^\vee) = 0$ for $0 \leqslant i \leqslant e$.

Assume $e = 2$. Let h be a defining equation of X. Since $h \in H^0(\mathbb{P}^n, \mathcal{I}_C(3))$, we can write $h = g(z_1^2 - z_0z_2) + h_3z_3 + \cdots + h_nz_n$ with $g \in k[z_0, z_1, z_2]_1$ and $h_i \in H^0(\mathbb{P}^n, \mathcal{O}(2))$. From the formula (23), it follows

$$\begin{bmatrix} (\varphi^2 \circ \gamma_X^*)(z_0^\vee) \\ (\varphi^2 \circ \gamma_X^*)(z_1^\vee) \\ (\varphi^2 \circ \gamma_X^*)(z_2^\vee) \end{bmatrix} = \begin{bmatrix} -\varphi^1(g)t^2 \\ 2\varphi^1(g)st \\ -\varphi^1(g)s^2 \end{bmatrix}.$$

Since $(\varphi^2 \circ \gamma_X^*)(z_i^\vee) = 0$ for $0 \leqslant i \leqslant 2$, we obtain $\varphi^1(g) = 0$. Since $g \in k[z_0, z_1, z_2]_1$, it follows $g = 0$. Hence $h \in H^0(\mathbb{P}^n, \mathcal{I}_L(3))$.

Assume $e = 3$. Then we can write $h = g_0(z_1z_2 - z_0z_3) + g_1(z_1^2 - z_0z_2) + g_2(z_2^2 - z_1z_3) + h_4z_4 + \cdots + h_nz_n$ with $g_i \in k[z_0, z_1, z_2, z_3]_1$ and $h_i \in H^0(\mathbb{P}^n, \mathcal{O}(2))$. From the equality (23) again, it follows

$$\begin{bmatrix} (\varphi^2 \circ \gamma_X^*)(z_0^\vee) \\ (\varphi^2 \circ \gamma_X^*)(z_1^\vee) \\ (\varphi^2 \circ \gamma_X^*)(z_2^\vee) \\ (\varphi^2 \circ \gamma_X^*)(z_3^\vee) \end{bmatrix} = \begin{bmatrix} -\varphi^1(g_0)t^3 - \varphi^1(g_1)st^2 \\ \varphi^1(g_0)st^2 + 2\varphi^1(g_1)s^2t - \varphi^1(g_2)t^3 \\ \varphi^1(g_0)s^2t - \varphi^1(g_1)s^3 + 2\varphi^1(g_2)st^2 \\ -\varphi^1(g_0)s^3 - \varphi^1(g_2)s^2t \end{bmatrix}. \tag{24}$$

Setting $g_i = a_{i,0}z_0 + a_{i,1}z_1 + a_{i,2}z_2 + a_{i,3}z_3$ with $a_{i,j} \in k$, we can represent the above vector (24) with respect to the basis $(s^6, s^5t, \dots, t^6)$ by the following matrix A,

$$\begin{bmatrix} 0 & 0 & -a_{1,0} & -a_{1,1} - a_{0,0} & -a_{1,2} - a_{0,1} & -a_{1,3} - a_{0,2} & -a_{0,3} \\ 0 & 2a_{1,0} & 2a_{1,1} + a_{0,0} & 2a_{1,2} + a_{0,1} - a_{2,0} & 2a_{1,3} + a_{0,2} - a_{2,1} & a_{0,3} - a_{2,2} & -a_{2,3} \\ a_{1,0} & a_{1,1} - a_{0,0} & a_{1,2} - a_{0,1} + 2a_{2,0} & a_{1,3} - a_{0,2} + 2a_{2,1} & -a_{0,3} + 2a_{2,2} & 2a_{2,3} & 0 \\ -a_{0,0} & -a_{0,1} - a_{2,0} & -a_{0,2} - a_{2,1} & -a_{0,3} - a_{2,2} & -a_{2,3} & 0 & 0 \end{bmatrix}.$$

Since $(\varphi^2 \circ \gamma_X^*)(z_i^\vee) = 0$ for $0 \leqslant i \leqslant 3$, it follows $A = 0$. As a result, we have $a_{0,1} = -a_{1,2} = -a_{2,0}$, $a_{0,2} = -a_{1,3} = -a_{2,1}$, and other $a_{i,j} = 0$, even if $\operatorname{char}(k) = 2$. Hence $g_0(z_1z_2 - z_0z_3) + g_1(z_1^2 - z_0z_2) + g_2(z_2^2 - z_1z_3) = 0$, which implies $h \in H^0(\mathbb{P}^n, \mathcal{I}_L(3))$. □

Lemma 3.15. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d, and let $C \subset X$ be a smooth rational curve of degree e such that X is smooth along C. Assume one of the following conditions:* (i) *$(e, d) = (2, 2), (3, 2)$ and $n \geqslant 3$; or* (ii) *$(e, d, n) = (2, 3, 3), (3, 3, 3), (3, 3, 4)$. Then we have $H^1(N_{C/X}) = 0$.*

Proof. We have $f^*N_{C/\mathbb{P}^n} \simeq f^*N_{L/\mathbb{P}^n} \oplus f^*N_{C/L} \simeq \mathcal{O}_{\mathbb{P}^1}(e)^{\oplus n-e} \oplus \mathcal{O}_{\mathbb{P}^1}(e+2)^{\oplus e-1}$. We consider the following exact sequence on $\mathbb{P}^1$,

$$0 \to f^*N_{C/X} \xrightarrow{\tau} f^*N_{C/\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^1}(e)^{\oplus n-e} \oplus \mathcal{O}_{\mathbb{P}^1}(e+2)^{\oplus e-1} \to f^*N_{X/\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^1}(de) \to 0. \tag{25}$$

Let $f^*N_{C/X} = \bigoplus_{i=1}^{n-2} \mathcal{O}(b_i)$ be the splitting on $\mathbb{P}^n$ with $b_1 \leqslant \dots \leqslant b_{n-2}$, and let $\beta := \sum_{i=1}^{n-2} b_i$. Here we have $\beta = (n+1-d)e - 2$, and have $b_{n-2} \leqslant e+2$ since τ is injective. To prove $H^1(N_{C/X}) = 0$, it is sufficient to show $b_1 \geqslant -1$.

If $n = 3$, then we have $b_1 = \beta = (4-d)e - 2$. If $n = 4$, then we have $b_1 = \beta - b_2 \geqslant ((5-d)e - 2) - (e+2) = (4-d)e - 4$. Hence in the case (ii), and in the case $(e, d) = (2, 2), (3, 2)$ with $n = 3, 4$, the assertion follows.

We consider the case (i) with $n \geqslant 5$. If an integer i_0 with $1 \leqslant i_0 \leqslant n-2$ satisfies $b_{i_0} \geqslant e+1$, then since the morphism τ in (25) is injective, so is $\mathcal{O}(b_{i_0}) \oplus \cdots \oplus \mathcal{O}(b_{n-2}) \to \mathcal{O}(e+2)^{\oplus e-1}$, and then the following inequality must hold:

$$\sum_{i=i_0}^{n-2} (b_i + 1) \leqslant (e-1)(e+3). \tag{26}$$

Suppose $(e, d) = (2, 2)$. Then $b_{n-2} \leqslant 4$. If $b_{n-3} \geqslant 3$, then $b_{n-3} + b_{n-2} + 2 \geqslant 8$, which contradicts the inequality (26). Hence we have $b_{n-3} \leqslant 2$, which implies $b_1 = \beta - \sum_{i=2}^{n-2} b_i \geqslant ((n-1)2 - 2) - (4 + 2(n-4)) = 0$. Suppose $(e, d) = (3, 2)$. Then $b_{n-3} \leqslant b_{n-2} \leqslant 5$. If $b_{n-4} \geqslant 4$, then $b_{n-4} + b_{n-3} + b_{n-2} + 3 \geqslant 15$, which contradicts the inequality (26). Hence we have $b_{n-4} \leqslant 3$, which implies $b_1 \geqslant ((n-1)3 - 2) - (10 + 3(n-5)) = 0$. $\square$

Proof of Proposition 3.3. (i) The case $d = 2$ follows from Lemma 3.15(i)

(ii) Suppose that the linear subspace L spanned by C is not contained in X. We show that $H^1(N_{C/X}) = 0$ by using the induction on n. If (e, d, n) is equal to $(2, 3, 3), (3, 3, 3)$, or $(3, 3, 4)$, then $H^1(N_{C/X}) = 0$, since the condition (ii) of Lemma 3.15 is satisfied. Assume either $(e, d) = (2, 3)$ and $n \geqslant 4$, or $(e, d) = (3, 3)$ and $n \geqslant 5$. From Lemma 3.14, the image $\gamma_X(C)$ is not contained in L^*; hence $\#(\gamma_X(C) \cap L^*) < \infty$. Since $\dim L^* = n - e - 1 \geqslant 1$, there exists a hyperplane $M \subset \mathbb{P}^n$ such that $M \in L^* \setminus \gamma_X(C) \subset (\mathbb{P}^n)^\vee$. Then the hypersurface $M \cap X$ in $M \simeq \mathbb{P}^{n-1}$ is smooth along C. By induction hypothesis, it follows $H^1(N_{C/M\cap X}) = 0$. From the exact sequence $0 \to N_{C/M\cap X} \to N_{C/X} \to N_{M\cap X/X}|_C \simeq \mathcal{O}_C(1) \to 0$, we have $H^1(N_{C/X}) = 0$.

(iii) Suppose that the hypersurface X is general. Let $C \subset X$ be a smooth rational curve of degree $e = 2$ or 3, and let $L \subset \mathbb{P}^n$ be the e-dimensional linear subspace spanned by C. Here we assume that $L \subset X$, since the case $L \not\subset X$ was already seen in (ii).

Let us consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(L, N_{L/X}) & \longrightarrow & H^0(L, N_{L/\mathbb{P}^n}) & \xrightarrow{v_L} & H^0(L, N_{X/\mathbb{P}^n}|_L) \simeq H^0(L, \mathcal{O}(3)) \\
 & \downarrow & & \downarrow & & \downarrow w \\
0 \longrightarrow & H^0(C, N_{L/X}|_C) & \longrightarrow & H^0(C, N_{L/\mathbb{P}^n}|_C) & \xrightarrow{v_C} & H^0(C, N_{X/\mathbb{P}^n}|_C) \simeq H^0(C, \mathcal{O}(3)).
\end{array}$$

We show that the k-linear map v_C is surjective, as follows: We denote by $F_e(X) \subset G(e, \mathbb{P}^n)$ the space of e-dimensional linear subspaces of $\mathbb{P}^n$ lying in X. Then $L \in F_e(X)$. Since X is general and since $F_e(X) \neq \emptyset$, it follows from [**16**], §1 and Théorème (2.1) that the scheme $F_e(X)$ is smooth and has the expected dimension; hence we have

$$h^0(L, N_{L/X}) = \dim_k T_L F_e(X) = (e+1)(n-e) - \binom{3+e}{e}.$$

Thus $h^0(L, N_{L/X}) = h^0(L, N_{L/\mathbb{P}^n}) - h^0(L, N_{X/\mathbb{P}^n}|_L)$, which implies that v_L is surjective. Since w is also surjective, so is the k-linear map v_C.

Since v_C is surjective and since $H^1(C, N_{L/\mathbb{P}^n}|_C) = 0$, we have $H^1(C, N_{L/X}|_C) = 0$. From the exact sequence $0 \to N_{C/L} \to N_{C/X} \to N_{L/X}|_C \to 0$, we obtain $H^1(N_{C/X}) = 0$. □

Here we have a corollary, which will be used in the next section:

Corollary 3.16. *For an irreducible and reduced conic $C \subset \mathbb{P}^n$, we have $Z^0_C = \emptyset$ if $d = 2$, and have* $\operatorname{codim}(Z^0_C, I_C) \geqslant 3$ *if $d = 3$.*

PROOF. Assume $d = 2$. For any hypersurface X satisfying $(X, C) \in I^0$, the condition (i) of Proposition 3.3 holds. Hence we have $H^1(N_{C/X}) = 0$, which implies that $H^0(N_{C/\mathbb{P}^n}) \to H^0(N_{X/\mathbb{P}^n}|_C)$ is surjective. From Lemma 2.7, it follows that $d_{(X,C)}p_H$ is surjective, that is, $(X, C) \notin Z^0_C$. Since X is arbitrary, we have $Z^0_C = \emptyset$.

Next, assume $d = 3$. Let $L \subset \mathbb{P}^n$ be the linear subspace spanned by C, and let W_C be the set of $(X, C) \in I_C \cap I^0$ such that X contains L. For any hypersurface X satisfying $(X, C) \in I^0 \setminus W_C$, the condition (iii) of Proposition 3.3 holds; hence we have $H^1(N_{C/X}) = 0$. Thus Lemma 2.7 implies $(X, C) \notin Z^0_C$. Therefore $Z^0_C \subset W_C$. Since $\operatorname{codim}(I_C, H \times \{ C \}) = h^0(\mathcal{O}_C(3)) = 7$ and $\operatorname{codim}(W_C, H \times \{ C \}) = h^0(\mathcal{O}_L(3)) = 10$, we get the statement. □

Now we come to the proof of Theorem I(b).

PROOF OF THEOREM I(b). The case $e = 1$ is nothing but Theorem A(b). We assume $e \geqslant 2$. Let $\mathcal{R} \subset \operatorname{Hilb}^{et+1}(\mathbb{P}^n/k)$ be the space of rational curves of degree e in $\mathbb{P}^n$ with $n \geqslant 3$, and assume either $e \leqslant 3$ and $d \geqslant 1$, or $e \geqslant 4$ and $d \geqslant 2e-3$. Then the condition (2.i) is satisfied. We also assume $d \geqslant 2$, since the case $d = 1$ (i.e., $X \simeq \mathbb{P}^{n-1}$) follows immediately. Here it follows from Corollary 3.5 that $I^0 \subset I$ is a dense subset.

Suppose $e = 2$ or 3, and suppose $d = 2$ or 3. In the case $d = 2$, since the condition (i) of Proposition 3.3 is satisfied for each $(X, C) \in I^0$, we have $H^1(N_{C/X}) = 0$. In the case $d = 3$, for general $(X, C) \in I^0$, the hypersurface X does not contain the linear subspace $L \subset \mathbb{P}^n$ spanned by C, because of $h^0(\mathbb{P}^n, \mathcal{I}_C(d)) > h^0(\mathbb{P}^n, \mathcal{I}_L(d))$. Since the condition (ii) of Proposition 3.3 is satisfied, we have $H^1(N_{C/X}) = 0$. Thus, in both cases $d = 2, 3$, it follows from Lemma 2.7 that $d_{(X,C)}p_H$ is surjective; hence the subset $p_H(I) \subset H$ is dense. For a general hypersurface $X \in H$, it follows $R_e(X) \simeq p_H^{-1}(X) \neq \emptyset$, and it follows from Proposition 3.3 again that $H^1(N_{C/X}) = 0$ for all $C \in R_e(X)$. Hence $R_e(X)$ is smooth and has the expected dimension μ.

Suppose $e \geqslant 2$ and $d \geqslant \max\{ 2e-3, 4 \}$. Then Proposition 3.2 implies that $\operatorname{codim}(Z^0, I) \geqslant \mu+1$. It follows from $\mu \geqslant 0$ that p_H is smooth on the non-empty subset $I^0 \setminus Z^0$. In particular, the image $p_H(I)$ is dense in H. Since we have $\dim I = \dim H + \mu$ as in Lemma 2.1, it follows

$$\dim Z^0 = \dim I - \operatorname{codim}(Z^0, I) \leqslant \dim H - 1 < \dim H,$$

which implies that $p_H(Z^0)$ is not dense in H. Hence, for a general smooth hypersurface $X \in H$, we obtain $p_H^{-1}(X) \subset I^0 \setminus Z^0$; thus $p_H^{-1}(X) \simeq R_e(X)$ is smooth and has the expected dimension μ. □

4. Connectedness of the space of conics on a hypersurface

As in the condition (2.ii) in §2, we set $\mathcal{R} = \mathrm{Hilb}^{2t+1}(\mathbb{P}^n/k)$, which is a proper smooth variety over k. In this section, we will prove that the projection $p_H : I \to H$ has connected fibers in the case $\mu \geqslant 1$, by showing $\mathrm{codim}(Z, I) \geqslant 2$.

We denote by $\mathcal{U} \subset \mathcal{R}$ the space of irreducible and reduced conics in $\mathbb{P}^n$, which was already studied as the case (2.i) with $e = 2$. Let B_1 be the set of $C \in \mathcal{R}$ such that C is a union of two lines $l_1, l_2 \subset \mathbb{P}^n$ with $l_1 \neq l_2$ intersecting in one point, and let $B_2 := \{\, C \in \mathcal{R} \mid \mathrm{red}(C) \subset \mathbb{P}^n \text{ is a line}\,\}$. Then we have $\mathcal{R} = \mathcal{U} \cup B_1 \cup B_2$.

Lemma 4.1. *We have* $\mathrm{codim}(B_1, \mathcal{R}) = 1$ *and* $\mathrm{codim}(B_2, \mathcal{R}) = 3$.

Proof. As in Remark 2.4, we consider the morphism $\pi : \mathcal{R} \to G(2, \mathbb{P}^n)$ which sends each C to the linear plane $L \subset \mathbb{P}^n$ spanned by C. Then the fiber $\pi^{-1}(L) \simeq \mathrm{Hilb}^{2t+1}(L/k)$ is of dimension 5 for any $L \in G(2, \mathbb{P}^n)$. Here $\pi^{-1}(L) \cap B_2$ is isomorphic to $L^\vee$, the space of lines in L, which is of dimension 2. On the other hand, we have a finite surjective morphism $L^\vee \times L^\vee \setminus \Delta \to \pi^{-1}(L) \cap B_1$ by sending (l_1, l_2) to $C = l_1 \cup l_2$, where $\Delta = \{\,(l, l) \mid l \in L^\vee\,\}$. Hence $\dim(\pi^{-1}(L) \cap B_1) = 4$.

Since the codimension of $\pi^{-1}(L) \cap B_1$ (resp. $\pi^{-1}(L) \cap B_2$) in $\mathcal{R}$ is equal to 3 (resp. 1) for each L, the statement follows. □

Since $\mathrm{codim}(p_{\mathcal{R}}^{-1}(B_1), I) = \mathrm{codim}(B_1, \mathcal{R}) = 1$, we have $\mathrm{codim}(Z \cap p_{\mathcal{R}}^{-1}(B_1), I) \geqslant 2$ by showing the following lemma:

Lemma 4.2. *Assume $\mu \geqslant 1$. For each $C \in B_1$, there exists a hypersurface $X \subset \mathbb{P}^n$ containing C such that $(X, C) \in I \setminus Z$.*

Proof. Let $C = l_1 \cup l_2$ with lines $l_1, l_2 \subset \mathbb{P}^n$ intersecting in one point, and let $L \subset \mathbb{P}^n$ be the linear plane spanned by C. By choosing homogeneous coordinates $(s, t, u, z_3, \ldots, z_n)$ on $\mathbb{P}^n$, we may assume $L = (z_3 = \cdots = z_n = 0)$ in $\mathbb{P}^n$, and assume $l_1 = (u = 0)$ and $l_2 = (t = 0)$ in L. Since $C = (tu = 0)$ in L, we have $H^0(C, \mathcal{O}(i)) \simeq (k[s,t,u]/(tu))_i$ with $i \geqslant 0$, and have $N^\vee_{C/\mathbb{P}^n} = \mathcal{I}_C/\mathcal{I}_C^2 = \mathcal{O}_C(2) \cdot \overline{tu} \oplus \bigoplus_{i=3}^n \mathcal{O}_C(1) \cdot \bar{z}_i$.

Now, in a similar way to the proof of Proposition 2.15, we will give an element $\alpha \in H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d)) \simeq \mathrm{Hom}_{\mathcal{O}_C}(N_{C/\mathbb{P}^n}, \mathcal{O}_C(d))$ such that the k-linear map

$$H^0(\alpha) : H^0(C, N_{C/\mathbb{P}^n}) \simeq H^0(C, \mathcal{O}(2)) \oplus H^0(C, \mathcal{O}(1))^{\oplus n-2} \to H^0(C, \mathcal{O}(d))$$

is surjective, as follows. We define polynomials $\xi_1^i, \xi_2^i \in H^0(C, \mathcal{O}(d-1))$ by

$$\xi_1^i := s^{d-3i-2}t^{3i+1} + s^{d-3i-1}u^{3i},\ \xi_2^i := s^{d-3i-1}t^{3i} + s^{d-3i-2}u^{3i+1}.$$

Since $tu = 0$ in $H^0(C, \mathcal{O}(i))$, the k-linear space $\xi_1^i \cdot H^0(C, \mathcal{O}(1)) + \xi_2^i \cdot H^0(C, \mathcal{O}(1))$ gives the following 6 monomials of $H^0(C, \mathcal{O}(d))$:

$$s^{d-3i}t^{3i},\ s^{d-3i-1}t^{3i+1},\ s^{d-3i-2}t^{3i+2},\ s^{d-3i}u^{3i},\ s^{d-3i-1}u^{3i+1},\ s^{d-3i-2}u^{3i+2}.$$

In addition, $s^{d-2} \cdot H^0(C, \mathcal{O}(2))$ gives 5 monomials $s^d, s^{d-1}t, s^{d-2}t^2, s^{d-1}u, s^{d-2}u^2$. Then we can take an element

$$\alpha \in H^0(C, N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_C(d)) = H^0(C, \mathcal{O}(d-2)) \cdot \overline{tu} \ \oplus \bigoplus_{i=3}^n H^0(C, \mathcal{O}(d-1)) \cdot \bar{z}_i,$$

as follows: Since $\mu = 3n-2d-2$, it follows from the assumption $\mu \geqslant 1$ that inequality $2d/3 \leqslant n-1$ holds. Here we take integers d' and r such that $d = 3d'+r$ with $3 \leqslant r \leqslant 5$. If $d = 3d'+3$, then it follows $2d'+2 \leqslant n-1$; thus we can define

$$\alpha := (s^{d-2}, \xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2, \dots, \xi_1^{d'}, \xi_2^{d'}, t^{d-1}+u^{d-1}, *, \dots, *).$$

If $d = 3d'+4$ or $3d'+5$, then it follows $2d'+3 \leqslant n-1$; thus we can define

$$\alpha := (s^{d-2}, \xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2, \dots, \xi_1^{d'}, \xi_2^{d'}, t^{d-1}+su^{d-2}, st^{d-2}+u^{d-1}, *, \dots, *).$$

Since $\alpha \cdot H^0(C, N_{C/\mathbb{P}^n})$ gives all monomials of $H^0(C, \mathcal{O}(d))$, the k-linear map $H^0(\alpha)$ is surjective.

It follows from $H^1(L, \mathcal{I}^2_{C/L}) \simeq H^1(L, \mathcal{O}_L(-4)) = 0$ that $\delta_{C/L}$ is surjective. Therefore Lemma 2.12 implies that δ_C is surjective, and hence there exists $h \in H^0(\mathbb{P}^n, \mathcal{I}_C(d))$ such that $\alpha = \delta_C(h)$. Let $X \subset \mathbb{P}^n$ be the hypersurface defined by h. Since $H^0(\delta_C(h)) = H^0(\alpha)$ is surjective, so is the k-linear map $d_{(X,C)}p_H$, due to Proposition 2.10. Hence we have $(X, C) \notin Z$. □

Here we have:

Proposition 4.3. *Assume $\mu \geqslant 1$ and $n \geqslant 4$. Then $p_H : I \to H$ has connected fibers.*

Proof. We denote by $I_S = p_{\mathcal{R}}^{-1}(S)$ for a subset $S \subset \mathcal{R}$. From Lemma 4.1, we have $\operatorname{codim}(B_2, \mathcal{R}) = 3$; hence it follows that $\operatorname{codim}(Z \cap I_{B_2}, I) \geqslant \operatorname{codim}(I_{B_2}, I) = 3$. From $\operatorname{codim}(B_1, \mathcal{R}) = 1$ and Lemma 4.2, we have $\operatorname{codim}(Z \cap I_{B_1}, I) \geqslant 2$. Since $\mu \geqslant 1$, it follows from Proposition 3.2 and Corollary 3.16 that $\operatorname{codim}(Z^0 \cap I_{\mathcal{U}}, I) \geqslant 2$. From Corollary 3.5, we have $\operatorname{codim}((I \setminus I^0) \cap I_{\mathcal{U}}, I) \geqslant 2$. Since $Z = (Z \cap I_{B_1}) \cup (Z \cap I_{B_2}) \cup (Z^0 \cap I_{\mathcal{U}}) \cup (Z \cap (I \setminus I^0) \cap I_{\mathcal{U}})$, we have

$$\operatorname{codim}(Z, I) \geqslant 2.$$

Now we take the Stein factorization,

$$p_H : I \xrightarrow{q_1} E \xrightarrow{q_2} H.$$

The variety I is irreducible as in Lemma 2.1, and is proper over k since $\mathcal{R}$ is so. Hence E is irreducible and proper over k. Suppose that q_2 is not étale. Then from [**25**], p57, Theorem 1, the ramification locus has an irreducible component E_1 which is of codimension 1 in E. Thus we get a non-smooth locus $q_1^{-1}E_1$ of p_H of codimension 1, which is absurd. Hence q_2 is étale; it is in fact an isomorphism because H is simply connected. Therefore p_H has connected fibers. □

Proof of Theorem I(c). We assume $d \geqslant 2$, since the case $d = 1$ follows immediately. Suppose $n = 3$. Since $\mu \geqslant 1$, we have $d = 2$ or 3. For a quadric $X \subset \mathbb{P}^3$, there exists an open embedding $R_2(X) \to (\mathbb{P}^3)^\vee$ which sends each conic $C \in R_2(X)$ to the linear plane spanned by C; hence $R_2(X)$ is connected. The case $(d, n) = (3, 3)$ is excepted (see Proposition 4.4).

Next, assume $n \geqslant 4$. Then p_H has connected fibers due to Proposition 4.3. Suppose $p_H(Z) \neq H$. Then $p_H^{-1}(X)$ is smooth for general $X \in H$. Suppose $p_H(Z) = H$. Then it follows $\operatorname{codim}(p_H^{-1}(X) \cap Z, p_H^{-1}(X)) \geqslant 2$ for general $X \in H$. Thus $p_H^{-1}(X)$ is normal since it is regular in codimension 1 and a local complete intersection of I.

As a consequence, we find that $p_H^{-1}(X)$ is a normal irreducible variety for general X. Therefore $R_2(X)$ is connected since it is isomorphic to an open subset of $p_H^{-1}(X)$. $\square$

Finally, we check the sharpness of Theorem I(c). First we investigate the exceptional case $(d,n) = (3,3)$ in Theorem I(c). We find that $R_2(X)$ is disconnected for general X as follows.

Proposition 4.4. *Let $X \subset \mathbb{P}^3$ be a smooth cubic. Then $R_2(X)$ has 27 connected components. On the other hand,* $\mathrm{Hilb}^{2t+1}(X/k)$ *is connected.*

PROOF. Let $\{ l_i \}$ be the 27 lines lying in X, and let $l_i^* \subset (\mathbb{P}^3)^\vee$ be the set of linear planes containing l_i. Then $\{ l_i^* \}$ are lines in $(\mathbb{P}^3)^\vee$, and the union $\bigcup l_i^* \subset (\mathbb{P}^3)^\vee$ is a connected subvariety. We set

$$U = \Big(\bigcup l_i^*\Big) \setminus \Big(\bigcup_{i,j} l_i^* \cap l_j^*\Big) \subset (\mathbb{P}^3)^\vee,$$

which is a disconnected open subset of $\bigcup l_i^*$. For $C \in \mathrm{Hilb}^{2t+1}(X/k)$, the intersection of X and the linear plane spanned by C is equal to the union of C and l_i for some i. Thus we have an isomorphism $\mathrm{Hilb}^{2t+1}(X/k) \to \bigcup l_i^*$ by sending C to the linear plane spanned by C, which induces $R_2(X) \simeq U$. $\square$

Next we give a special X with $\mu \geqslant 1$ such that $R_2(X)$ is disconnected.

Example 4.5. For a general hypersurface $X' \subset \mathbb{P}^4$ of degree 5, the schemes $R_1(X')$ and $R_2(X')$ are finite sets, because the expected dimensions of these schemes are equal to zero in the case $(d,n) = (5,4)$. Let $\{ C_i' \}$ be the conics lying in X', and let $\{ l_j' \}$ be the lines lying in X'.

Let us consider $\pi_x : \mathbb{P}^5 \setminus \{ x \} \to \mathbb{P}^4$, a projection from a point $x \in \mathbb{P}^5$. We set $X = \overline{\pi_x^{-1}(X')} \subset \mathbb{P}^5$, the cone of X' with vertex x. For a conic $C \subset \mathbb{P}^5$, we obtain that $\pi_x(C)$ is a line or a conic. Thus $R_2(X)$ is isomorphic to the disjoint union of $R_2\big(\overline{\pi_x^{-1}(C_i')}\big)$ and $R_2\big(\overline{\pi_x^{-1}(l_j')}\big)$.

However, we can show the following result for the Hilbert scheme $\mathrm{Hilb}^{2t+1}(X/k)$, which contains $R_2(X)$ as an open subset.

Proposition 4.6. *Assume $\mu \geqslant 1$. Then* $\mathrm{Hilb}^{2t+1}(X/k)$ *is connected for any X if $n \geqslant 4$, and for any smooth X if $n = 3$.*

PROOF. Suppose $n \geqslant 4$. Then p_H has connected fibers due to Proposition 4.3. Since $\mathrm{Hilb}^{2t+1}(X/k) \simeq p_H^{-1}(X)$, the result follows. Suppose $n = 3$. In the case $(d,n) = (2,3)$, we have $\mathrm{Hilb}^{2t+1}(X/k) \simeq (\mathbb{P}^3)^\vee$. The case $(d,n) = (3,3)$ has been seen in Proposition 4.4. $\square$

CHAPTER II

Gauss map of rank zero

1. Bundles of principal parts

For a line bundle $\mathcal{L}$ on a projective variety X, we denote by $\mathcal{P}^1_X(\mathcal{L})$ the bundle of principal parts of $\mathcal{L}$ of first order ([**27**, §16], [**54**, §2]), which is equipped with a natural exact sequence,

$$0 \to \Omega^1_X \otimes \mathcal{L} \to \mathcal{P}^1_X(\mathcal{L}) \to \mathcal{L} \to 0 \qquad (\xi).$$

A generically surjective homomorphism $a^1 : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \otimes \mathcal{O}_X \to \mathcal{P}^1_X(\mathcal{O}_X(1))$ is associated to a projective variety X in $\mathbb{P}^N$. The Gauss map γ of X is formally defined to be the rational map $X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$ associated with a^1 by the universality of $\mathbb{G}(n, \mathbb{P}^N)$, where $n := \dim X$.

If a vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a_1)^{r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)^{r_m}$, then $[a_1^{r_1}, \ldots, a_m^{r_m}]$ is called the *splitting type* of $\mathcal{E}$. Note that, according to a theorem of A. Grothendieck ([**32**, V, Exercise 2.6]), every vector bundle on $\mathbb{P}^1$ splits into a direct sum of line bundles, as above. By abuse of notation, a vector bundle of splitting type $[a_1^{r_1}, \ldots, a_m^{r_m}]$ is denoted by the same symbol, for simplicity.

Lemma 1.1. *For a line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$ on $\mathbb{P}^1$, we have*

$$\mathcal{P}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(a)) = \begin{cases} [a, a-2], & \text{if } p | a, \\ [a-1^2], & \text{otherwize.} \end{cases}$$

Proof. See [**38**, (1.2)]. □

Proposition 1.2. *Let X be a projective variety, let $f : \mathbb{P}^1 \to X$ be an unramified morphism, and denote by N_f the dual of the kernel of the natural homomorphism $f^* : f^*\Omega^1_X \to \Omega^1_{\mathbb{P}^1}$. Assume that X is smooth along $f(\mathbb{P}^1)$, and $N_f^\vee = [-1^{r_{-1}}, 0^{r_0}, \ldots, i^{r_i}, \ldots]$. Then for an embedding $\iota : X \hookrightarrow \mathbb{P}^M$, we have*

$$f^*\mathcal{P}^1_X(\iota^*\mathcal{O}_{\mathbb{P}^M}(1)) = \begin{cases} [a-2, a-1^{r_{-1}}, a^{r_0+1}, a+1^{r_1}, a+2^{r_2}, \ldots, a+i^{r_i}, \ldots], & \text{if } p|a, \\ [a-1^{r_{-1}+2}, a^{r_0}, a+1^{r_1}, a+2^{r_2}, \ldots, a+i^{r_i}, \ldots], & \text{otherwise,} \end{cases}$$

where $a := \deg f^*\iota^*\mathcal{O}_{\mathbb{P}^M}(1)$.

Proof. This follows from Lemmas 1.1 and 1.3 below. □

Lemma 1.3. *With the same assumption as in Proposition 1.2, for a line bundle $\mathcal{L}$ on X, we have a natural, splitting exact sequence,*

$$0 \to N_f^\vee \otimes f^*\mathcal{L} \to f^*\mathcal{P}^1_X(\mathcal{L}) \to \mathcal{P}^1_{\mathbb{P}^1}(f^*\mathcal{L}) \to 0.$$

Proof. A homomorphism $f^*\mathcal{P}^1_X(\mathcal{L}) \to \mathcal{P}^1_{\mathbb{P}^1}(f^*\mathcal{L})$ is naturally induced, and is surjective by the assumption on f. Using the sequences (ξ) for $\mathcal{L}$ on X and for $f^*\mathcal{L}$ on $\mathbb{P}^1$, one obtains the exact sequence above, which splits since $\mathrm{Ext}^1(\mathcal{P}^1_{\mathbb{P}^1}(f^*\mathcal{L}), N_f^\vee \otimes f^*\mathcal{L}) = 0$ by Lemma 1.1 and the assumption on $N_f^\vee$. □

Proposition 1.4. *Let X be a projective variety, let $f : \mathbb{P}^1 \to X$ be a morphism, and assume that X is smooth along $f(\mathbb{P}^1)$. If X satisfies* (GMRZ)*, then the splitting type of $f^*\mathcal{P}^1_X(\iota^*\mathcal{O}_{\mathbb{P}^M}(1))$ is divisible by p.*

Proof. Denote by $\mathcal{Q}$ the universal quotient bundle of $H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1)) \otimes \mathcal{O}_{\mathbb{G}(n,\mathbb{P}^M)}$. Then, $\mathcal{P}^1_X(\iota^*\mathcal{O}_{\mathbb{P}^M}(1)) \simeq \gamma^*\mathcal{Q}$ locally around $f(\mathbb{P}^1)$ by the definition of the Gauss map, and one may assume that $\dim \gamma(f(\mathbb{P}^1)) = 1$: Indeed, if not, $f^*\mathcal{P}^1_X(\iota^*\mathcal{O}_{\mathbb{P}^M}(1))$ is trivial, and the conclusion is obvious. Let L' be the normalisation of $\gamma(f(\mathbb{P}^1))$, and let $\gamma' : \mathbb{P}^1 \to L'$ be the induced morphism from γ. Then it follows that

$$f^*\mathcal{P}^1_X(\iota^*\mathcal{O}_{\mathbb{P}^M}(1)) \simeq \gamma'^*\mathcal{Q}_{L'},$$

where $\mathcal{Q}_{L'}$ is the pull-back of $\mathcal{Q}$ to L'. Since $d\gamma$ is identically zero, so is $d\gamma'$; hence γ' has degree divisible by p. Since the splitting type of $f^*\mathcal{P}^1_X(\iota^*\mathcal{O}_{\mathbb{P}^M}(1))$ is equal to that of $\mathcal{Q}_{L'}$ over $L' \simeq \mathbb{P}^1$ multiplied by $\deg \gamma'$, the conclusion follows. □

Proof of Theorem II.1. According to Proposition 1.2, if both r_{-1} and r_0 were positive, then $a-1$ and a would be divisible by p by Proposition 1.4. If both r_0 and r_1 were positive, then a and $a+1$ would be divisible by p. Similarly for any $i \geq 2$, if both r_{i-1} and r_i were positive, then $a+i-1$ and $a+i$ would be divisible by p. Anyway this is a contradiction. Moreover, using Propositions 1.2 and 1.4, we see that if $r_{-1} > 0$, then $p|a-1$. If $r_0 > 0$, then $p|a-2$ and $p|a$; hence $p = 2$. Furthermore we see that $r_i > 0$ implies $p|i+1$ for any odd $i \geq 1$, and that $r_i > 0$ implies $p = 2$ or $p|i+1$ for any even $i \geq 2$. This completes the proof. □

2. Conormal bundles

Lemma 2.1. *Let L be a projective line in $\mathbb{P}^N$. Then we have:*

(a) $\Omega^1_{\mathbb{P}^N}|_L = [-2, -1^{N-1}]$.

(b) *a natural exact sequence,* $0 \to N^\vee_{L/\mathbb{P}^N} \to \Omega^1_{\mathbb{P}^N}|_L \to \Omega^1_L \to 0$ *splits.*

(c) $N^\vee_{L/\mathbb{P}^N} = [-1^{N-1}]$.

Proof. Restricting to L the Euler sequence on $\mathbb{P}^N$, $0 \to \Omega^1_{\mathbb{P}^N}(1) \to H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \otimes \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(1) \to 0$, we see that $\Omega^1_{\mathbb{P}^N}|_L = [-2, -1^{N-1}]$. Since $\mathrm{Hom}_{\mathbb{P}^1}(\Omega^1_{\mathbb{P}^N}|_L, \Omega^1_L) = \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2), \Omega^1_L) = K$, the surjection $\Omega^1_{\mathbb{P}^N}|_L \to \Omega^1_L$ splits. □

Lemma 2.2. *Let X be a projective variety in $\mathbb{P}^N$, let L be a projective line in X, and assume that X is smooth along L. Then we have:*

(a) *a natural exact sequence,* $0 \to N^\vee_{L/X} \to \Omega^1_X|_L \to \Omega^1_L \to 0$ *splits.*

(b) $N^\vee_{L/X} = [a_1, \dots, a_r]$ *with $a_j \geq -1$ for any j, that is, $N^\vee_{L/X}(1)$ is spanned.*

Proof. Since $\Omega^1_{\mathbb{P}^N}|_L \to \Omega^1_L$ factors through $\Omega^1_X|_L$, the assertion (a) follows from Lemma 2.1(b). Since $\Omega_{\mathbb{P}^N}|_L \to \Omega_X|_L$ is surjective, so is $N^\vee_{L/\mathbb{P}^N} \to N^\vee_{L/X}$; hence (b) follows. □

Lemma 2.3. *Let X be a projective variety with a morphism π onto a variety Y, let y be a smooth point of Y, and assume that π is smooth along the fibre $X_y := \pi^{-1}(y)$. Then $N_{X_y/X} = [0^m]$, where $m := \dim Y$.*

PROOF. By the assumption we have natural exact sequences of vector bundles, $0 \to T_{X_y} \to T_X|_{X_y} \to N_{X_y/X} \to 0$ and $0 \to T_{X/Y}|_{X_y} \to T_X|_{X_y} \to \pi^* T_Y|_{X_y} \to 0$. Comparing these sequences via the canonical isomorphism $T_{X_y} \simeq T_{X/Y}|_{X_y}$, we see that $N_{X_y/X} \simeq \pi^* T_Y|_{X_y}$, which is isomorphic to a trivial bundle $t_{Y,y} \otimes_K \mathcal{O}_{X_y}$, where $t_{Y,y}$ is the Zariski tangent space to Y at y. □

Lemma 2.4. *With the same assumption as in Lemma 2.3, assume moreover that the fibre X_y is isomorphic to a projective space $\mathbb{P}^l$, and let L be a projective line in X_y. Then we have $N^\vee_{L/X} = [-1^{l-1}, 0^m]$.*

PROOF. It follows from Lemmas 2.1(c) and 2.3 that a natural exact sequence, $0 \to N_{L/X_y} \to N_{L/X} \to N_{X_y/X}|_L \to 0$ splits; hence the conclusion follows. □

Lemma 2.5. *Let X be a Grassmann variety $\mathbb{G}(l, l+m)$ of l-dimensional subspaces of an $(l+m)$-dimensional vector space $(l, m \geq 1)$, and let L be a projective line in X via the Plücker embedding. Then we have $N^\vee_{L/X} = [-1^{l+m-2}, 0^{(l-1)(m-1)}]$.*

PROOF. Let $0 \to \mathcal{S} \to \mathcal{O}_X^{l+m} \to \mathcal{Q} \to 0$ be the natural exact sequence on $X = \mathbb{G}(l, l+m)$, with a universal sub-bundle $\mathcal{S}$ of rank l and a universal quotient $\mathcal{Q}$ of rank m. Restricting to L, we see that $\mathcal{Q}|_L = [0^{m-1}, 1]$ since $\mathcal{Q}|_L$ is spanned and $\deg \mathcal{Q}|_L = \deg L = 1$. Taking the dual of the sequence above, we obtain $\mathcal{S}|_L = [-1, 0^{l-1}]$ as well. Using a well-known fact $\Omega^1_X \simeq \mathcal{Q}^\vee \otimes \mathcal{S}$ ([**26**, (1.10)]), we have $\Omega^1_X|_L = [-2, -1^{(l-1)+(m-1)}, 0^{(l-1)(m-1)}]$; hence the conclusion follows from Lemma 2.2(a). □

Lemma 2.6. *Let X be a smooth quadric hypersurface in $\mathbb{P}^N$ $(N \geq 3)$, and let L be a projective line in X. Then we have $N^\vee_{L/X} = [-1^{N-3}, 0]$.*

PROOF. Restricting to L a natural exact sequence, $0 \to N^\vee_{X/\mathbb{P}^N}(1) \to \Omega^1_{\mathbb{P}^N}(1)|_X \to \Omega^1_X(1) \to 0$, we see that $\deg \Omega^1_X(1)|_L = 0$ by Lemma 2.1(a) and $N_{X/\mathbb{P}^N} \simeq \mathcal{O}_X(2)$; hence $\deg N^\vee_{L/X}(1) = 1$ by the sequence, $0 \to N^\vee_{L/X}(1) \to \Omega^1_X(1)|_L \to \Omega^1_L(1) \to 0$. According to Lemma 2.2(b), $N^\vee_{L/X}(1)$ is spanned; hence we have $N^\vee_{L/X}(1) = [0^{N-3}, 1]$. □

Lemma 2.7. *Let X be a smooth cubic hypersurface in $\mathbb{P}^N$ $(N \geq 3)$, and let L be a projective line in X. Then we have $N^\vee_{L/X} = [-1^{N-3}, 1]$ or $[-1^{N-4}, 0^2]$.*

PROOF. Similarly to the quadric case above, we see that $N^\vee_{L/X}(1)$ is spanned of degree 2. Therefore $N^\vee_{L/X}(1)$ is either $[0^{N-3}, 2]$ or $[0^{N-4}, 1^2]$. □

Example 2.8. Let X be an n-fold product $(\mathbb{P}^1)^n$ of $\mathbb{P}^1$ in $p = 2$, set

$$I_k := \{(a_1, \dots, a_n) \in \{0, 1, 2\}^n | \#\{j | a_j = 1\} = k\},$$

and let $\iota : X \dashrightarrow \mathbb{P}^M$ be a rational map defined by

$$(1 : y_1) \times \cdots \times (1 : y_n) \mapsto (y_1^{a_1} \cdots y_n^{a_n})_{(a_1, \dots, a_n) \in I_0 \cup I_1},$$

where $M+1 = 2^{n-1}(n+2)$. Then by a direct computation as in [**20**, Proof of Proposition] one can verify that ι gives an embedding of X with Gauss map of rank zero; hence $(\mathbb{P}^1)^n$ in $p=2$ satisfies (GMRZ).

Proof of Theorem II.2. Each only-if-part of (1–4) follows from Lemmas 2.4, 2.5, 2.6, 2.7 and Theorem II.1, where we note that every X in question contains a projective line L. The if-parts of a and c follow from Example 2.8, and that of b follows from [**21**, Example 3.1]. □

3. Absence of minimal free rational curves

Proof of Theorem II.3. It follows from [**45**, IV, 2.11] that a minimal free f is unramified. Theorem II.1 implies $N_f^\vee = [-1^{n-1}]$ or $[0^{n-1}]$.

Suppose $N_f^\vee = [-1^{n-1}]$. Then we have $\deg(-f^*K_X) = n+1$, and it follows from Theorem II.1 that $p \mid a-1$. We show $a \neq 1$ as follows: Assume $a = 1$, and identify X with $\iota(X) \subseteq \mathbb{P}^M$. Then $L := f(\mathbb{P}^1)$ is a line in $\mathbb{P}^M$. We fix a point $x = f(o) \in L$ with $o \in \mathbb{P}^1$, where x is a smooth point of X. Since $h^1((f^*T_X)(-1)) = 0$, it follows from [**45**, II, 1.7] that $\mathrm{Hom}(\mathbb{P}^1, X; o \mapsto x)$ is smooth at f. For an irreducible component $V \subseteq \mathrm{Hom}(\mathbb{P}^1, X; o \mapsto x)$ containing f, we consider the evaluation morphism $F : \mathbb{P}^1 \times V \to X$. Since $f^*T_X = [2, 1^{n-1}]$, it follows from [**45**, II, 3.10] that $\mathrm{rk}\, d_{(o,f)}F = n$; hence F is dominant. On the other hand, setting $E := F^*\mathcal{O}_{\mathbb{P}^M}(1)$, we see from [**45**, II. 3.9.2] that the image of a morphism $g \in V$ is a line in X passing through x, which implies that X is a cone with vertex x. Since X is non-linear by our convention, X is singular at x. Thus we reach a contradiction.

If $N_f^\vee = [0^{n-1}]$, then we have $\deg(-f^*K_X) = 2$; hence it follows from Proposition 1.2 that $p = 2$ and $p \mid a$. □

Remark 3.1. Both cases (1–2) in Theorem II.3 actually occur:

(a) According to [**21**, Example 3.1], $\mathbb{P}^n$ satisfies (GMRZ), and we have $T_{\mathbb{P}^n}|_L = [1^{n-1}, 2]$ for each line $L \subset \mathbb{P}^n$.

(b) Let $X = (\mathbb{P}^1)^n$ with $p = 2$, which satisfies (GMRZ) by Example 2.8. Let us consider an embedding $f : \mathbb{P}^1 \to X$ such that $f(\mathbb{P}^1)$ is a product of $\mathbb{P}^1$ and a point in $(\mathbb{P}^1)^{n-1}$. Then f is minimal free with $f^*T_X = [0^{n-1}, 2]$.

Theorem 3.2. *Assume $p > 0$, and let X be a Fermat hypersurface of degree $ep+1$ in $\mathbb{P}^N$ with $e \in \mathbb{N}$. Then X satisfies* (GMRZ)*, and we have:*

(a) *X has no minimal free line, or equivalently, no free line.*

(b) *If $N > e(p+1)$, then X has no minimal free rational curve.*

(c) *If $N \geqslant 2ep+1$, then X has a free $f : \mathbb{P}^1 \to X$ with* $\deg f^*\mathcal{O}_X(1) = ep$.

Thus a Fermat hypersurface $X \subseteq \mathbb{P}^N$ of degree $ep+1$ with $N \geqslant 2ep+1$ gives a counter-example for Theorem A in each characteristic $p > 0$.

Lemma 3.3. *Let X be as in Theorem 3.2, suppose that X has a minimal free $f : \mathbb{P}^1 \to X$, and set $a := \deg f^*\mathcal{O}_X(1)$. Then one of the following hold:*

(a) *$\deg(-f^*K_X) = N$, and there exist positive integers e', e'' such that $e = e'e''$, $a = e'p+1 \geqslant 3$ and $N = ep + e''$.*

(b) $\deg(-f^*K_X) = p = a = 2$ *and* $N = 2e+1$.

Proof. Since $-K_X = \mathcal{O}_X(N-ep)$, we have $\deg(-f^*K_X) = a(N-ep)$. Applying Theorem II.3, we see that one of the statements (1–2) there holds.

If $\deg(-f^*K_X) = N$ and $e'p = a-1$ with $e' \geqslant 1$, then it follows $N = aep/(a-1) = ep + e/e'$. Here we have $e' \mid e$, and set $e'' := e/e'$. If $\deg(-f^*K_X) = p = 2$ and $2a' = a$ with $a' \geqslant 1$, then we have $2a'(N-2e) = 2$; hence $a'(N-2e) = 1$, which implies $a' = 1$ and $N = 2e+1$. □

Proof of Theorem 3.2. (a) The result follows immediately from Theorem II.3.

(b) In the case of $N > e(p+1)$, neither (a) nor (b) in Lemma 3.3 occurs.

(c) Set $F := x_0^{ep+1} + x_1^{ep+1} + \cdots + x_N^{ep+1}$, and assume that X is defined by $F = 0$. Let us consider a morphism,

$$f : \mathbb{P}^1 \to \mathbb{P}^N : (s:t) \mapsto (s^{ep} : s^{ep-1}t : \cdots : t^{ep} : \xi s^{ep} : \xi s^{ep-1}t : \cdots : \xi t^{ep} : 0 : \cdots : 0),$$

with $\xi^{ep+1} + 1 = 0$ ($\xi \in K$), and set $C := f(\mathbb{P}^1)$. Then C is smooth and contained in X.

To prove that f is free, we show $H^1(\mathbb{P}^1, f^*N_{C/X} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))) = 0$. From a natural exact sequence, $0 \to f^*N_{C/X} \to f^*N_{C/\mathbb{P}^N} \xrightarrow{\varepsilon} f^*N_{X/\mathbb{P}^N} \simeq \mathcal{O}_{\mathbb{P}^1}(ep(ep+1)) \to 0$, we obtain an exact sequence,

$$\begin{aligned} H^0(\mathbb{P}^1, f^*N_{C/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) &\xrightarrow{H^0(\varepsilon\otimes\mathcal{O}_{\mathbb{P}^1}(-1))} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(ep(ep+1)-1)) \\ &\to H^1(\mathbb{P}^1, f^*N_{C/X} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \to H^1(\mathbb{P}^1, f^*N_{C/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)). \end{aligned}$$

Since $H^1(\mathbb{P}^1, f^*N_{C/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ by $f^*N_{C/\mathbb{P}^N} = [ep^{N-ep}, ep+2^{ep-1}]$ [**38**, (3.5)], it is sufficient to show that $H^0(\varepsilon \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ is surjective. Set $V := H^0(\mathbb{P}^N, \mathcal{O}(1))$. From the Euler sequence we obtain a diagram as follows:

$$\begin{array}{ccc} V \otimes f^*\mathcal{O}_{\mathbb{P}^N}(1) \simeq V \otimes \mathcal{O}_{\mathbb{P}^1}(ep) & \longrightarrow & f^*T_{\mathbb{P}^N} \\ & & \downarrow \\ & & f^*N_{C/\mathbb{P}^N} \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}^1}(ep(ep+1)), \end{array}$$

and a composite map of the maps above, denoted by $\tilde{\varepsilon}$, is given explicitly by

$$(f^*(\partial F/\partial x_i))_{i=0}^N = (f^*x_0^{ep}, f^*x_1^{ep}, \ldots, f^*x_N^{ep}) = (s^{(ep)^2}, s^{ep(ep-1)}t^{ep}, \ldots, t^{(ep)^2}, \ldots).$$

Therefore an induced K-linear map,

$$H^0(\tilde{\varepsilon} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) : V \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(ep-1)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(ep(ep+1)-1)),$$

is surjective, so is $H^0(\varepsilon \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$. □

Remark 3.4. Let X be a Fermat hypersurface of degree p^r+1 in $\mathbb{P}^N$. It follows from [**15**, pp. 50–51] that $N_{L/X} = [1-p^r, 1^{N-3}]$ for each line $L \subseteq X$, from which one can deduce that Theorem 3.2(a) holds for this X.

Remark 3.5. For a Fermat cubic surface X in $\mathbb{P}^3$ with $p=2$, both cases (a–b) in Lemma 3.3 (hence in Theorem II.3) actually occur: First we have $-K_X = \mathcal{O}_X(1)$. For a twisted cubic curve $C_3 \subseteq X$ with a parametrisation $f_3 : \mathbb{P}^1 \to X$, we have

$\deg(-f_3^*K_X) = 3$; hence (a) occurs with f_3. For a conic $C_2 \subseteq X$ with a parametrisation $f_2 : \mathbb{P}^1 \to C_2$, we have $\deg(-f_2^*K_X) = 2$; hence (b) occurs with f_2.

4. General conics on general hypersurfaces

First we have the following:

Proposition 4.1. *A general hypersurface X in $\mathbb{P}^N$ of degree d with $3 \leqslant d \leqslant 2N-3$ satisfies* (GMRZ) *only if $p = 2$ and either $d = 2N-3$ or $d = N-1$.*

Proof. From [**45**, V, (4.4.2)], for a general line $L \subseteq X$, we have

$$N_{L/X}^\vee = \begin{cases} [0^{2N-3-d}, 1^{d-N+1}], & \text{if } N-1 \leqslant d \leqslant 2N-3, \\ [-1^{N-1-d}, 0^{d-1}], & \text{if } d \leqslant N-1. \end{cases}$$

Hence Theorem II.1 implies either $d = 2N-3$ or $d = N-1$. If $d = 2N-3$ (resp. $d = N-1$), then it follows $r_1 > 0$ (resp. $r_0 > 0$); hence we have $p = 2$ as well. □

To complete the proof of Theorem II.4, we study normal bundles of general conics on X. Let $\mathcal{R}$ be the set of (irreducible reduced) conics in $\mathbb{P}^N$. Here $\mathcal{R}$ is an open subvariety of $\mathrm{Hilb}^{2t+1}(\mathbb{P}^N/K)$, the Hilbert scheme attached to the Hilbert polynomial $2t+1$. For an integer $d \geqslant 1$, we set $\mathcal{H} := |\mathcal{O}_{\mathbb{P}^N}(d)|$, and

$$I := \{\, (X, C) \in \mathcal{H} \times \mathcal{R} \mid C \subseteq X \,\},$$

which is a projective space bundle over $\mathcal{R}$, with projections $p_{\mathcal{H}} : I \to \mathcal{H}$ and $p_{\mathcal{R}} : I \to \mathcal{R}$. We moreover set $I^0 := \{\, (X, C) \in I \mid X \text{ is smooth along } C \,\}$, and

$$\mu_\xi := 3N - 2d - 2 + (N-2)\xi,$$

where we note that $\mu_\xi = \chi(f^*N_{C/X} \otimes \mathcal{O}_{\mathbb{P}^1}(\xi))$ for any $(X, C) \in I^0$.

Fix a conic C, and take an embedding $f : \mathbb{P}^1 \to \mathbb{P}^N$ with $f(\mathbb{P}^1) = C$. From the exact sequence, $0 \to \mathcal{I}_C^2 \to \mathcal{I}_C \to N_{C/\mathbb{P}^N}^\vee \to 0$ on $\mathbb{P}^N$, we obtain the following K-linear map,

$$\delta_C : H^0(\mathbb{P}^N, \mathcal{I}_C(d)) \to \mathsf{D} := \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(f^*N_{C/\mathbb{P}^N}, f^*\mathcal{O}_{\mathbb{P}^N}(d)),$$

which gives each $X \in p_{\mathcal{H}}(p_{\mathcal{R}}^{-1}(C) \cap I^0)$ a natural homomorphism of normal bundles,

$$\delta_C(X) : f^*N_{C/\mathbb{P}^N} \to f^*N_{X/\mathbb{P}^N} \simeq f^*\mathcal{O}_{\mathbb{P}^N}(d).$$

In addition, we have a decomposition, $f^*N_{C/X} = \bigoplus_{i=1}^{N-2} \mathcal{O}_{\mathbb{P}^1}(b_i(C/X))$ for some integers $b_i(C/X)$ determined by $(X, C) \in I^0$. Then, we set

$$I_{[\geqslant\xi]} := \{\, (X, C) \in I^0 \mid \min\{\, b_i(C/X) \,\} \geqslant \xi \,\},$$
$$I_{[\leqslant\xi]} := \{\, (X, C) \in I^0 \mid \max\{\, b_i(C/X) \,\} \leqslant \xi \,\},$$

where we note that $I_{[\geqslant\xi]}$ (resp. $I_{[\leqslant\xi]}$) is an open subset of I by virtue of the upper semi continuity of $-\min\{\, b_i(C/X) \,\}$ (resp. $\max\{\, b_i(C/X) \,\}$) for (X, C) ([**45**, II, (3.9.2)]).

Lemma 4.2. *The K-linear map δ_C is surjective.*

Proof. This follows from Chapter I, Proposition 2.13. □

Lemma 4.3. *The morphism $p_{\mathcal{H}}$ is smooth on the open subset $I_{[\geqslant -1]} \subseteq I$.*

Proof. This follows from Chapter I, Lemma 2.7 □

Proposition 4.4. (a) $I_{[\geqslant \xi]}$ *is not empty if* $\mu_{-\xi-1} \geqslant 0$ *and* $\xi \leqslant 2$.
(b) $I_{[\leqslant \xi]}$ *is not empty if* $\mu_{-\xi-1} \leqslant 0$ *and* $\xi \leqslant 2d$.

Proof. (a) Assume $\xi \leqslant 2$, and fix $C \in \mathfrak{R}$. Since $h^1(f^*N_{C/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1)) = 0$, it follows that for $X \in p_{\mathcal{H}}(p_{\mathfrak{R}}^{-1}(C))$, $\min(b_i(C/X)) \geqslant \xi$ (*i.e.*, $h^1(f^*N_{C/X} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1)) = 0$) if and only if the K-linear map,

$$H^0(\mathbb{P}^1, f^*N_{C/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1)) \xrightarrow{H^0(\delta_C(X) \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1))} H^0(\mathbb{P}^1, f^*N_{X/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1))$$

is surjective. Since $h^0(f^*N_{C/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1)) - h^0(f^*N_{X/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1)) = \mu_{-\xi-1} \geqslant 0$, there exists a homomorphism $\alpha \in \mathsf{D}$ such that $H^0(\alpha \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1))$ is surjective: Indeed, taking account of $f^*N_{C/\mathbb{P}^N} = f^*N_{C/Y} \oplus f^*N_{Y/\mathbb{P}^N}|_C = [4, 2^{N-2}]$ with $Y = \langle C \rangle \subseteq \mathbb{P}^N$, one can easily verify the surjectivity, by writing α explicitly in terms of an $(N-1)$-tuple of homogeneous polynomials in s, t, where $(s:t)$ is a system of homogeneous coordinates of $\mathbb{P}^1$. It follows from Lemma 4.2 that there exists $X \in \mathcal{H}$ such that $\delta_C(X) = \alpha$, which implies $(X, C) \in I_{[\geqslant \xi]}$.

(b) Assume $\xi \leqslant 2d$, and fix $C \in \mathfrak{R}$. For $X \in p_{\mathcal{H}}(p_{\mathfrak{R}}^{-1}(C))$, since $h^1(f^*N_{X/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1)) = 0$, it follows that $\max\{b_i(C/X)\} \leqslant \xi$ (*i.e.*, $h^0(f^*N_{C/X} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1)) = 0$) if and only if $H^0(\delta_C(X) \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1))$ is injective. As in (a), $h^0(f^*N_{X/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1)) - h^0(f^*N_{C/\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1)) \geqslant -\mu_{-\xi-1} \geqslant 0$ implies the existence of $\alpha \in \mathsf{D}$ such that $H^0(\alpha \otimes \mathcal{O}_{\mathbb{P}^1}(-\xi-1))$ is injective, hence of $X \in \mathcal{H}$ with $\delta_C(X) = \alpha$ by Lemma 4.2, so that $(X, C) \in I_{[\leqslant \xi]}$. □

Corollary 4.5. *Assume* $\mu_0 = 3N - 2d - 2 \geqslant 0$. *Then for a general hypersurface* X *in* $\mathbb{P}^N$ *of degree* d, *there exists a conic* C *lying in* X. *Moreover for a general conic* $C \subseteq X$, *we have:*
(a) $\max\{ b_i(C/X) \} \leqslant 1$ *if* $\mu_{-2} = \mu_0 - 2(N-2) \leqslant 0$,
(b) $\min\{ b_i(C/X) \} \geqslant 0$ *if* $\mu_{-1} = \mu_0 - (N-2) \geqslant 0$.
Hence if $N - 2 \leqslant \mu_0 \leqslant 2(N-2)$ (*i.e.*, $-N/2 + N + 1 \leqslant d \leqslant N$), *then* $f^*N^{\vee}_{C/X} = [-1^{2(N-d)}, 0^{N-2-2(N-d)}]$.

Proof. Since $\mu_0 \geqslant 0$, it follows from Lemma 4.3 and Proposition 4.4(a) that the morphism $p_{\mathcal{H}}$ is smooth on the non-empty open subset $I_{[\geqslant -1]}$. In particular $p_{\mathcal{H}}$ is dominant; hence we find $C \in p_{\mathfrak{R}}(p_{\mathcal{H}}^{-1}(X))$ if $X \in \mathcal{H}$ is general. Assume $\mu_{-2} \leqslant 0$. Then it follows from Proposition 4.4(b) that $I_{[\leqslant 1]}$ is non-empty. Since $p_{\mathcal{H}}(I_{[\leqslant 1]})$ is dense in $\mathcal{H}$, we have a conic $C \in p_{\mathfrak{R}}(p_{\mathcal{H}}^{-1}(X) \cap I_{[\leqslant 1]})$. Hence the statement of (a) is proved. The statement of (b) follows in a similar way. □

Proof of Theorem II.4. Let $X \subseteq \mathbb{P}^N$ be a general hypersurface of degree $d \geqslant 3$ such that X satisfies (GMRZ). From Proposition 4.1, it is sufficient to show that the case of $p = 2$ and $d = N - 1$ does not occur.

Assume $p = 2$ and $N = d + 1 \geqslant 4$. It follows from Corollary 4.5 that $f^*N^{\vee}_{C/X} = [-1^2, 0^{N-4}]$ for a general conic $C \subseteq X$. Hence Theorem II.1 implies $N = 4$ and $2|a-1$, where we set $a := \deg f^*\iota^*\mathcal{O}_{\mathbb{P}^M}(1)$ for an embedding $\iota : X \hookrightarrow \mathbb{P}^M$ with Gauss map of rank zero. On the other hand, from the Lefschetz theorem [**32**, III, Exercise 11.6 (c)], it follows $\operatorname{Pic} X = \operatorname{Pic} \mathbb{P}^4$ for $X \subseteq \mathbb{P}^4$; hence a is divisible by $2 = \deg f^*(\mathcal{O}_{\mathbb{P}^4}(1)|_X)$. This is a contradiction. □

5. Characterisation of a cubic hypersurface with (GMRZ)

Let X be a smooth cubic hypersurface in $\mathbb{P}^N$ with $N \geq 3$. We denote the Gauss map of $X \subseteq \mathbb{P}^N$ by $\gamma_0 : X \to \mathbb{G}(N-1, \mathbb{P}^N) = \check{\mathbb{P}}^N$. Let $F_x \subseteq \mathbb{G}(1, \mathbb{P}^N)$ be the algebraic set which parametrises lines in X passing through a point $x \in X$, and set

$$Y_x := \bigcup_{L \in F_x} L \subseteq X.$$

We will characterise a smooth cubic hypersurface with (GMRZ). First of all, we note the following:

Lemma 5.1. *If L is a projective line in X with $N^\vee_{L/X} = [-1^{N-3}, 1]$, then the image $\gamma_0(L)$ is a projective line in $\check{\mathbb{P}}^N$ with $\gamma_0^*\mathcal{O}_{\check{\mathbb{P}}^N}(1)|_L \simeq \mathcal{O}_L(2)$.*

Proof. For the Gauss map γ_0, we have an exact sequence

$$0 \to \gamma_0^*\Omega^1_{\check{\mathbb{P}}^N}(1)|_L \to H^0(\check{\mathbb{P}}^N, \mathcal{O}_{\check{\mathbb{P}}^N}(1)) \otimes \mathcal{O}_L \to \gamma_0^*\mathcal{O}_{\check{\mathbb{P}}^N}(1)|_L \to 0$$

by the Euler sequence on $\check{\mathbb{P}}^N$. We consider their global sections:

$$0 \to H^0(L, \gamma_0^*\Omega^1_{\check{\mathbb{P}}^N}(1)|_L) \to H^0(\check{\mathbb{P}}^N, \mathcal{O}_{\check{\mathbb{P}}^N}(1)) \xrightarrow{\tau} H^0(L, \gamma_0^*\mathcal{O}_{\check{\mathbb{P}}^N}(1)|_L).$$

Then, the restriction $\gamma_0|_L$ to L is corresponding to the linear system defined by the image of τ. Since $\mathcal{P}^1_X(\mathcal{O}_X(1)) \simeq \gamma_0^*\Omega^1_{\check{\mathbb{P}}^N}(1)^\vee$, it follows from Lemmas 1.1, 1.3 and 2.1(c) that $\gamma_0^*\Omega^1_{\check{\mathbb{P}}^N}(1)|_L = [0^{N-1}, -2]$. Hence, $\gamma_0^*\mathcal{O}_{\check{\mathbb{P}}^N}(1)|_L$ has degree 2 and τ rank 2, which implies that $\gamma_0(L)$ is a projective line. □

Proposition 5.2. *We assume that $N \geq 5$, $p = 2$ and $N^\vee_{L/X} = [-1^{N-3}, 1]$ for any projective line $L \subseteq X$. Then, the Gauss map γ_0 of X in $\mathbb{P}^N$ is of rank zero.*

Proof. A standard dimension-counting argument shows that for any $x \in X$, every irreducible component of F_x has dimension at least $N - 4$ if $N \geq 4$ ([**45**, V.4.6.1]); hence we have

$$\dim Y'_x \geq N - 3, \tag{27}$$

for every irreducible component Y'_x of Y_x. Denote by $d_x\gamma_0$ the differential of γ_0 at $x \in X$, by r the rank of $d_x\gamma_0$ for a general x, and let U be the open subset of X such that $d_x\gamma_0$ has rank r for any $x \in U$.

Suppose that the rank r is not zero. Then, we have $r \geq 2$: Indeed, $d_x\gamma_0$ is given by a certain Hessian matrix ([**42**, (3.3.15)]), which is skew-symmetric since it is symmetric and the diagonal elements are all zero in $p = 2$; hence r must be even ([**9**, §5, n°1, Corollaire 3]). We define M_x to be the linear subspace in $\mathbb{P}^N$ containing x such that its Zariski tangent space at x coincides with the kernel of $d_x\gamma_0$. For any $x \in U$, since $d_x\gamma_0$ has kernel of dimension at most $N - 3$ by $r \geq 2$, we have

$$\dim M_x \leq N - 3. \tag{28}$$

Now, assume that $Y_x \nsubseteq M_x$ for some $x \in X$. Then, by the definition of M_x, there exists a line $L \in F_x$ such that the restriction $\gamma_0|_L$ is unramified at x. Moreover it follows from Lemma 5.1 that $\gamma_0|_L$ has separable degree 2. Therefore, $\gamma_0(x_L) = \gamma_0(x)$ for some point $x_L \in L \setminus \{x\}$. Since we see from 27 that such a line L is movable in Y_x if $N \geq 5$, there exist infinitely many $x_L \in X$ with $\gamma_0(x_L) = \gamma_0(x)$. On the other hand, γ_0 is finite since X is a smooth hypersurface. This is a contradiction.

Thus $Y_x \subseteq M_x$ for any $x \in X$; hence, it follows from 27 and 28 that

$$M_x = Y_x$$

for any point $x \in U$, which is linear of dimension $N - 3$ and contained in X. Then, by Lemma 5.3 below, we obtain a contradiction if $N = 5$.

For the case $N \geq 6$, one can easily deduce a contradiction from the above, as follows: Since $M_x \subseteq T_y X$ for any $y \in M_x$ by the linearity of M_x, we have

$$\gamma_0(M_x) \subseteq M_x^*$$

in $\check{\mathbb{P}}^N$, where M_x^* denotes the set of all hyperplanes containing M_x. This is a contradiction to the finiteness of γ_0 when $N \geq 6$: Indeed, we have $\dim M_x^* = 2 < N - 3 = \dim M_x$. □

Lemma 5.3. *For a smooth cubic hypersurface X in $\mathbb{P}^5$ (in arbitrary characteristic), there does not exist a non-empty open subset U of X such that Y_x is a linear space of dimension 2 for any point $x \in U$.*

Proof. Assume that there exists such U. Firstly, for any $x \in U$ and for any $y \in Y_x \cap U$, we have:

$$\text{(29)} \qquad \text{(a)} \quad \gamma_0(Y_x) = Y_x^*, \qquad \text{(b)} \quad Y_y = Y_x.$$

Indeed, since $Y_x \subseteq X$ is linear of dimension 2, we have $\gamma_0(Y_x) \subseteq Y_x^*$ and $\dim Y_x^* = 2$; hence (a) follows from the finiteness of γ_0. Next we have $Y_x \subseteq Y_y$: Indeed, if $z \in Y_x$, then the line $\overline{yz}$ passing through y and z is contained in $Y_x \subseteq X$; hence $z \in \overline{yz} \subseteq Y_y$. Then, (b) follows from $\dim Y_x = \dim Y_y$.

We note secondly the following elementary fact: If X is an irreducible hypersurface in $\mathbb{P}^N$, then for a smooth point x of X and for a hyperplane H in $\mathbb{P}^N$, we have

$$\text{(30)} \qquad H = T_x X \Leftrightarrow x \in \mathrm{Sing}(H \cap X).$$

Since X is smooth cubic and γ_0 is finite, it follows from 30 that for each $x \in X$,

$$Z_x := X \cap T_x X$$

is an irreducible cubic hypersurface in $T_x X \simeq \mathbb{P}^4$ with only finitely many singular points. Denote by γ_{Z_x} the Gauss map of Z_x, which satisfies $\gamma_{Z_x}(y) = T_x X \cap T_y X$ for each smooth point y of Z_x. Then the image $\gamma_{Z_x}(Y_x)$ has positive dimension by (29.a): Indeed, if $\dim \gamma_{Z_x}(Y_x) = 0$, then $T_x X \cap T_y X$ would be a fixed linear space of dimension 3 for any smooth point y of Z_x contained in Y_x, hence for a general $y \in Y_x$. But, $\dim \gamma_0(Y_x) = 2$ by (29.a). This is a contradiction.

For $x \in U$ and $y \in Y_x \cap U$ with $T_y X \neq T_x X$, set

$$Z_{xy} := Z_x \cap T_y X = X \cap T_x X \cap T_y X.$$

Since $\dim \gamma_{Z_x}(Y_x) > 0$, it follows from 30 that Z_{xy} is smooth at a general point of $Y_x \subseteq Z_{xy}$. Therefore, we have a decomposition,

$$Z_{xy} = Q \cup Y_x$$

with some quadric hypersurface Q in $T_xX \cap T_yX \simeq \mathbb{P}^3$ such that $Q \not\supseteq Y_x$ as sets. Since $x \in \operatorname{Sing} Z_x$, we have $x \in \operatorname{Sing} Z_{xy}$; hence $x \in Q \cap Y_x$. This implies that Q is irreducible, reduced and singular: Indeed, if Q is not irreducible or not reduced, then Q is a union of linear spaces or a linear space as a set, hence there exists a line in Q passing through x but not contained in Y_x. This contradicts the definition of Y_x. Thus Q is irreducible and reduced. Moreover if Q is smooth, then we have a decomposition, $Q \cap Y_x = L_1 \cup L_2$ with two lines $L_1 \neq L_2$ satisfying $L_1 \cap L_2 = \{x\}$: Indeed, there exist exactly two lines contained in Q passing through x, which must be contained also in Y_x by its definition. Now, it follows from (29.b) that $y \in Q \cap Y_y = Q \cap Y_x$. Applying the same argument above to y, we have $\{y\} = L_1 \cap L_2 = \{x\}$. This is a contradiction.

Thus we may assume that Q is a cone over a conic C with a vertex z. Here we see that $z \neq x$ by the same argument above: Indeed, there exists a line in Q passing through z but not contained in Y_x. Therefore we may assume moreover that $x \in C$; hence, $C \cap U$ is non-empty. If $w \in C \cap U$, then $Y_w \subseteq T_zX$: Indeed, Y_w is linear and $z \in Y_w$ by $\overline{wz} \subseteq Q \subseteq X$. Set

$$W := \bigcup_{w \in C \cap U} Y_w \subseteq X \cap T_zX.$$

Since $X \cap T_zX$ is irreducible, the closure $\overline{W}$ of W coincides with $X \cap T_zX$. If we consider the projection $\pi_z : T_xX \simeq \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from z, then we see that $\overline{W}$ is a cone over a cubic surface $\pi_z(W \setminus \{z\})^- \subseteq \mathbb{P}^3$ with vertex z: Indeed, Y_w is a linear space containing z. Moreover, $\pi_z(W \setminus \{z\})^-$ is singular, because $\pi_z(W \setminus \{z\})$ contains infinitely many lines by $\dim Y_w = 2$. Therefore the singular locus of $\overline{W} = X \cap T_zX$ has dimension at least 1; hence by 30, this contradicts the finiteness of γ_0. ☐

Theorem 5.4. *Let X be a smooth cubic hypersurface in $\mathbb{P}^N$ with $N \geq 3$ in $p = 2$. Then, the Gauss map γ_0 of $X \subseteq \mathbb{P}^N$ is of rank zero if and only if X is projectively equivalent to the Fermat cubic hypersurface.*

Proof. The if-part is easily verified by a direct computation ([**42**, Exemple 3.4]). We prove the only-if part. Let F be a homogeneous cubic polynomial defining X, and denote the partial derivatives as follows: $F_i := \partial F / \partial x_i$, $F_{ij} := (F_i)_j = \partial^2 F / \partial x_i \partial x_j$, and so on. To prove the assertion it suffices to show

$$F_{ij} = 0, \tag{31}$$

as a polynomial for any i, j: Indeed, 31 implies that there exist linear polynomials $L_0, \dots, L_N$ such that

$$F = x_0 L_0^2 + \cdots + x_N L_N^2.$$

According to an argument in [**8**, Théorème, (iv) ⇒ (v)], F is projectively equivalent to a Fermat polynomial, as it is asserted.

To prove 31 we show firstly that

$$G(ijk) := F_iF_{jk} + F_jF_{ik} + F_kF_{ij} = 0 \tag{32}$$

on X for any i, j, k. If $F_k = 0$ on X, then F_k is divisible by F, hence $F_k = 0$ as a polynomial. Therefore, $F_{ik} = 0$, $F_{jk} = 0$, and $G(ijk) = 0$, as is required. For the case $F_k \neq 0$ on X, it suffices to show that for any i, j, there exists some $l \notin \{i, j, k\}$ such that 32 holds on a canonical affine open subset U_l of X defined by $x_l \neq 0$. Renumbering the indices, we may assume that $l = 0 < i, j \leq N = k$ without loss of generality. Set $y_i := x_i/x_0$ and $f(y_1, \ldots, y_N) := F(1, y_1, \ldots, y_N)$. Then $y_1, \ldots, y_{N-1}$ form a system of local coordinates by virtue of the assumption $f_N(y_1, \ldots, y_N) = F_N(1, y_1, \ldots, y_N) \neq 0$ on U_0. If $i = N$ or $j = N$, then 32 holds on U_0 since $f_{NN} = 0$ in $p = 2$. Thus it suffices to consider the case $0 < i, j < N$. Taking the partial derivative of $f = 0$ on U_0 by y_i then by y_j, we obtain $f_i + f_N \frac{\partial y_N}{\partial y_i} = 0$, and

$$\left(f_{ij} + f_{iN}\frac{\partial y_N}{\partial y_j}\right) + \left(f_{Nj} + f_{NN}\frac{\partial y_N}{\partial y_j}\right)\frac{\partial y_N}{\partial y_i} + f_N\frac{\partial^2 y_N}{\partial y_i \partial y_j} = 0.$$

Here we note again that $f_{NN} = 0$ in $p = 2$, and that $\frac{\partial^2 y_N}{\partial y_i \partial y_j} = 0$ for any i, j since γ_0 is of rank zero (see [**42**, (3.3.15)]). Combining these equations, we obtain

$$f_if_{jN} + f_jf_{iN} + f_Nf_{ij} = 0.$$

Homogenising the above, we obtain $G(ijN) = 0$ on U_0 for any i, j with $0 < i, j < N$. This complete the proof of 32.

Now we see from 32 that $aF = G(ijk)$ for some $a \in K$, by comparing the degrees. Taking the partial derivative by x_l, we obtain

$$aF_l = F_{il}F_{jk} + F_iF_{jkl} + F_{jl}F_{ik} + F_jF_{ikl} + F_{kl}F_{ij} + F_kF_{ikl}, \tag{33}$$

as polynomials. It follows from 32 that for a point $x \in X$ if $F_i(x) = F_j(x) = 0$, then $F_{ij}(x) = 0$, by the smoothness of X. Moreover it follows from 33 that if $F_i(x) = F_j(x) = F_k(x) = 0$, then $aF_l(x) = 0$. Since X is smooth and $N \geq 3$, we find $a = 0$. Setting $l := i$ in 33, we see that $F_iF_{ijk} = 0$ as a polynomial for any i, j, k; hence $F_{ijk} = 0$. By virtue of Euler's formula, we finally obtain 31. □

Remark 5.5. R. Pardini [**53**] and A. Hefez [**33**] obtained formulae with the same form as the key claim 31 in the proof of Theorem 5.4 under certain more general conditions on the degree and singularities of X ([**33**, (7.4)], [**53**, (2.1)]), and deduced a canonical form of F as well ([**33**, §9], [**44**, I, (14)], [**53**, §§2–3]). However, those results are proved under the assumption $p > 2$, hence do not cover our result in $p = 2$. In fact, 31 does not hold in $p = 2$ unless X is smooth, although the result of Hefez [**33**, (7.4)] is valid even for a singular X if it is regular in codimension one. A cubic surface X defined by $F = wx^2 + wyz + z^3$ in $\mathbb{P}^3$, for instance, has Gauss map γ_0 of rank zero with only a finite number of singular points, but $F_{wy} \neq 0$.

Now we prove Theorem II.5 in the case where X is of dimension $\geqslant 4$ (i.e., $N \geqslant 5$).

Proof of Theorem II.5 ($N \geqslant 5$). Denote by γ_0 the Gauss map of the embedding of X in $\mathbb{P}^N$ as a cubic hypersurface, as before. For the if-part, it is easily verified by a direct computation that γ_0 is of rank zero; hence X satisfies (GMRZ). For the only-if-part, it follows from Theorem II.1 and Lemma 2.7 that $N_{L/X}^{\vee} \simeq [-1^{N-3}, 1]$ for any projective line $L \subseteq X$. Then, γ_0 is of rank zero by Proposition 5.2; hence X is projectively equivalent to a Fermat by Theorem 5.4. □

6. Cubic 3-fold

In this section, we prove Theorem II.5 with $N = 4$ in several steps. Let $X \subset \mathbb{P}^4$ be a smooth cubic 3-fold. We recall that $F := \{ L \in \mathbb{G}(1, \mathbb{P}^4) \mid L \subset X \}$ is the set of lines on X, and denote by $U \subset F \times X$ the universal family over F with projections

$$u : U \to F \quad \text{and} \quad v : U \to X.$$

For a projective line $L \subset X$, the splitting type of the normal bundle $N_{L/X}$ is equal to either $[0, 0]$ or $[-1, 1]$, as in Lemma 2.7. This implies that $\dim F = 2$, and hence v is generically finite.

Proposition 6.1. *For a smooth cubic 3-fold X, we have $\deg(v) = 6$. In particular, if $p = 2$, then the separable degree of v is equal to either 3 or 6.*

Proof. The statement follows from [**1**, (1.7)]. □

Recall that $\gamma_0 : X \to (\mathbb{P}^4)^{\vee}$ is the Gauss map of the original embedding $X \subset \mathbb{P}^4$, where γ_0 is a finite morphism. We denote by $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^4}(1)|_X$. Since X is cubic, it follows $\gamma_0^*(\mathcal{O}_{(\mathbb{P}^4)^{\vee}}(1)) = \mathcal{O}_X(2)$.

Proposition 6.2. *Let $X \subset \mathbb{P}^4$ satisfy (GMRZ). Then, for any line L on X, it follows that $N_{L/X} = [-1, 1]$ and that the image $\gamma_0(L)$ is equal to a line in $(\mathbb{P}^4)^{\vee}$.*

Remark 6.3. Under the assumption of Proposition 6.2, we immediately have $p = 2$ due to Theorem II.2

In order to prove Proposition 6.2, we need to show the following:

Lemma 6.4. *Under the assumption of* Proposition 6.2*, one of the following properties holds:*

(a) *We have $N_{L/X} = [-1, 1]$ for any line L on X.*
(b) *We have $N_{L/X} = [0^2]$ for any line L on X.*

Proof. As in Remark 6.3, we have $p = 2$. Let ι be an embedding whose Gauss map is of rank zero. From the Lefschetz theorem, it follows that $\operatorname{Pic} X$ is isomorphic to $\operatorname{Pic} \mathbb{P}^N$ and is generated by $\mathcal{O}_X(1)$; hence there exists an integer a such that $\iota^*\mathcal{O}_{\mathbb{P}^M}(1) = \mathcal{O}_X(a)$. Let $L \subset X$ be a line. If $N_{L/X} = [1, -1]$, then from Propositions 1.2 and 1.4, we have $2 \mid a - 1$. If $N_{L/X} = [0^2]$, then from Propositions 1.2 and 1.4 again, we have $2 \mid a$. Since the properties $2 \mid a - 1$ and $2 \mid a$ do not hold at the same time, the statement follows. □

Now, we denote by $F_x := u(v^{-1}(x)) \subset F$ the set of $L \in F$ such that $x \in L$, where the Zariski tangent space $t_L F_x$ at $L \in F_x$ is isomorphic to $H^0(N_{L/X}(-1))$.

PROOF OF PROPOSITION 6.2. Assume that the property b of Lemma 6.4 holds. Then v is a finite morphism. The reason is the following: Let $x \in X$, let $L \in F_x$, and let V be an irreducible component of F_x containing L. Then since $h^1(N_{L/X}(-1)) = h^0(N_{L/X}(-1)) = 0$, we have $V = \{ L \}$, which implies the finiteness of v.

Next, we show that v is a smooth morphism, as follows: For each point $(L, x) \in F$, we have an exact sequence of Zariski tangent spaces,

$$0 \to t_{(L,x)} v^{-1}(x) \to t_{(L,x)} U \xrightarrow{d_{(L,x)} v} t_x X$$

Since $v^{-1}(x) \simeq F_x$, it follows that $t_{(L,x)} v^{-1}(x)$ is of dimension $H^0(N_{L/X}(-1)) = 0$. Since $\dim U = \dim X = 3$, we have that $d_{(L,x)} v$ is surjective. Hence v is smooth.

As a result, we have that the morphism v is étale. By [**23**, Cor. 2], the hypersurface X is simply-connected. Therefore v is isomorphism, which contradicts Proposition 6.1. Thus we have $N_{L/X} = [-1, 1]$ for any $L \in F$. Then it follows from Lemma 5.1 that $\gamma_0(L)$ is equal to a line in $(\mathbb{P}^n)^\vee$. □

Proposition 6.5. *Let X be as in* Proposition 6.2. *Then $\gamma_0|_L$ is inseparable for any line $L \subset X$.*

To prove Proposition 6.5, we show the following result. Here, for a linear subspace $A \subset \mathbb{P}^4$, we denote by $A^* \subset (\mathbb{P}^4)^\vee$ the subset of $H \in (\mathbb{P}^4)^\vee$ such that $A \subset H$.

Proposition 6.6. *Let X be as in* Proposition 6.2*, and assume that $\gamma_0|_L$ is separable for a general line $L \subset X$. Then, for the 2-plane $M \subset \mathbb{P}^4$ satisfying $\gamma_0(L) = M^* \subset (\mathbb{P}^4)^\vee$, we have a line $R \subset X$ such that $M \cap X = L \cup R$. Moreover, for a general point $x \in L$, we have a line $L^{(x)} \subset X$ and two distinct points $x_2, x_3 \in L^{(x)}$ such that $L^{(x)} \cap L = \emptyset$, $L^{(x)} \cap R \neq \emptyset$, and $\gamma_0(x_i) = \gamma_0(x)$ with $i = 2, 3$.*

Lemma 6.7. *Let $X \subset \mathbb{P}^4$ be a smooth cubic 3-fold. Let $L \subset X$ be a line, and let $M \subset \mathbb{P}^4$ be a 2-plane such that $\gamma_0(L) = M^*$ in $(\mathbb{P}^4)^\vee$. Then $M \cap X$ contains L multiply. Thus, set-theoretically, $M \cap X$ is equal to either*

(a) *the line L, or*
(b) *a union $L \cup R$ with some line R.*

PROOF. Since $\gamma_0(L) \subset L^*$, we have $L \subset M$. By assumption, we have $M \subset T_x X$ for every $x \in L$; thus $M \cap X$ is singular at every point of L. Since X is cubic, the assertion follows. □

From [**56**, §1], we have the following basic properties of a singular cubic surface:

Lemma 6.8. *Let $S \subset \mathbb{P}^3$ be a singular cubic surface which is not a cone.*

(a) *Every singular point of S is a double point.*
(b) *For distinct singular points $P, Q \in S$, the line $\overline{PQ}$ is contained in S.*
(c) *The singular locus of S is equal to either a line of singularities or a set of finitely many double points.*
(d) *Assume that the singular locus of S is equal to a set of finitely many double points. Then, no three double points are colinear, no four double points are coplanar, and the number of double points is less than or equal to four.*

Proof of Proposition 6.6. We denote by X_0 the set of points $x \in X$ such that F_x is a finite set and that $\gamma_0|_L$ is separable and unramified at x for any line $L \in F_x$. Then, since v is generically finite and since $\gamma_0|_L$ is separable for general $L \in F$, the following subset of X is of codimension $\geqslant 1$:

$$\bigcup_{L \in F^i} L \cup \bigcup_{L \in F^s} \{ x \in L \mid \gamma_0|_L : \text{ramified at } x \} \cup \{ x \in X \mid \dim(v^{-1}(x)) > 0 \}$$

where $F^i := \{ L \in F \mid \gamma_0|_L : \text{inseparable} \}$ and $F^s := \{ L \in F \mid \gamma_0|_L : \text{separable} \}$. Thus the subset X_0 is dense in X.

Let $L = L_1 \subset X$ be a general line, and let $x = x_0 \in L \cap X_0$ be a point. From Proposition 6.1, we have at least two lines $L_2, L_3 \subset X$ passing through x. Then there exist three points $x_i \in L_i$ with $1 \leqslant i \leqslant 3$ not equal to x such that $\gamma_0(x_i) = \gamma_0(x)$. For the embedded tangent space $H = T_xX$, the cubic surface $S := H \cap X$ contains three lines L_i with $1 \leqslant i \leqslant 3$ and is singular at four points x_i with $0 \leqslant i \leqslant 3$. Here, by the definition of X_0, the cubic S is not a cone. In addition, since γ_0 is a finite morphism, S is singular at finitely many points.

As in Lemma 6.8, the line $L^{(x)} := \overline{x_2x_3}$ is contained in S. Now we set $M \subset \mathbb{P}^4$ as the 2-plane satisfying $\gamma_0(L) = M^*$, where we have $L \subset M$ as in Lemma 6.7. Since $L^{(x)}$ and M are contained in T_xX, we have $L^{(x)} \cap M \neq \emptyset$. Since the four points x_i are not coplanar, two lines L and $L^{(x)}$ are disjoint, which implies that the intersection point of $L^{(x)}$ and M is not contained in L. Hence we have $M \cap X \neq L$. From Lemma 6.7, there exists a line R satisfying $X \cap M = L \cup R$, and R contains the intersection point of $L^{(x)}$ and M. $\square$

Proof of Proposition 6.5. Let $h \in H^0(\mathbb{P}^4, \mathcal{O}(3))$ be the defining polynomial of X, and let $z_0, \dots, z_4$ be homogeneous coordinates on $\mathbb{P}^4$. Then γ_0 is expressed by polynomials $\partial h / \partial z_i|_X \in H^0(X, \mathcal{O}(2))$ with $0 \leqslant i \leqslant 4$.

If $\gamma_0|_L$ is separable for some $L \in F$, then so is $\gamma_0|_L$ for general $L \in F$. This is because $\gamma_0|_L$ is separable for $L \in F$ if and only if $\partial h/\partial z_i|_L \in H^0(L, \mathcal{O}(2))$ is not contained in the subset $\{ f^2 \mid f \in H^0(L, \mathcal{O}(1)) \}$ with some i.

We fix a general line $L \subset X$, and suppose that $\gamma_0|_L$ is separable. Let U be the subset of $x \in L$ such that there exists a line $L^{(x)}$ stated in Proposition 6.6. Then we consider the following locus of X,

$$Y := \overline{\bigcup_{x \in U} L^{(x)}}.$$

Let $x \in L$ be a general point. Then $\gamma_0(L^{(x)})$ intersects with $\gamma_0(L)$ and with $\gamma_0(R)$. Thus the image $\gamma_0(Y)$ is equal to the 2-plane $R^* \subset (\mathbb{P}^4)^\vee$, which is spanned by lines $\gamma_0(L)$ and $\gamma_0(R)$. Since $\gamma_0^{-1}(\gamma_0(x)) = \operatorname{Sing}(X \cap T_xX)$, it follows from Lemma 6.8d that $\gamma_0^{-1}(\gamma_0(x)) \cap Y$ is equal to a set of 4 points $\{ x, x_1, x_2, x_3 \}$, where $x, x_1 \in L$ and $x_2, x_3 \in L^{(x)}$. Thus the separable degree of $\gamma_0|_Y$ is equal to 4. Since

$$Y \cdot \gamma_0^*(\mathcal{O}_{(\mathbb{P}^4)^\vee}(1))^2 = (\gamma_0)_*(Y) \cdot \mathcal{O}_{(\mathbb{P}^4)^\vee}(1)^2 = \deg(\gamma_0|_Y) \cdot \deg(\gamma_0(Y)) = \deg(\gamma_0|_Y)$$

and since the left hand side of the above formula is equal to $Y \cdot \mathcal{O}_{\mathbb{P}^4}(2)^2 = 2^2 \deg Y$, it follows that $\deg Y$ is equal to the inseparable degree of $\gamma_0|_Y$. Thus $\deg Y = 2^a$ with some integer $a \geqslant 0$. Since $Y \subset X$ is a divisor and since $\operatorname{Pic} X = \operatorname{Pic} \mathbb{P}^N$, we have $3 \mid \deg Y$, a contradiction. Thus the assertion follows. □

Proof of Theorem II.5 ($N = 4$). Let $X \subset \mathbb{P}^4$ be as in Proposition 6.2. From Theorem 5.4, it is sufficient to prove $\operatorname{rk} d\gamma_0 = 0$, where recall that γ_0 is the Gauss map of the original embedding $X \subset \mathbb{P}^4$.

From Proposition 6.5, it follows that $\gamma_0|_L$ is inseparable for any line L in X. Now we show $\operatorname{rk} d\gamma_0 = 0$, as follows. Let $x \in X$ be a general point. From Proposition 6.1, we find at least three distinct lines $L_i \subset X$ with $1 \leqslant i \leqslant 3$ passing through x. Since $\gamma_0|_{L_i}$ is inseparable, we have $\operatorname{rk} d_x\gamma_0|_{L_i} = 0$, that is, $d_x\gamma_0(t_xL_i) = 0$ for the Zariski tangent space $t_xL_i \subset t_xX$. Since $L_1 \neq L_2$, we have $\operatorname{rk} d_x\gamma_0 \leqslant 1$. Suppose that $\operatorname{rk} d_x\gamma_0 \neq 0$. Then, as in the proof of Proposition 5.2, by considering the Hessian matrix of $d_x\gamma_0$, it follows from $p = 2$ that we find $\operatorname{rk} d_x\gamma_0 \geqslant 2$, a contradiction. Thus $\operatorname{rk} d_x\gamma_0 = 0$. □

7. blowing-ups of varieties satisfying (GMRZ)

In §7.1, we will prove Theorem II.6, which is precisely described as follows:

Theorem 7.1. *Let Y be a projective variety and let $\tilde{Y} \to Y$ be the blow-up at a point $P \in Y$. Assume that Y satisfies* (GMRZ)*, and assume $p = 2$. Then $\tilde{Y}$ satisfies* (GMRZ)*.*

Note that, in the theorem, we need not assume the smoothness of Y at the point P. On the other hand, we can determine a situation that blowing-ups satisfy (GMRZ), as follows:

Corollary 7.2. *Let Y be a projective variety of dimension $\geqslant 2$ satisfying* (GMRZ)*, and let $Z = \bigcup Z_i \subset Y$ be a reduced closed subvariety of codimension $\geqslant 2$ with the irreducible components $\{ Z_i \}$ such that $(Z_i)_{\mathrm{reg}} \cap Y_{\mathrm{reg}} \neq \emptyset$ for each i. Then the blowing-up $\mathrm{BL}_Z Y$ of Y along Z satisfies* (GMRZ) *if and only if $p = 2$ and Z is a set of finitely many points.*

Since a smooth cubic surface in $\mathbb{P}^3$ is given by the blowing-up of $\mathbb{P}^2$ at 6 points, it follows from Theorem 7.1 that we have:

Corollary 7.3. *Every smooth cubic surface satisfies* (GMRZ) *if $p = 2$.*

In §7.2, we will give construction of projective varieties which satisfy (GMRZ) by using Theorem 7.1. As a result, we have:

Proposition 7.4. *Let L' be any function field of dimension* 1 *over the ground field of characteristic $p = 2$. For any purely transcendental extension L with finite transcendence degree over L', there exists an smooth projective variety X satisfying* (GMRZ) *such that $K(X) = L$.*

In addition, as a generalization of Corollary 7.3, we have:

Theorem 7.5. *A smooth projective rational surface X satisfies* (GMRZ) *if and only if either $p = 2$ or $p > 0$ and $X \simeq \mathbb{P}^2$.*

Remark 7.6 (curves)**.** Every rational or elliptic smooth projective curve satisfies (GMRZ) for any $p > 0$. This follows from the study of inseparable Gauss maps for rational curves (Kaji [**39**, Ex. 4.1], Rathmann [**55**, Ex. 2.13]), and for elliptic curves (Kaji [**40**, Thm. 5.1] [**41**, Thm. 0.1]).

7.1. Blowing-up. In order to prove Theorem 7.1, we first study the blowing-up of projective space $\mathbb{P}^m$ in characteristic $p > 0$. Let us consider the following composite morphism,

$$F : \mathbb{P}^m \xrightarrow{\Gamma_{\mathrm{Frob}_p}} \mathbb{P}^m \times \mathbb{P}^m \hookrightarrow \mathbb{P}^M,$$

where Γ_{Frob_p} is the graph morphism of the Frobenius morphism $\mathrm{Frob}_p : \mathbb{P}^m \to \mathbb{P}^m$, and $\mathbb{P}^m \times \mathbb{P}^m \hookrightarrow \mathbb{P}^M$ is the Segre embedding with $M = (m+1)^2 - 1$. Let $P \in \mathbb{P}^m$, $P_1 := F(P)$, and let $\pi_{P_1} : \mathbb{P}^M \dashrightarrow \mathbb{P}^{M-1}$ be the projection from a point P_1. Now we set

$$X_0 := (\pi_{P_1} \circ F)(\mathbb{P}^m \setminus \{ P \}) \text{ and } X := \overline{X_0} \subset \mathbb{P}^{M-1}. \tag{34}$$

We denote by $\tilde{\mathbb{P}} := \mathrm{BL}_P(\mathbb{P}^m)$. By resolving indeterminacy of $\pi_{P_1} \circ F : \mathbb{P}^m \dashrightarrow X$, we have a morphism $\varphi : \tilde{\mathbb{P}} \to X$. Here we have the following results.

Proposition 7.7. *Let $X \subset \mathbb{P}^{M-1}$ be as in* (34) *above. Then the following holds.*

(a) *X_0 is isomorphic to $\mathbb{P}^m \setminus \{ P \}$ and the Gauss map of the embedding $X \hookrightarrow \mathbb{P}^{M-1}$ is of rank zero for any $p > 0$.*
(b) *X is isomorphic to $\tilde{\mathbb{P}}$ if and only if $p = 2$.*

Thus the blowing-up of $\mathbb{P}^m$ at one point satisfies (GMRZ) *if $p = 2$.*

Remark 7.8. In (b) we in fact show that $\mathrm{Sing}(X) = \varphi(E)$ in the case $p \geqslant 3$, where $E \subset \tilde{\mathbb{P}}$ is the exceptional divisor.

PROOF. (a) By changing coordinates $(x_0, x_1, \dots, x_m)$ on $\mathbb{P}^m$, we may assume $P = (1, 0, \dots, 0)$ and assume that F is given by $(x_i)_{0 \leqslant i \leqslant m} \mapsto (x_i^p x_j)_{0 \leqslant i,j \leqslant m}$. Then $\pi_{P_1} \circ F$ is given by

$$(x_i)_{0 \leqslant i \leqslant m} \mapsto (x_i^p x_j)_{0 \leqslant i,j \leqslant m, (i,j) \neq (0,0)}.$$

On the open subset $\{ x_u = 1 \} \subset \mathbb{P}^N \setminus \{ P \}$ with $1 \leqslant u \leqslant n$, the sub-parameters $(x_0, \dots, x_{u-1}, 1, x_{u+1} \dots x_m)$ are appeared in the right hand side of the above description of $\pi_{P_1} \circ F$. Thus $\{ x_u = 1 \}$ is isomorphic to its image in X for each u; hence we have $\mathbb{P}^N \setminus \{ P \} \simeq X_0$. Since $x_i^p x_j$ vanishes by the operators

$$\{ \partial^2 / \partial x_v \partial x_w \}_{0 \leqslant v,w \leqslant m,\, v \neq u,\, w \neq u},$$

it follows from [**22**, Lem. 2.1] that the Gauss map of $X \hookrightarrow \mathbb{P}^{M-1}$ is of rank zero.

(b) First, we give the coordinates of the morphism $\varphi : \tilde{\mathbb{P}} \to X \subset \mathbb{P}^{M-1}$ as follows: Let $\Gamma_{\pi_P} : \mathbb{P}^m \setminus \{ P \} \to \mathbb{P}^m \times \mathbb{P}^{m-1}$ be the graph morphism of the projection $\pi_P : \mathbb{P}^m \dashrightarrow \mathbb{P}^{m-1}$, where Γ_{π_P} is given by

$$(x_0, x_1, \dots, x_m) \mapsto ((x_0, x_1, \dots, x_m), (x_1, \dots, x_m)).$$

Then $\tilde{\mathbb{P}}$ is equal to the closure of $\Gamma_{\pi_P}(\mathbb{P}^m \setminus \{ P \})$. Let $(y_1, \dots, y_m)$ be the set of coordinates on $\mathbb{P}^{m-1}$. Let us consider a morphism $\Phi = ((\varphi^1_{i,j})_{0 \leqslant i \leqslant m, 1 \leqslant j \leqslant m}, (\varphi^2_i)_{1 \leqslant i \leqslant m}) :$

$\mathbb{P}^m \times \mathbb{P}^{m-1} \to \mathbb{P}^{M-1}$ defined by

$$\varphi^1_{i,j}((x_0, x_1, \dots, x_m), (y_1, \dots, y_m)) = x_i^p y_j \quad \text{for } 0 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant m,$$
$$\varphi^2_i((x_0, x_1, \dots, x_m), (y_1, \dots, y_m)) = x_0 x_i^{p-1} y_i \quad \text{for } 1 \leqslant i \leqslant m,$$

where $M = (m+1)^2 - 1$. Then we have $\Phi \circ \Gamma_{\pi_P} = \pi_{P_1} \circ F : \mathbb{P}^N \setminus \{ P \} \to X_0$, and have $\varphi = \Phi|_{\tilde{\mathbb{P}}}$. Note that, we have two isomorphisms $\varphi|_{\tilde{\mathbb{P}} \setminus E} : \tilde{\mathbb{P}} \setminus E \to X_0$ and $\varphi|_E : E \to \varphi(E)$, where $E := \{ P \} \times \mathbb{P}^{m-1} \subset \tilde{\mathbb{P}}$ is the exceptional divisor. Therefore φ is a bijective morphism.

Now suppose $p = 2$. We will show that φ is isomorphic, as follows: Let $p_2 : \tilde{\mathbb{P}} \to \mathbb{P}^{m-1}$ be the second projection, and let $U_i := \{ y_i \neq 0 \} \subset \mathbb{P}^{m-1}$ be the standard open subset. Then it is sufficient to show that φ is isomorphic on $p_2^{-1}(U_i)$ for all $1 \leqslant i \leqslant m$. By symmetry, we may assume $i = 1$ and set $U := U_1$. We have

$$\tilde{\mathbb{P}} = \{ ((x_i), (y_j)) \in \mathbb{P}^m \times \mathbb{P}^{m-1} \mid x_i y_j = x_j y_i \text{ for } 1 \leqslant i, j \leqslant m \}$$

Here, in the case $y_1 = 1$, we have equalities $x_i = x_1 y_i$ for $2 \leqslant i \leqslant m$, and have an isomorphism $\psi : \mathbb{P}^1 \times U \to p_2^{-1}(U)$ defined by

$$((x_0, x_1), (1, y_2, y_3, \dots, y_m)) \mapsto ((x_0, x_1, x_1 y_2, x_1 y_3, \dots, x_1 y_m), (1, y_2, y_3, \dots, y_m)).$$

Let $V = \{ x_0 \neq 0 \} \subset \mathbb{P}^1$. Then $E \cap p_2^{-1}(U) \subset \psi(V \times U)$. Here we have

$$((\varphi^1_{i,j} \circ \psi|_{V \times U})((1, x_1), (1, y_2, \dots, y_m)))_{i,j} = (1, y_2, \dots, y_m, *, \dots, *),$$
$$((\varphi^2_i \circ \psi|_{V \times U})((1, x_1), (1, y_2, \dots, y_m)))_i = (x_1, x_1 y_2^2, \dots, x_1 y_m^2).$$

Thus $\Phi \circ \psi|_{V \times U}$ is isomorphic to its image; hence so is $\varphi|_{p_2^{-1}(U)}$.

Suppose $p \geqslant 3$. As above, we consider the morphism $\varphi \circ \psi|_{V \times U}$. In this case, it is obtained by,

$$((\varphi^1_{i,j} \circ \psi|_{V \times U})((1, x_1), (1, y_2, \dots, y_m)))_{i,j} = (1, y_2, \dots, y_m, x_1^p, x_1^p y_2, \dots, x_1^p y_m, *, \dots, *),$$
$$((\varphi^2_i \circ \psi|_{V \times U})((1, x_1), (1, y_2, \dots, y_m)))_i = (x_1^{p-1}, x_1^{p-1} y_2^p, \dots, x_1^{p-1} y_m^p).$$

Thus $\varphi \circ \psi$ is not isomorphic at each point of $\{ (1, 0) \} \times U$. By symmetry, φ is not isomorphic for each point of E. Here, we show that $\varphi(E)$ is the singular locus of X, as follows: Assume that X is smooth at a point of $\varphi(E)$. Then, by symmetry, X is smooth at every point of $\varphi(E)$; hence X is a smooth variety. Since φ is bijective, the Zariski main theorem implies that φ is isomorphic, a contradiction.

Thus X is a singular variety with $\mathrm{Sing}(X) = \varphi(E)$. In particular, it follows that X is not isomorphic to $\tilde{\mathbb{P}}$. □

Proof of Theorem 7.1. Assume $p = 2$, and let $\iota : Y \hookrightarrow \mathbb{P}^m$ be an embedding whose Gauss map is of rank zero. We take a general point $Q \in Y$. By changing coordinates on $\mathbb{P}^m$, we may assume that $P = (1, 0, 0, \dots, 0)$ and $Q = (0, 1, 0, \dots, 0)$ in $\mathbb{P}^m$. Then, as in [**22**, (2.1) and Lem. 2.1], we have local coordinates of Y around Q:

$$(f_0, 1, z_2, \dots, z_n, z_{n+1}, f_{n+2}, \dots, f_m)$$

where $(z_2, \dots, z_n, z_{n+1})$ are the local parameters, and $\{ f_i \}$ are polynomials contained in the maximal ideal of $\mathcal{O}_{X,Q}$ such that $\partial^2 f_i / \partial z_j \partial z_k = 0$ for each i.

As in Proposition 7.7, let $\tilde{\mathbb{P}} := \mathrm{BL}_P(\mathbb{P}^m)$ and let $\varphi : \tilde{\mathbb{P}} \to \mathbb{P}^{M-1}$ be the morphism given by resolving indeterminacy of

$$\pi_{P_1} \circ F : \mathbb{P}^m \dashrightarrow \mathbb{P}^{M-1} : (x_i)_{0 \leqslant i \leqslant m} \mapsto (x_i^p x_j)_{0 \leqslant i,j \leqslant m, (i,j) \neq (0,0)}. \tag{35}$$

Here φ is an embedding because of $p = 2$. Let $\tilde{Y} \subset \tilde{\mathbb{P}}$ be the blowing-up of Y at P, and let $\tilde{Q} \in \tilde{Y}$ be the point corresponding to $Q \in Y$. Then

$$\varphi|_{\tilde{Y}} : \tilde{Y} \to \mathbb{P}^{M-1}$$

gives an embedding. The local coordinates of $\varphi(\tilde{Y}) \subset \mathbb{P}^{M-1}$ around the point $\varphi(\tilde{Q}) = (\pi_{P_1} \circ F)(Q)$ is given by

$$(\pi_{P_1} \circ F)(f_0, 1, z_2, \dots, z_n, z_{n+1}, f_{n+2}, \dots, f_m), \tag{36}$$

By using the parametrization (35), we find that (36) consists of the local parameters $(1, z_2, \dots, z_n, z_{n+1})$ and local functions vanishing by operators $\{ \partial^2/\partial z_v \partial z_w \}_{2 \leqslant v,w \leqslant n+1}$. Thus it follows from [**22**, Lem. 2.1] that the Gauss map of $\varphi(\tilde{Y}) \subset \mathbb{P}^{M-1}$ is of rank zero. □

Next we consider the only-if-part of Corollary 7.2.

Lemma 7.9. *Let Y be an n-dimensional projective variety, let $Z \subset Y$ be a closed subvariety of codimension $\geqslant 2$, and let $P \in Z_{\mathrm{reg}} \cap Y_{\mathrm{reg}}$. If Z is of dimension m at P, we have $N_{L/\mathrm{BL}_Z(Y)} = [1^{n-m-2}, 0^m, -1]$, where $q : \mathrm{BL}_Z Y \to Y$ is the projection, and $L \subset q^{-1}(P) \simeq \mathbb{P}^{n-m-1}$ is a projective line.*

Proof. Let Z' be an irreducible component of Z containing P. Here $q^{-1}(Z'_{\mathrm{reg}} \cap Y_{\mathrm{reg}})$ is a $\mathbb{P}^{n-m-1}$-bundle over $Z'_{\mathrm{reg}} \cap Y_{\mathrm{reg}}$. As in Lemma 2.4, we have $N_{L/q^{-1}(Z)} = [1^{n-m-2}, 0^m]$. Since $N_{q^{-1}(Z)/\mathrm{BL}_Z(Y)}|_L = [-1]$, we have $N_{L/\mathrm{BL}_Z(Y)} = [1^{n-m-2}, 0^m, -1]$. □

Corollary 7.10. *Under the assumption of* Lemma 7.9, $\mathrm{BL}_Z Y$ *satisfies* (GMRZ) *only if $p = 2$ and $m = 0$ (i.e., $\{ P \}$ is an irreducible component of Z).*

Proof. The statement follows from Lemma 7.9 and Theorem II.1. □

Proof of Corollary 7.2. If $p = 2$ and Z is a set of finitely many points, then Theorem 7.1 implies that $\mathrm{BL}_Z Y$ satisfies (GMRZ).

Conversely, suppose that $\mathrm{BL}_Z Y$ satisfies (GMRZ). For an irreducible component Z_i of Z, we can take a point $P \in Z_i \cap Z_{\mathrm{reg}} \cap Y_{\mathrm{reg}}$. By applying Corollary 7.10, we have $p = 2$ and $m = 0$ (i.e., $Z_i = \{ P \}$). □

7.2. Construction of varieties satisfying (GMRZ)**.** For a smooth projective variety $Y \subset \mathbb{P}^M$, and for an embedding $\mathbb{P}^M \hookrightarrow \mathbb{P}^{M+1}$ with a point $P \in \mathbb{P}^{M+1} \setminus \mathbb{P}^M$, we set

$$\mathrm{R}(Y) := \mathrm{BL}_P \,\mathrm{Cone}(P, Y),$$

the smooth projective variety ruled over Y defined as the blowing up of the cone $\mathrm{Cone}(P, Y) \subset \mathbb{P}^{M+1}$ at the vertex P.

Lemma 7.11. *Assume $p = 2$. Let Y be a smooth projective variety satisfying* (GMRZ)*, and let $\iota : Y \hookrightarrow \mathbb{P}^M$ be an embedding whose Gauss map is of rank zero. Then the ruled variety* $\mathrm{R}(\iota(Y))$ *satisfies* (GMRZ).

Proof. We set $Y_P := \mathrm{Cone}(P, \iota(Y)) \subset \mathbb{P}^{M+1}$ and denote by ι_P its embedding in $\mathbb{P}^{M+1}$. Then we have the following commutative diagram,

$$\begin{array}{ccccc} Y_P \setminus \{ P \} & \xrightarrow{\gamma_{\iota_P}} & \mathrm{im}(\gamma_{\iota_P}) & \subset & \mathbb{G}(\dim(Y)+1, \mathbb{P}^{M+1}) \\ \big\downarrow{\scriptstyle \pi_P} & & \big\downarrow{\scriptstyle \simeq} & & \\ Y & \xrightarrow{\gamma_\iota} & \mathrm{im}(\gamma_\iota) & \subset & \mathbb{G}(\dim(Y), \mathbb{P}^M), \end{array}$$

where $\pi_P : \mathbb{P}^{M+1} \setminus \{ P \} \to \mathbb{P}^M$ denotes the projection from P. Since γ_ι is of rank zero, so is γ_{ι_P}. Hence Theorem 7.1 implies that $\mathrm{R}(\iota(Y))$, the blowing-up of Y_P, satisfies (GMRZ). □

Corollary 7.12. *Assume $p = 2$, and let C be a smooth projective curve, and let $\iota : C \hookrightarrow \mathbb{P}^N$ be an arbitrary embedding. Then the ruled surface* $\mathrm{R}(\iota(C))$ *satisfies* (GMRZ).

Proof. From [**40**, Cor. 2.2 and 2.3], since C is a curve, it follows from $p = 2$ that the Gauss map of ι is of rank zero. Therefore, from Lemma 7.11, the ruled surface $\mathrm{R}(\iota(C))$ satisfies (GMRZ). □

Proof of Proposition 7.4. For any function field L' of dimension 1 over the ground field, we find a smooth projective curve C with $K(C) = L'$. Then, as above, the Gauss map of any embedding $\iota : C \hookrightarrow \mathbb{P}^N$ is of rank zero. Let $Y_1 := C$. From Lemma 7.11, we inductively have that $Y_i := R(\iota_{i-1}(Y_{i-1}))$ satisfies (GMRZ) for any $i > 1$ if $p = 2$, where ι_{i-1} is an embedding whose Gauss map is of rank zero. Here $K(Y_i)$ is purely transcendental extension over L'. □

Now, in order to prove Theorem 7.5, we study minimal rational surfaces:

Proposition 7.13. *A Hirzebruch surface $\Sigma_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ with $e \geqslant 0$ satisfies* (GMRZ) *if and only if $p = 2$.*

Proof. If Σ_e satisfies (GMRZ), then Theorem II.2(a) implies $p = 2$.

Next, we assume $p = 2$. If $e = 0$, then $\Sigma_e \simeq \mathbb{P}^1 \times \mathbb{P}^1$; hence, in this case, the statement follows from Theorem II.2(c). Thus, we assume $e > 0$. For a rational normal curve $C \subset \mathbb{P}^e$ of degree e, and for an embedding $\mathbb{P}^e \hookrightarrow \mathbb{P}^{e+1}$ with a point $P \in \mathbb{P}^{e+1} \setminus \mathbb{P}^e$, the surface Σ_e is isomorphic to the blowing-up of the cone $\mathrm{Cone}(P, C) \in \mathbb{P}^{e+1}$ at the vertex P. Therefore, as in Lemma 7.12, we find that Σ_e satisfies (GMRZ). □

Corollary 7.14. *A relative minimal rational surface X satisfies* (GMRZ) *if and only if either $p = 2$ or $p > 0$ and $X \simeq \mathbb{P}^2$.*

Proof. A relative minimal rational surface X is isomorphic to $\mathbb{P}^2$ or Σ_e with $e \geqslant 0, e \neq 1$. In the case $X = \Sigma_e$, the assertion follows from Proposition 7.13. On the other hand, $\mathbb{P}^2$ satisfies (GMRZ) for any $p > 0$ as in [**21**, Ex. 3.1]. □

Proof of Theorem 7.5. Let X be a smooth rational surface. Then X is given by a chain of blow-ups of points $X = X_1 \to X_2 \to \cdots \to X_r$ with a relative minimal rational surface X_r. Thus the assertion follows from Corollaries 7.2 and 7.14. □

CHAPTER III

Defining ideal of the Segre locus in arbitrary characteristic

1. Calculation of the defining ideal of the total Segre locus

Let $X \subset \mathbb{P}^N$ be as in Theorem III. We assume that X is of codimension $\geqslant 2$, because if X is a hypersurface, then the total Segre locus is determined immediately.

We set $X_L := \overline{\pi_L(X \setminus L)}$, the closure of the image in $\mathbb{P}^{N-\dim(L)-1}$, where $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^{N-\dim(L)-1}$ is the projection from a linear subspace $L \subset \mathbb{P}^N$. In particular, we set $x_L := \pi_L(x)$ for a point $x \in \mathbb{P}^N \setminus L$. The cone $\mathrm{Cone}_L(X)$ of X with vertex L is given by the closure of the preimage $\pi_L^{-1}(X_L) \subset \mathbb{P}^N$, where we have $\deg(X_L) = \deg(\mathrm{Cone}_L(X))$.

Definition 1.1. We set $\mathrm{Loc}_e(X) := \{\, z \in \mathbb{P}^N \mid \deg(X_z) \leqslant e \,\}$ for an integer e.

Here $\mathfrak{S}^{\mathrm{tot}}(X)$ is contained in $\mathrm{Loc}_e(X)$ for some $e < \deg(X)$ in the case when X is not a cone. This is because if $z \in \mathbb{P}^N$ satisfies that $\pi_{z|X}$ is generically finite and is not birational onto its image, then we have $\deg(X_z) < \deg(X)$.

In §1.1, we construct a matrix $\mathbf{\Lambda}(e)$ consisting of iterative higher derivations $D_{\boldsymbol{i}}$, which defines $\mathrm{Loc}_e(X)$ set-theoretically as a determinant variety if $\mathrm{codim}(X, \mathbb{P}^N) = 2$ (Theorem 1.7). In addition, we see examples of actual calculation of the matrix $\mathbf{\Lambda}(e)$ (Examples 1.4 and 1.9). In §1.2, we show that each irreducible component of $\mathfrak{S}^{\mathrm{tot}}(X)$ is equal to an irreducible component of $\mathrm{Loc}_e(X)$ with some $e < \deg(X)$ (Proposition 1.10). In §1.3, we generalize the argument of §2.1 for the case of $\mathrm{codim}(X, \mathbb{P}^N) \geqslant 2$. (Theorem 1.13).

1.1. Determinantal ideal defining $\mathrm{Loc}_e(X)$ set-theoretically for X of codimension two. We use the following notation: Let $x_0, x_1, \dots, x_N$ be a set of homogeneous coordinates on $\mathbb{P}^N$. We denote by $x^{\boldsymbol{i}} = x_0^{i_0} x_1^{i_1} \cdots x_N^{i_N}$ the monomial of multidegree $\boldsymbol{i} = (i_0, i_1, \dots, i_N) \in \mathbb{Z}_{\geqslant 0}^{N+1}$, by $|\boldsymbol{i}| := \sum_{l=0}^N i_l$, and by

$$\boldsymbol{I}_s := \{\, \boldsymbol{i} \in \mathbb{Z}_{\geqslant 0}^{N+1} \mid |\boldsymbol{i}| = s \,\}.$$

For each integer l with $0 \leqslant l \leqslant N$, we set $\boldsymbol{\omega}_l = (\omega_{l,0}, \omega_{l,1}, \dots, \omega_{l,N}) \in \boldsymbol{I}_1$ to satisfy $\omega_{l,l} = 1$ and $\omega_{l,m} = 0$ if $m \neq l$. In arbitrary characteristic, the *iterative higher derivation* [**46**, p209] of polynomials is defined as the operator induced by

$$D_{\boldsymbol{i}} x^{\boldsymbol{j}} := \binom{\boldsymbol{j}}{\boldsymbol{i}} x^{\boldsymbol{j}-\boldsymbol{i}}, \tag{37}$$

where $\binom{\boldsymbol{j}}{\boldsymbol{i}} = \binom{j_0}{i_0}\binom{j_1}{i_1} \cdots \binom{j_N}{i_N}$. Note that, in the characteristic zero case, we have

$$D_{\boldsymbol{i}} f = \frac{1}{i_0! i_1! \cdots i_N!} \Big(\frac{\partial}{\partial x_0}\Big)^{i_0} \Big(\frac{\partial}{\partial x_1}\Big)^{i_1} \cdots \Big(\frac{\partial}{\partial x_N}\Big)^{i_N} (f)$$

for a polynomial f.

Definition 1.2. Let e be an integer. Then, for a homogeneous polynomial $f \in H^0(\mathbb{P}^N, \mathcal{O}(e))$ of degree e and for an integer $s \leqslant e$, we set the following column vector,

$$\lambda_{e-s}(f) := \begin{bmatrix} D_{(e-s)\boldsymbol{\omega}_0} f \\ D_{(e-s-1)\boldsymbol{\omega}_0+\boldsymbol{\omega}_1} f \\ \vdots \\ D_{\boldsymbol{i}} f \\ \vdots \\ D_{(e-s)\boldsymbol{\omega}_N} f \end{bmatrix} \quad (\boldsymbol{i} \in \boldsymbol{I}_{e-s}).$$

Note that $\lambda_0(f) = D_0(f) = f$. In addition, we set the following column vector,

$$\boldsymbol{\lambda}(e)(f) := \lambda_{e-1}(f), \qquad \text{if } p > e \text{ or } p = 0,$$

$$\boldsymbol{\lambda}(e)(f) := \begin{bmatrix} \lambda_{e-1}(f) \\ \lambda_{e-p}(f) \\ \lambda_{e-2p}(f) \\ \vdots \\ \lambda_{e-\lfloor e/p \rfloor \cdot p}(f) \end{bmatrix}, \qquad \text{if } p \leqslant e.$$

Here p is the characteristic of the base field k, and $\lfloor e/p \rfloor := \max\{\, \alpha \in \mathbb{Z} \mid \alpha \leqslant e/p \,\}$.

For a variety $Y \subset \mathbb{P}^N$, we denote by $\operatorname{Vert}(Y) \subset \mathbb{P}^N$ the *maximal vertex* of Y, which is the locus of points $z \in Y$ such that $\pi_{z|_Y}$ is not generically finite.

Lemma 1.3. *Let $F := (f = 0) \subset \mathbb{P}^N$ be a hypersurface of degree e defined by $f \in H^0(\mathbb{P}^N, \mathcal{O}(e))$. Then* $\operatorname{Vert}(F)$ *is equal to the locus $\{\, z \in \mathbb{P}^N \mid \boldsymbol{\lambda}(e)(f)|_z = \mathbf{0} \,\}$, where* $\mathbf{0}$ *is the zero vector.*

Example 1.4. We see what Lemma 1.3 means, by considering an example that $F \subset \mathbb{P}^3$ is a hypersurface of degree e defined by a polynomial $f = x_1^e - x_2 x_3^{e-1}$. Here $\operatorname{Vert}(F)$ must be equal to a point $P = (1, 0, 0, 0)$ since f does not have the variable x_0.

In characteristic zero, the vector $\boldsymbol{\lambda}(e)(f)$ is defined by $\lambda_{e-1}(f)$, which consists of linear polynomials $D_{(e-1)\boldsymbol{\omega}_0} f, D_{(e-2)\boldsymbol{\omega}_0+\boldsymbol{\omega}_1} f, \dots, D_{(e-1)\boldsymbol{\omega}_N} f$. Here we have

$$(38) \qquad D_{(e-1)\boldsymbol{\omega}_1} f = e x_1,\ D_{\boldsymbol{\omega}_2+(e-2)\boldsymbol{\omega}_3} f = -(e-1)x_3,\ D_{(e-1)\boldsymbol{\omega}_3} = -x_2,$$

and $D_{\boldsymbol{i}} f = 0$ for other $\boldsymbol{i} \in \boldsymbol{I}_{e-1}$. Thus, by Lemma 1.3, we can compute that $\operatorname{Vert}(F)$ is equal to $(x_1 = x_3 = x_2 = 0) = \{\, P \,\}$.

Next we study the positive characteristic case. Here $\boldsymbol{\lambda}(e)(f)$ is given by $\lambda_{e-1}(f)$ and $\lambda_{e-\alpha p}(f)$ with $1 \leqslant \alpha \leqslant \lfloor e/p \rfloor$. If $p \nmid e-1$ and $p \nmid e$, then we can calculate $\operatorname{Vert}(F)$ in the same way as above. If $p \mid e-1$, then the polynomial $-(e-1)x_3$ in (38) vanishes. To complete this polynomial, we focus on $\lambda_1(f)$, which is a sub-vector of $\boldsymbol{\lambda}(e)(f)$ because of $e - ((e-1)/p)p = 1$. Here $\lambda_1(f)$ gives a polynomial

$$D_{\boldsymbol{\omega}_2} f = -x_3^{e-1}.$$

Thus, Lemma 1.3 implies that $\operatorname{Vert}(F)$ is equal to $(x_3^{e-1} = x_1 = x_2 = 0) = \{\, P \,\}$. If $p \mid e$, then $e x_1$ in (38) vanishes. On the other hand, $\boldsymbol{\lambda}(e)(f)$ has a sub-vector $\lambda_0(f)$ consisting of the polynomial

$$D_0 f = f.$$

Thus, $\mathrm{Vert}(F)$ is again equal to $(f = x_3 = x_2 = 0) = \{P\}$.

Lemma 1.5. *Let $f \in H^0(\mathbb{P}^N, \mathcal{O}(e))$ be a homogeneous polynomial with $f = \sum_{\boldsymbol{j}\in\boldsymbol{I}_e} f_{\boldsymbol{j}} x^{\boldsymbol{j}}$ ($f_{\boldsymbol{j}} \in k$). Then we have $D_{\boldsymbol{i}} f = \sum_{s=0}^{N} (i_s+1) f_{\boldsymbol{i}+\boldsymbol{\omega}_s} x_s$ for each $\boldsymbol{i} \in \boldsymbol{I}_{e-1}$.*

Proof. We have $D_{\boldsymbol{i}} f = \sum_{\boldsymbol{j}\in\boldsymbol{I}_e} f_{\boldsymbol{j}} D_{\boldsymbol{i}} x^{\boldsymbol{j}} = \sum_{s=0}^{N} f_{\boldsymbol{i}+\boldsymbol{\omega}_s} D_{\boldsymbol{i}} x^{\boldsymbol{i}+\boldsymbol{\omega}_s}$. Since $D_{\boldsymbol{i}} x^{\boldsymbol{i}+\boldsymbol{\omega}_s}$ is equal to $(i_s+1)x_s$, we get the assertion. □

Proof of Lemma 1.3. Let $z \in \mathbb{P}^N$. By a suitable coordinate change on $\mathbb{P}^N$, we may assume $z = (1, 0, \dots, 0)$. If $z \in \mathrm{Vert}(F)$, then f does not have the variable x_0. Hence $D_{\boldsymbol{i}} f|_z = 0$ for each $\boldsymbol{i}$, that is to say, $\boldsymbol{\lambda}(e)(f)|_z = \mathbf{0}$.

Conversely, suppose that $\boldsymbol{\lambda}(e)(f)|_z = \mathbf{0}$, and let $f = \sum_{\boldsymbol{j}\in\boldsymbol{I}_e} f_{\boldsymbol{j}} x^{\boldsymbol{j}}$ with $f_{\boldsymbol{j}} \in k$. For each $\boldsymbol{i} \in \boldsymbol{I}_{e-1}$, Lemma 1.5 implies that

$$(i_0+1) f_{\boldsymbol{i}+\boldsymbol{\omega}_0}|_z = D_{\boldsymbol{i}} f|_z = 0.$$

Hence if $p \nmid (i_0+1)$, then we have $f_{\boldsymbol{i}+\boldsymbol{\omega}_0}|_z = 0$. Therefore $f_{\boldsymbol{j}}|_z = 0$ if $\boldsymbol{j} \in \boldsymbol{I}_e$ satisfies $j_0 > 0$ and $p \nmid j_0$. On the other hand, for $\boldsymbol{i} \in \boldsymbol{I}_{e-\alpha p}$ satisfying $i_0 = 0$, we have $D_{\boldsymbol{i}} f = \sum_{\boldsymbol{u}\in\boldsymbol{I}_{\alpha p}} f_{\boldsymbol{i}+\boldsymbol{u}} D_{\boldsymbol{i}} x^{\boldsymbol{i}+\boldsymbol{u}}$, and hence

$$f_{\boldsymbol{i}+\alpha p\boldsymbol{\omega}_0}|_z = D_{\boldsymbol{i}} f|_z = 0.$$

This implies that $f_{\boldsymbol{j}}|_z = 0$ if $\boldsymbol{j} \in \boldsymbol{I}_e$ satisfies $j_0 > 0$ and $p \mid j_0$. Therefore f does not have the variable x_0, that is to say, $z \in \mathrm{Vert}(F)$. □

Definition 1.6. Let $\{h_1, \dots, h_r\}$ be a basis of the k-vector space $H^0(\mathbb{P}^N, \mathcal{I}_X(e))$, where $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^N}$ is the ideal sheaf of $X \subset \mathbb{P}^N$. For $s \leqslant e$, we set the following matrix:

$$\begin{aligned}\Lambda_{e-s} :&= \begin{bmatrix}\lambda_{e-s}(h_1) & \lambda_{e-s}(h_2) & \cdots & \lambda_{e-s}(h_r)\end{bmatrix}\\ &= \begin{bmatrix} D_{(e-s)\boldsymbol{\omega}_0} h_1 & D_{(e-s)\boldsymbol{\omega}_0} h_2 & \dots & D_{(e-s)\boldsymbol{\omega}_0} h_r \\ D_{(e-s-1)\boldsymbol{\omega}_0+\boldsymbol{\omega}_1} h_1 & D_{(e-s-1)\boldsymbol{\omega}_0+\boldsymbol{\omega}_1} h_2 & \dots & D_{(e-s-1)\boldsymbol{\omega}_0+\boldsymbol{\omega}_1} h_r \\ \cdots & \cdots & \cdots & \cdots \\ D_{\boldsymbol{i}} h_1 & D_{\boldsymbol{i}} h_2 & \dots & D_{\boldsymbol{i}} h_r \\ \cdots & \cdots & \cdots & \cdots \\ D_{(e-s)\boldsymbol{\omega}_N} h_1 & D_{(e-s)\boldsymbol{\omega}_N} h_2 & \dots & D_{(e-s)\boldsymbol{\omega}_N} h_r \end{bmatrix} \quad (\boldsymbol{i} \in I_{e-s}),\end{aligned}$$

In addition, we set $\boldsymbol{\Lambda}(e) := \begin{bmatrix}\boldsymbol{\lambda}(e)(h_1) & \boldsymbol{\lambda}(e)(h_2) & \cdots & \boldsymbol{\lambda}(e)(h_r)\end{bmatrix}$. In this setting, it follows that $\boldsymbol{\Lambda}(e) = \Lambda_{e-1}$ if either $p > e$ or $p = 0$, and that

$$\boldsymbol{\Lambda}(e) = \begin{bmatrix} \Lambda_{e-1} \\ \Lambda_{e-p} \\ \Lambda_{e-2p} \\ \vdots \\ \Lambda_{e-\lfloor e/p\rfloor\cdot p} \end{bmatrix} \quad \text{if } p \leqslant e.$$

Now we denote by $Z_s(\boldsymbol{\Lambda}(e))$ the zero set of the $s \times s$ minors of $\boldsymbol{\Lambda}(e)$ for an integer s.

Theorem 1.7. *Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety of codimension $\geqslant 2$, and let $r := h^0(\mathbb{P}^N, \mathcal{I}_X(e))$ as above. Then we have $\mathrm{Loc}_e(X) \subset Z_r(\boldsymbol{\Lambda}(e))$. Moreover, the equality holds if X is of codimension 2 in $\mathbb{P}^N$ and is not a cone.*

Remark 1.8. The submatrix Λ_{e-1} of $\boldsymbol{\Lambda}(e)$ plays a central role in the actual calculation of $\mathrm{Loc}_e(X)$, where Λ_{e-1} consists of linear polynomials. In the positive characteristic case, some entries of Λ_{e-1} may vanish. Then, instead of these entries, we focus on polynomials of degree > 1 appeared in submatrices $\{\Lambda_{e-\alpha p}\}$ (see Examples 1.4 and 1.9).

In the following example, we see how Theorem 1.7 is used to calculate $\mathrm{Loc}_e(X)$.

Example 1.9. Let $X \subset \mathbb{P}^3$ be a space rational curve of degree e^2 defined by two polynomials,

$$h_1 = x_1^e - x_2 x_3^{e-1} \quad \text{and} \quad h_2 = x_0^e - x_1 x_3^{e-1}.$$

Note that X is parametrized by a morphism $\mathbb{P}^1 \to \mathbb{P}^3 : (s,1) \mapsto (s, s^e, s^{e^2}, 1)$. Here we have:

(a) $\mathrm{Loc}_e(X) = \{(1,0,0,0),(0,0,1,0)\}$ if either $p \nmid e$ or $p = 0$,
(b) $\mathrm{Loc}_e(X) = (x_3 = x_0^e x_2 - x_1^{e+1} = 0)$ if $p \mid e$.

Proof. Here, the polynomials h_1 and h_2 give a basis of $H^0(\mathbb{P}^N, \mathcal{I}_X(e))$. Note that h_1 is the polynomial f studied in Example 1.4. From Theorem 1.7, the locus $\mathrm{Loc}_e(X)$ is equal to $Z_2(\boldsymbol{\Lambda}(e))$, the zero set of 2×2 minors of $\boldsymbol{\Lambda}(e)$. The submatrix corresponding to the nonzero part of Λ_{e-1} is equal to

$$A_1 := \begin{bmatrix} D_{(e-1)\boldsymbol{\omega}_0} h_1 & D_{(e-1)\boldsymbol{\omega}_0} h_2 \\ D_{(e-1)\boldsymbol{\omega}_1} h_1 & D_{(e-1)\boldsymbol{\omega}_1} h_2 \\ D_{\boldsymbol{\omega}_1+(e-2)\boldsymbol{\omega}_3} h_1 & D_{\boldsymbol{\omega}_1+(e-2)\boldsymbol{\omega}_3} h_2 \\ D_{\boldsymbol{\omega}_2+(e-2)\boldsymbol{\omega}_3} h_1 & D_{\boldsymbol{\omega}_2+(e-2)\boldsymbol{\omega}_3} h_2 \\ D_{(e-1)\boldsymbol{\omega}_3} h_1 & D_{(e-1)\boldsymbol{\omega}_3} h_2 \end{bmatrix} = \begin{bmatrix} 0 & e x_0 \\ e x_1 & 0 \\ 0 & -(e-1)x_3 \\ -(e-1)x_3 & 0 \\ -x_2 & -x_1 \end{bmatrix}.$$

(a) Suppose $p \nmid e$ and $p \nmid e-1$. Then A_1 gives three non-zero minors:

$$\begin{vmatrix} 0 & -(e-1)x_3 \\ -(e-1)x_3 & 0 \end{vmatrix}, \quad \begin{vmatrix} e x_1 & 0 \\ -x_2 & -x_1 \end{vmatrix}, \quad \begin{vmatrix} 0 & e x_0 \\ -x_2 & -x_1 \end{vmatrix},$$

which are equal to $-(e-1)^2 x_3^2$, $-e x_1^2$, $e x_0 x_2$. Thus we find that $Z_2(\boldsymbol{\Lambda}(e))$ is contained in $Z_2(A_1) = \{(1,0,0,0),(0,0,1,0)\}$.

Suppose $p \nmid e$ and $p \mid e-1$. Then the minor $-(e-1)^2 x_3^2$ of the matrix A_1 vanishes. Instead of this, we consider Λ_1, which is a submatrix of $\boldsymbol{\Lambda}(e)$ because of $e - ((e-1)/p)\cdot p = 1$. Here Λ_1 contains a submatrix

$$A_2 := \begin{bmatrix} D_{\boldsymbol{\omega}_1} h_1 & D_{\boldsymbol{\omega}_1} h_2 \\ D_{\boldsymbol{\omega}_2} h_1 & D_{\boldsymbol{\omega}_2} h_2 \end{bmatrix} = \begin{bmatrix} e x_1^{e-1} & -x_3^{e-1} \\ -x_3^{e-1} & 0 \end{bmatrix},$$

which gives a minor $-x_3^{2(e-1)}$. Combining A_2 and A_1, we find that $Z_2(\boldsymbol{\Lambda}(e))$ is contained in the set $\{(1,0,0,0),(0,0,1,0)\}$ as above.

In addition, the opposite inclusion holds, as follows: For each $\boldsymbol{i}$, $D_{\boldsymbol{i}} h_1$ does not have the variable x_0 since so is h_1. This implies that $D_{\boldsymbol{i}} h_1|_{(1,0,0,0)} = 0$. Hence every 2×2 minors of $\boldsymbol{\Lambda}(e)|_{(1,0,0,0)}$ is equal to zero, that is, $(1,0,0,0) \in Z_2(\boldsymbol{\Lambda}(e))$. In a similar way, we have $D_{\boldsymbol{i}} h_2|_{(0,0,1,0)} = 0$ and have $(0,0,1,0) \in Z_2(\boldsymbol{\Lambda}(e))$.

(b) Suppose $p \mid e$. Then entries ex_0 and ex_1 of A_1 vanish. Instead of these, we focus on Λ_0, which is a submatrix of $\mathbf{\Lambda}(e)$ because of $e - (e/p) \cdot p = 0$. Here Λ_0 consists of $D_0 h_i = h_i$ with $i = 1, 2$. Thus A_1 and Λ_0 gives the following submatrix of $\mathbf{\Lambda}(e)$:

$$A_3 := \begin{bmatrix} D_{\boldsymbol{\omega}_1+(e-2)\boldsymbol{\omega}_3}h_1 & D_{\boldsymbol{\omega}_1+(e-2)\boldsymbol{\omega}_3}h_2 \\ D_{\boldsymbol{\omega}_2+(e-2)\boldsymbol{\omega}_3}h_1 & D_{\boldsymbol{\omega}_2+(e-2)\boldsymbol{\omega}_3}h_2 \\ D_{(e-1)\boldsymbol{\omega}_3}h_1 & D_{(e-1)\boldsymbol{\omega}_3}h_2 \\ D_0 h_1 & D_0 h_2 \end{bmatrix} = \begin{bmatrix} 0 & -(e-1)x_3 \\ -(e-1)x_3 & 0 \\ -x_2 & -x_1 \\ x_1^e - x_2 x_3^{e-1} & x_0^e - x_1 x_3^{e-1} \end{bmatrix}.$$

Therefore $Z_2(\mathbf{\Lambda}(e))$ is contained in $Z_2(A_3) = (x_3 = x_0^e x_2 - x_1^{e+1} = 0)$. In addition, the opposite inclusion holds, as follows: For $\boldsymbol{i} \neq 0$ with $|\boldsymbol{i}| < e$, the polynomial $D_{\boldsymbol{i}} h_1$ (resp. $D_{\boldsymbol{i}} h_2$) does not have the variable x_1 (resp. x_0) because of $p \mid e$. Thus, on the locus $(x_3 = 0)$, the nonzero entries of $\mathbf{\Lambda}(e)|_{(x_3=0)}$ are given by the third and fourth row vectors of $A_3|_{(x_3=0)}$. This implies that $Z_2(\mathbf{\Lambda}(e)) = Z_2(\mathbf{\Lambda}(e)) \cap (x_3 = 0)$ is equal to $(x_3 = x_0^e x_2 - x_1^{e+1} = 0)$. □

Proof of Theorem 1.7. Suppose $z \in \mathrm{Loc}_e(X)$. Then $X_z \subset \mathbb{P}^{N-1}$ is a variety of degree $\leqslant e$. Thus there exists a polynomial $f \in \pi_z^* H^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e)) \subset H^0(\mathbb{P}^N, \mathcal{I}_X(e))$. Here we have $f = \sum_{j=1}^r a_j h_j$ with some $a_1, \ldots, a_r \in k$. It follows from Lemma 1.3 that

$$\sum_{j=1}^r a_j \boldsymbol{\lambda}(e)(h_j)|_z = \boldsymbol{\lambda}(e)(f)|_z = \mathbf{0}.$$

Since the r column vectors of $\mathbf{\Lambda}(e)|_z$ are linearly dependent, every $r \times r$ minor of $\mathbf{\Lambda}(e)|_z$ is equal to zero, i.e., $z \in Z_r(\mathbf{\Lambda}(e))$.

Suppose $\mathrm{codim}(X, \mathbb{P}^N) = 2$, and let $z \in Z_r(\mathbf{\Lambda}(e))$. Then there exist $a_1, \ldots, a_r \in k$ such that $\sum_{j=1}^r a_j \boldsymbol{\lambda}(e)(h_j)|_z = \mathbf{0}$. Setting $f = \sum_{j=1}^r a_j h_j \in H^0(\mathbb{P}^N, \mathcal{I}_X(e))$, we have $\boldsymbol{\lambda}(e)(f)|_z = \mathbf{0}$. From Lemma 1.3, the hypersurface $F := (f = 0) \subset \mathbb{P}^N$ is a cone with vertex z. Since $X_z \subset \mathbb{P}^{N-1}$ coincides with an irreducible component of F_z, we obtain $\deg(X_z) \leqslant e$; hence $z \in \mathrm{Loc}_e(X)$. □

1.2. Irreducible component of the total Segre locus. We denote by $\mu_{z,X}$ the *multiplicity* of X at a point $z \in \mathbb{P}^N$, i.e., the intersection multiplicity of $X \cap L$ along z for a general linear subspace $L \simeq \mathbb{P}^{\mathrm{codim}(X)}$ containing z. Here we set $\mu_{z,X} = 0$ if $z \notin X$.

Proposition 1.10. *Let $X \subset \mathbb{P}^N$ be as in Theorem III, and assume that X is not a cone. For an irreducible component Z of $\mathfrak{S}^{\mathrm{tot}}(X)$, there exist integers $e, m < \deg(X)$ such that $\deg(X_z) = e$ and $\mu_{z,X} = m$ for general $z \in Z$ and that Z is an irreducible component of* $\mathrm{Loc}_e(X)$.

In order to prove Proposition 1.10, we need two basic lemmas.

Lemma 1.11. *Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety. For a point $z \in \mathbb{P}^N \setminus \mathrm{Vert}(X)$, the following hold.*

(a) *We have* $\deg(X) - \mu_{z,X} = \deg(\pi_{z|X}) \cdot \deg(X_z)$.

(b) *Let $w \in \mathbb{P}^N \setminus \mathrm{Cone}_z(X)$ satisfy that $\pi_{w|X}$ is birational. Then we have $\mu_{z_w, X_w} = \mu_{z,X}$.*

(c) *For an integer m, the set of points $z \in \mathbb{P}^N$ satisfying $\mu_{z,X} \leqslant m$ is open in $\mathbb{P}^N$.*

PROOF. (a) Let $c := \operatorname{codim}(X, \mathbb{P}^N)$. We fix a general $(c-2)$-dimensional linear subspace $L_1 \subset \mathbb{P}^{N-1}$ such that $L_1 \cap X_z = \emptyset$, and set $L := \overline{\pi_z^{-1}(L_1)} \subset \mathbb{P}^N$. Then the projection $\pi_{L|X} : X \dashrightarrow \mathbb{P}^{N-c}$ factors into $\pi_{z|X} : X \dashrightarrow X_z \subset \mathbb{P}^{N-1}$ followed by $\pi_{L_1|X_z} : X_z \dashrightarrow \mathbb{P}^{N-c}$. Therefore we have

$$\deg(\pi_{L|X}) = \deg(\pi_{z|X}) \cdot \deg(\pi_{L_1|X_z}) \text{ and } \deg(\pi_{L_1|X_z}) = \deg(X_z).$$

For a general point $x \in \mathbb{P}^{N-c}$ and for a c-dimensional linear subspace $M_x := \overline{\pi_L^{-1}(x)} \subset \mathbb{P}^N$, it follows that $\pi_{L|X}^{-1}(x) = X \cap M_x \setminus \{ z \}$. Since $X \cap M_x \setminus \{ z \}$ has length $\deg(X) - \mu_{z,X}$, we obtain $\deg(\pi_{L|X}) = \deg(X) - \mu_{z,X}$. Thus the result follows.

(b) From (a), it follows that

$$\deg(X) - \mu_{z,X} = \deg(\pi_{z|X}) \cdot \deg(X_z) \text{ and } \deg(X_w) - \mu_{z_w,X_w} = \deg(\pi_{z_w|X_w}) \cdot \deg(X_{\overline{zw}}).$$

Since $w_z \notin X_z$, we have $\deg(X_z) = \deg(\pi_{w_z|X_z}) \cdot \deg(X_{\overline{zw}})$. Since $\pi_{w|X}$ is birational, we have $\deg(X) = \deg(X_w)$ and $\deg(\pi_{z_w|X_w}) = \deg(\pi_{z|X}) \cdot \deg(\pi_{w_z|X_z})$. Hence we obtain $\mu_{z_w,X_w} = \mu_{z,X}$.

(c) We show the result by induction on $c = \operatorname{codim}(X, \mathbb{P}^N)$. Suppose $c = 1$. For each $z \in \mathbb{P}^N$, changing coordinates, we may assume $z = (1, 0, \dots, 0)$. For the defining equation $f \in H^0(\mathbb{P}^N, \mathcal{O}(e))$ of X with $e = \deg(X)$, we have $f = x_0^{e-s} f_s + x_0^{e-s-1} f_{s+1} + \cdots + f_e$ with $s \leqslant e$, $f_j \in k[x_1, \dots, x_N]_j$, and $f_s \neq 0$. Then it follows $\mu_{z,X} = s$; hence we have the result by using $D_{\boldsymbol{i}}$ defined in equation (37) in §1.1. Suppose $c > 1$, and let $z \in \mathbb{P}^N$ satisfy $\mu_{z,X} \leqslant m$. We show that there exists an open neighborhood U of z such that every $x \in U$ satisfies $\mu_{x,X} \leqslant m$. Let $w \in \mathbb{P}^N \setminus \operatorname{Cone}_z(X)$ be a general point such that $\pi_{w|X} : X \to X_w \subset \mathbb{P}^{N-1}$ is birational. Let $V \subset \mathbb{P}^{N-1}$ be the set of points $y \in \mathbb{P}^{N-1}$ such that $\mu_{y,X_w} \leqslant m$. By induction hypothesis, the subset V is open in $\mathbb{P}^{N-1}$. From (b), it follows that $\mu_{z_w,X_w} = \mu_{z,X}$; thus we have $z_w \in V$. Let

$$U' := \mathbb{P}^N \setminus \overline{\{ x \in \mathbb{P}^N \mid w \in \operatorname{Cone}_x(X) \}}.$$

Again from (b), we have $\mu_{x_w,X_w} = \mu_{x,X}$ for each $x \in U'$. Let $U := \pi_w^{-1}(V) \cap U'$, which contains the point z and satisfies that $\mu_{x,X} \leqslant m$ for any $x \in U$. ☐

Lemma 1.12. *Let X be as in* Lemma 1.11 *and assume that X is not a cone. Then the subset* $\operatorname{Loc}_e(X)$ *is closed in $\mathbb{P}^N$.*

PROOF. We prove the result by induction on $c = \operatorname{codim}(X, \mathbb{P}^N)$. Suppose $c = 2$. Then it follows from Theorem 1.7 that $\operatorname{Loc}_e(X)$ coincides with an determinantal variety, thus is closed in $\mathbb{P}^N$.

Suppose $c > 2$. For each $z \in \mathbb{P}^N$ with $\deg(X_z) > e$, it is sufficient to show that there exists an open neighborhood U of z such that $\deg(X_x) > e$ for all $x \in U$. For a point

$$w \in \mathbb{P}^N \setminus (\operatorname{Cone}_z(X) \cup \mathfrak{S}^{\text{out}}(X) \cup \pi_z^{-1}\mathfrak{S}^{\text{out}}(X_z)),$$

we have birational projections $\pi_{w|X}$ and $\pi_{w_z|X_z}$. Then $\deg(X) = \deg(X_w)$ and $\deg(\pi_{z|X}) = \deg(\pi_{z_w|X_w})$. By induction hypothesis, the subset $V = \mathbb{P}^{N-1} \setminus \operatorname{Loc}_e(X_w)$ is open. Let $U := \pi_w^{-1}(V)$, where we have $\deg(X_x) \geqslant \deg(X_{\overline{xw}}) > e$ for any $x \in U$. From

Lemma 1.11(b), we have $\mu_{z,X} = \mu_{z_w,X_w}$. From Lemma 1.11(a), we have $\deg(X_{\overline{zw}}) = (\deg(X_w) - \mu_{z_w,X_w})/\deg(\pi_{z_w|X_w}) = \deg(X_z) > e$. Thus $z \in U$, and hence the assertion follows. $\square$

PROOF OF PROPOSITION 1.10. For an irreducible component Z of $\mathfrak{S}^{\mathrm{tot}}(X)$, we take m to be the largest integer such that $\mu_{z,X} \geqslant m$ for any $z \in Z$, and take e to be the smallest integer such that $Z \subset \mathrm{Loc}_e(X)$. Let $z \in Z$ be a general point. From Lemma 1.11(c), we have $\mu_{z,X} = m$. From Lemma 1.12, we have $\deg(X_z) = e$. Note that $(\deg(X) - m)/e = \deg(\pi_{z|X}) > 1$ due to Lemma 1.11(a).

Let Z' be an irreducible component of $\mathrm{Loc}_e(X)$ containing Z, and let m' be the largest integer such that $\mu_{z',X} \geqslant m'$ for any $z' \in Z'$. For general $z' \in Z'$, Lemma 1.11(c) implies that $\mu_{z',X} = m'$. Since $m' \leqslant m$, it follows from Lemma 1.11(a) that

$$\deg(\pi_{z'}|_X) \geqslant (\deg X - m')/e \geqslant (\deg X - m)/e > 1.$$

Hence $Z' \subset \mathfrak{S}^{\mathrm{tot}}(X)$, which implies $Z = Z'$. $\square$

1.3. Determinantal ideal defining $\mathrm{Loc}_e(X)$ set-theoretically for X of any codimension. Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety of codimension $\geqslant 2$, and assume that X is not a cone. For an irreducible subvariety $Z \subset \mathbb{P}^N$ and for an integer e, we define $\bar{r}(Z, e)$ as the integer satisfying that $h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e)) = \bar{r}(Z, e)$ for general $z \in Z$. We also set $r(e) := h^0(\mathbb{P}^N, \mathcal{I}_X(e))$.

Recall that, from Lemma 1.12, the locus $\mathrm{Loc}_{e_0}(X)$ with an integer e_0 is a closed subset of $\mathbb{P}^N$. Now, we show the following generalized result of Theorem 1.7.

Theorem 1.13. *Let $X \subset \mathbb{P}^N$ be as above, let e_0 be an integer, and let $\{ Z_j \}_{j=1}^{j_0}$ be the irreducible components of $\mathrm{Loc}_{e_0}(X)$. Then we have*

$$\mathrm{Loc}_{e_0}(X) = \bigcup_{1 \leqslant j \leqslant j_0} \bigcap_{e \in \mathbb{N}} Z_{r(e) - \bar{r}(Z_j, e) + 1}(\mathbf{\Lambda}(e)).$$

In addition, there exists integers j_1 and e_1 with $1 \leqslant j_1 \leqslant j_0$ and $e_1 > 0$ such that $\mathrm{Loc}_{e_0}(X)$ is equal to $\bigcap_{e \geqslant e_1} Z_{r(e) - \bar{r}(Z_{j_1}, e) + 1}(\mathbf{\Lambda}(e))$.

Remark 1.14. The integer $\bar{r}(Z, e)$ is obtained as follows: Note that the Euler sequence $0 \to \Omega^1_{\mathbb{P}^N} \to V \otimes_k \mathcal{O}_{\mathbb{P}^N}(-1) \to \mathcal{O}_{\mathbb{P}^N} \to 0$ with $V := H^0(\mathbb{P}^N, \mathcal{O}(1))$ induces $\mathbb{P}(V) \times \mathbb{P}^N \dashrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^N}(1))$, a rational map of projective bundles over $\mathbb{P}^N$ which gives the projection

$$\pi_z : \mathbb{P}^N \simeq \mathbb{P}(V) \dashrightarrow \mathbb{P}^{N-1} \simeq \mathbb{P}(\Omega^1_{\mathbb{P}^N}(1) \otimes k(z))$$

for each $z \in \mathbb{P}^N$. Let us consider $\varphi : S^e(\Omega^1_{\mathbb{P}^N}(1)) \to S^e(V) \otimes_k \mathcal{O}_{\mathbb{P}^N}$, a injective homomorphism of e-th symmetric products. We regard $H^0(\mathbb{P}^N, \mathcal{I}_X(e))$ as a subspace of $S^e(V)$, and set $\mathcal{J}^e_X := \varphi^{-1}(H^0(\mathbb{P}^N, \mathcal{I}_X(e)) \otimes_k \mathcal{O}_{\mathbb{P}^N})$. Then $\mathcal{J}^e_X \otimes k(z)$ is isomorphic to $H^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e))$ for each $z \in \mathbb{P}^N$. Applying the semi-continuity theorem to the sheaf $\mathcal{J}^e_X|_Z$ on Z, we find $\bar{r}(Z, e)$ satisfying that $h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e))$ is greater than or equal to $\bar{r}(Z, e)$ for any $z \in Z$ and is equal to $\bar{r}(Z, e)$ for general $z \in Z$.

Proposition 1.15. *Let $X \subset \mathbb{P}^N$ be as above, let $e \in \mathbb{N}$, and let $r := r(e)$. For an integer $\bar{r}$, we have*

$$Z_{r - \bar{r} + 1}(\mathbf{\Lambda}(e)) = \{ z \in \mathbb{P}^N \mid h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e)) \geqslant \bar{r} \}.$$

Proof. We show the inclusion "$\supset$". Let $z \in \mathbb{P}^N$ satisfy $h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e)) \geqslant \bar{r}$. Then we find $\bar{r}$ polynomials $f_1, f_2, \dots, f_{\bar{r}} \in \pi_z^* H^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e)) \subset H^0(\mathbb{P}^N, \mathcal{I}_X(e))$ which are linearly independent. Here, the hypersurface in $\mathbb{P}^N$ defined by f_s is a cone with vertex z. Thus Lemma 1.3 implies that $\boldsymbol{\lambda}(e)(f_s)|_z = \mathbf{0}$ for each $1 \leqslant s \leqslant \bar{r}$. It follows that the matrix $\boldsymbol{\Lambda}(e)|_z$ is of rank $\leqslant r - \bar{r}$, as in the proof of Theorem 1.7. Hence $z \in Z_{r-\bar{r}+1}(\boldsymbol{\Lambda}(e))$. Conversely, "$\subset$" can be shown in a similar way. □

Corollary 1.16. *Let $Z \subset \mathbb{P}^N$ be an irreducible subvariety, and let $\bar{r} := \bar{r}(Z, e)$. Then we have $Z \subset Z_{r-\bar{r}+1}(\boldsymbol{\Lambda}(e))$.*

Lemma 1.17. *Let $Z \subset \mathbb{P}^N$ be an irreducible subvariety.* (a) *Then there exists a numeric polynomial P such that $P = P_{X_z}$ for general $z \in Z$, where P_{X_z} is the Hilbert polynomial of the subvariety $X_z \subset \mathbb{P}^{N-1}$.* (b) *In addition, $\bar{r}(Z, e) = h^0(\mathbb{P}^{N-1}, \mathcal{O}(e)) - P(e)$ for $e \gg 0$.*

Proof. (a) As in Remark 1.14, regarding X as a subvariety of $\mathbb{P}(V)$, we have the rational map $\pi : X \times Z \dashrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^N}(1)|_Z)$ which gives the projection $\pi_{z|X} : X \setminus \{ z \} \to \mathbb{P}^{N-1}$ for each $z \in Z$. Let $\mathcal{X} := \overline{\mathrm{im}(\pi)} \subset \mathbb{P}(\Omega^1_{\mathbb{P}^N}(1)|_Z)$ and let $q : \mathcal{X} \to Z$ be the projection. Then we have $q^{-1}(z) = X_z$ for each $z \in Z$. By [**49**, p57, Prop.], there exists an open subset $Z_0 \subset Z$ such that the morphism $q^{-1}(Z_0) \to Z_0$ is flat. Thus we have a polynomial P such that $P = P_{X_z}$ for each $z \in Z_0$.

(b) From (a), the polynomial $Q(e) := h^0(\mathbb{P}^{N-1}, \mathcal{O}(e)) - P(e) \in \mathbb{Q}[e]$ is equal to $\chi(\mathcal{I}_{X_z}(e))$ for any $z \in Z_0$. Thus, from [**49**, p101, Thm.], there exists an integer m depending only on Q such that X_z is m-regular in the sense of Castelnuovo-Mumford for any $z \in Z_0$. Then $Q(e) = h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e))$ for any $e \geqslant m - 1$ and for any $z \in Z_0$. On the other hand, for each integer e, we have a general point $z \in Z$ such that $\bar{r}(Z, e) = h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e))$. As a result, we have $Q(e) = \bar{r}(Z, e)$ for any $e \geqslant m - 1$. □

Proof of Theorem 1.13. We set $Z'_j := \bigcap_{e \in \mathbb{N}} Z_{r(e)-\bar{r}(Z_j, e)+1}(\boldsymbol{\Lambda}(e))$. From Corollary 1.16, we have that $Z_j \subset Z'_j$. Conversely, let us take a point $z \in Z'_j$. Then, from Proposition 1.15, we have $h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e)) \geqslant \bar{r}(Z_j, e)$ for $e \geqslant 0$. From Lemma 1.17(a), there exists a polynomial P_j such that $P_j = P_{X_w}$ for general $w \in Z_j$. For $e \gg 0$, it follows $P_{X_z}(e) = h^0(\mathbb{P}^{N-1}, \mathcal{O}(e)) - h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e))$, and hence Lemma 1.17(b) implies $P_{X_z}(e) \leqslant P_j(e) = P_{X_w}(e)$. Thus $\deg(X_z) \leqslant \deg(X_w) \leqslant e_0$. Therefore we have $Z'_j \subset \mathrm{Loc}_{e_0}(X)$, and the first assertion follows.

Let j_1 be an integer such that $P_{j_1}(e) \geqslant P_j(e)$ for $1 \leqslant j \leqslant j_0$ and $e \gg 0$. Then there exists an integer e_1 such that $P_{j_1}(e) \geqslant P_j(e)$ and $\bar{r}(Z_{j_1}, e) \leqslant \bar{r}(Z_j, e)$ for $1 \leqslant j \leqslant j_0$ and $e \geqslant e_1$, due to Lemma 1.17(b). We set $Z'' := \bigcap_{e \geqslant e_1} Z_{r(e)-\bar{r}(Z_{j_1}, e)+1}(\boldsymbol{\Lambda}(e))$. Then Proposition 1.15 implies that Z'' contains Z'_j with $1 \leqslant j \leqslant j_0$. Hence it follows $\mathrm{Loc}_{e_0}(X) \subset Z''$. Next, we take $z \in Z''$. Then $h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e)) \geqslant \bar{r}(Z_j, e)$ for $e \geqslant e_1$. Thus, in the same way as above, by taking general $w \in Z_{j_1}$, we have $\deg(X_z) \leqslant \deg(X_w) \leqslant e_0$. This implies $Z'' = \mathrm{Loc}_{e_0}(X)$. □

Corollary 1.16 and the following two related lemmas will be applied in the proof of Theorem 2.2.

Lemma 1.18. *Let $Y \subset \mathbb{P}^N$ be a projective variety of degree e, and let $f_1, f_2, \dots, f_{s_0}$ be a basis of $H^0(\mathbb{P}^N, \mathcal{I}_Y(e))$. Then Y coincides with $\bigcap_{s=1}^{s_0}(f_s = 0) \subset \mathbb{P}^N$, the intersection of hypersurfaces defined by f_s.*

Proof. We immediately have $Y \subset \bigcap_{s=1}^{s_0}(f_s = 0)$. Conversely, let $x \in \mathbb{P}^N \setminus Y$. We take a general linear subspace $L \subset \mathbb{P}^N \setminus \mathrm{Cone}_x(Y)$ of dimension $N - \dim(Y) - 2$. Then we have $x \notin \mathrm{Cone}_L(Y)$. Since L is general, $\mathrm{Cone}_L(Y)$ is a hypersurface of degree e, and hence its defining polynomial is contained in $H^0(\mathbb{P}^N, \mathcal{I}_Y(e))$. Thus it follows $\bigcap_{s=1}^{s_0}(f_s = 0) \subset \mathrm{Cone}_L(Y)$, which implies that $x \notin \bigcap_{s=1}^{s_0}(f_s = 0)$. □

Lemma 1.19. *Let $Y \subset \mathbb{P}^N$ be a cone with the maximal vertex $M := \mathrm{Vert}(Y)$, and let $\{ F_s \}_{s=1}^{s_0}$ be hypersurfaces in $\mathbb{P}^N$ such that $M \subset \bigcap_{s=1}^{s_0} \mathrm{Vert}(F_s)$ and $Y = \bigcap_{s=1}^{s_0} F_s$. Then $M = \bigcap_{s=1}^{s_0} \mathrm{Vert}(F_s)$.*

Proof. Let $z \in \bigcap_{s=1}^{s_0} \mathrm{Vert}(F_s)$. For any $y \in Y$ and for each s, since $y \in F_s$, the line $\overline{yz}$ is contained in F_s. This implies $\overline{yz} \subset Y$, which means $z \in M$. Thus the assertion follows. □

2. Linearity of the total Segre locus

2.1. Example of a non-linear total Segre locus. The following example shows that the linearity of $\mathfrak{S}^{\mathrm{tot}}(X)$ does not hold in general if the characteristic p is small.

Example 2.1 ([**19**]). Let ℓ be a prime number, and let $X \subset \mathbb{P}^3$ be a space rational curve of degree ℓ^2 defined by $h_1 = x_1^\ell - x_2 x_3^{\ell-1}$ and $h_2 = x_0^\ell - x_1 x_3^{\ell-1}$, which is parametrized by a morphism $\mathbb{P}^1 \to \mathbb{P}^3 : (s, 1) \mapsto (s, s^\ell, s^{\ell^2}, 1)$. Then the following holds:

(a) Suppose that either $p \neq \ell$ or $p = 0$. Then we have $\mathfrak{S}^{\mathrm{out}}(X) = \{ (1, 0, 0, 0) \}$. Moreover we have $\mathfrak{S}^{\mathrm{inn}}(X) = \{ (0, 0, 0, 1) \}$ if $\ell \geqslant 3$.
(b) Suppose $p = \ell$. Then $\mathfrak{S}^{\mathrm{out}}(X)$ is equal to a non-linear curve $(x_3 = x_0^\ell x_2 - x_1^{\ell+1} = 0)$. Moreover $\mathfrak{S}^{\mathrm{inn}}(X)$ is equal to the non-linear curve X if $\ell \geqslant 3$.
(c) If $\ell = 2$, then it follows $\mathfrak{S}^{\mathrm{inn}}(X) = \emptyset$ in arbitrary characteristic.

Proof. By using our method (§1), We determine the locus $\mathfrak{S}^{\mathrm{out}}(X)$ in the cases (a-b). Let $z \in \mathbb{P}^N \setminus X$ be a point. It follows from $\deg(X) = \ell^2$ and Lemma 1.11(a) that we have $\deg(X_z) = \ell$ if and only if $z \in \mathfrak{S}^{\mathrm{out}}(X)$. This implies that $\mathfrak{S}^{\mathrm{out}}(X)$ is equal to the closure of $\mathrm{Loc}_\ell(X) \setminus X$, where we already calculated $\mathrm{Loc}_\ell(X)$ in Example 1.9.

(a) Suppose $p \neq \ell$. Then we have $\mathrm{Loc}_\ell(X) = \{ (1, 0, 0, 0), (0, 0, 1, 0) \}$. Since $(0, 0, 1, 0) \in X$, it follows $\mathfrak{S}^{\mathrm{out}}(X) = \{ (1, 0, 0, 0) \}$.

(b) Suppose $p = \ell$. Then we have $\mathrm{Loc}_\ell(X) = (x_3 = x_0^\ell x_2 - x_1^{\ell+1} = 0)$, which is irreducible and not contained in X; hence $\mathfrak{S}^{\mathrm{out}}(X) = \mathrm{Loc}_\ell(X)$.

In a similar way, we can calculate the defining ideal of $\mathfrak{S}^{\mathrm{inn}}(X)$ in the case (a). In the case (b), since $\dim(\mathfrak{S}^{\mathrm{out}}(X)) = 1$, we have $X_z = \pi_z(\mathfrak{S}^{\mathrm{out}}(X)) \subset \mathbb{P}^2$ for general $z \in X$. Since $\deg(\pi_z(\mathfrak{S}^{\mathrm{out}}(X))) = \ell + 1$, we have $X = \mathfrak{S}^{\mathrm{inn}}(X)$ if $\ell \geqslant 3$. □

2.2. Proof of linearity of the total Segre locus. Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety of codimension $\geqslant 2$. We denote by $\mathrm{Str}(X) \subset \mathbb{P}^N$ the intersection of all the embedded tangent spaces to X at smooth points. Here each point of $\operatorname{Str}(X)$ is called a *strange point* of X. We have $z \in \operatorname{Str}(X)$ if and only if either $z \in \operatorname{Vert}(X)$ or $\pi_{z|X}$ is inseparable. Hence $\operatorname{Str}(X)$ is a linear subspace of dimension $< \dim(X)$ contained in $\mathfrak{S}^{\mathrm{tot}}(X)$.

Theorem 2.2. *Assume that X is not a cone, let $e_0 \in \mathbb{N}$, and let Z be an irreducible closed subset of $\mathfrak{S}^{\mathrm{tot}}(X)$, such that $Z \not\subset \operatorname{Str}(X)$ and that $\deg(X_z) = e_0$ for general $z \in Z$. Suppose either $p > e_0$ or $p = 0$. Then there exists a $(\dim(X)+1)$-dimensional variety $Y \subset \mathbb{P}^N$ of degree e_0 which contains X and satisfies*

$$Z \subset \operatorname{Vert}(Y) \subset \mathfrak{S}^{\mathrm{tot}}(X).$$

In particular, if Z is an irreducible component of $\mathfrak{S}^{\mathrm{tot}}(X)$, then Z is linear, and moreover is equal to $\operatorname{Vert}(Y)$.

For the proof, we maintain the following notations. We fix a general point $z \in Z \setminus \operatorname{Str}(X)$ such that $h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e)) = \bar{r}(Z, e)$ for each $e \leqslant e_0$, where $\bar{r}(Z, e)$ is the integer stated in Remark 1.14. Let us consider the cone

$$Y := \operatorname{Cone}_z(X) \subset \mathbb{P}^N$$

and the maximal vertex $M := \operatorname{Vert}(Y)$, where we have $\deg(Y) = e_0$.

We denote by $S(\mathbb{P}^N)$ the homogeneous coordinate ring of $\mathbb{P}^N$, by $I(V)$ the homogeneous ideal of a subvariety V, and by $I(V)_d$ the set of polynomials of degree d in $I(V)$. We denote by lex the lexicographical order of monomials, and by $\mathbf{mdeg} = \mathbf{mdeg}_{\mathrm{lex}}$ the multidegree of a polynomial ([**14**, Ch. 2, §2, Def. 3 and 7]).

Before giving the detail, in an example below, we see how our proof of "$Z \subset M$" works:

Example 2.3. Let $X \subset \mathbb{P}^3$ be the curve defined by h_1 and h_2 given in Example 2.1. We set $f = h_1$. For $z = (1, 0, 0, 0)$, we have $Y = \operatorname{Cone}_z(X) = (f = 0) \subset \mathbb{P}^3$. Suppose $p > \ell$ and suppose that $Z \subset \mathfrak{S}^{\mathrm{out}}(X)$ is an irreducible component containing z. Then we can show that Z is contained in (hence is equal to) $M = \operatorname{Vert}(Y) = \{ z \}$, in the following way.

We have that Z is contained in $\operatorname{Loc}_\ell(X)$, since so is $\mathfrak{S}^{\mathrm{out}}(X)$ as in the proof of Example 2.1. Here, Theorem 1.7 implies that $\operatorname{Loc}_\ell(X)$ is equal to $Z_2(\Lambda_{\ell-1})$, the zero set of 2×2 minors of $\Lambda_{\ell-1}$. For the index $(\ell-1)\boldsymbol{\omega}_0 = \mathbf{mdeg}(h_2) - \boldsymbol{\omega}_0$, we have $D_{(\ell-1)\boldsymbol{\omega}_0} f = 0$ and $D_{(\ell-1)\boldsymbol{\omega}_0} h_2 = \ell x_0$. Thus, for each $\boldsymbol{i} \in \boldsymbol{I}_{\ell-1}$, we have a 2×2 minor of $\Lambda_{\ell-1}$,

$$\begin{vmatrix} D_{\boldsymbol{i}} f & D_{\boldsymbol{i}} h_2 \\ D_{(\ell-1)\boldsymbol{\omega}_0} f & D_{(\ell-1)\boldsymbol{\omega}_0} h_2 \end{vmatrix} = \begin{vmatrix} D_{\boldsymbol{i}} f & D_{\boldsymbol{i}} h_2 \\ 0 & \ell x_0 \end{vmatrix} = D_{\boldsymbol{i}} f \cdot \ell x_0.$$

It follows $D_{\boldsymbol{i}} f \cdot \ell x_0 \in I(Z)$. Since $x_0 \notin I(Z)$, we obtain $D_{\boldsymbol{i}} f \in I(Z)$. Therefore it follows from Lemma 1.3 that we have $Z \subset M$.

In order to prove Theorem 2.2, it is essential to show the following result.

Proposition 2.4. *For each point $w \in \mathbb{P}^N \setminus Y$, there exists a hypersurface $F \subset \mathbb{P}^N$ defined by $f \in \pi_M^* I(Y_M)$ such that $w \notin F$ and $Z \subset \operatorname{Vert}(F)$.*

Remark 2.5. A polynomial f is contained in $\pi_M^* I(Y_M)$ if and only if $F = (f = 0) \subset \mathbb{P}^N$ is a cone which contains X and satisfies $M \subset \mathrm{Vert}(F)$. In other word, $\pi_M^* I(Y_M) = I(X) \cap \pi_M^* S(\mathbb{P}^{N-m-1})$, where $m := \dim M$.

First, we show two preparation lemmas.

Lemma 2.6. *Let $F \subset \mathbb{P}^N$ be a hypersurface of degree $e \leqslant e_0$ defined by a polynomial $f \in \pi_M^* I(Y_M)$, and let $h_{\bar{r}+1}, \dots, h_r \in I(X)_e$ be polynomials giving a basis of the quotient space $I(X)_e / \pi_z^* I(X_z)_e$, where $r := h^0(\mathbb{P}^N, \mathcal{I}_X(e))$ and by $\bar{r} := h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e))$. Suppose that there exist multi-indices $\boldsymbol{i}_{\bar{r}+1}, \dots, \boldsymbol{i}_r \in \boldsymbol{I}_{e-1}$ such that the column vector*

$$E = {}^t\begin{bmatrix} D_{\boldsymbol{i}_{\bar{r}+1}} & D_{\boldsymbol{i}_{\bar{r}+2}} & \dots & D_{\boldsymbol{i}_r} \end{bmatrix}$$

satisfies the following condition:

$$\det\left(E \cdot \begin{bmatrix} h_{\bar{r}+1} & h_{\bar{r}+2} & \dots & h_r \end{bmatrix}\right) \notin I(Z) \quad \text{and} \quad E \cdot f = \mathbf{0}. \tag{39}$$

Then we have $Z \subset \mathrm{Vert}(F)$.

Proof. By the choice of $z \in Z$, it follows from $e \leqslant e_0$ that we have $\bar{r} = \bar{r}(Z, e)$. Since $f \notin \pi_z^* I(X_z)_e$, we can regard $\{ f, h_{\bar{r}+1}, \dots, h_r \}$ as a subset of a basis of $I(X)_e$. We set ξ to be the determinant of the matrix $E \cdot \begin{bmatrix} h_{\bar{r}+1} & h_{\bar{r}+2} & \dots & h_r \end{bmatrix}$. For each $\boldsymbol{i} \in \boldsymbol{I}_{e-1}$, since $E \cdot f = \mathbf{0}$, we have

$$D_{\boldsymbol{i}} f \cdot \xi = \det\left(\begin{bmatrix} D_{\boldsymbol{i}} \\ E \end{bmatrix} \cdot \begin{bmatrix} f & h_{\bar{r}+1} & h_{\bar{r}+2} & \dots & h_r \end{bmatrix}\right),$$

which is a $(r - \bar{r} + 1) \times (r - \bar{r} + 1)$ minor of $\boldsymbol{\Lambda}(e) = \Lambda_{e-1}$. Thus Corollary 1.16 implies that $D_{\boldsymbol{i}} f \cdot \xi \in I(Z)$. Since $\xi \notin I(Z)$, we obtain $D_{\boldsymbol{i}} f \in I(Z)$. Hence, from Lemma 1.3, we have $Z \subset \mathrm{Vert}(F)$. $\square$

By changing coordinates $(x_0, x_1, \dots, x_N)$ on $\mathbb{P}^N$, we may assume $z = (1, 0, \dots, 0)$. We denote by $\deg(h, x_0)$ the degree of h for one variable x_0.

Lemma 2.7. *Let h be a homogeneous polynomial of degree e.*

(a) *Let $\boldsymbol{i} \in \boldsymbol{I}_{e-1}$ be a multi-index. Then the linear polynomial $D_{\boldsymbol{i}} h$ has the variable x_0 (i.e., $D_{\boldsymbol{i}} h \notin (x_1, \dots, x_N)$), only if $\boldsymbol{i} \leqslant_{\mathrm{lex}} \mathbf{mdeg}(h) - \boldsymbol{\omega}_0$. In particular, if h is monic, then $D_{\mathbf{mdeg}(h) - \boldsymbol{\omega}_0} h$ is expressed as $\deg(h, x_0) \cdot x_0 + g$ with $g \in (x_1, \dots, x_N)$.*

(b) *Recall that $z = (1, 0, \dots, 0)$ satisfies that $z \notin \mathrm{Str}(X)$ and that $\pi_z|_X$ is not birational onto its image. Assume $\deg(h, x_0) = 1$. Then h is equal to $x_0 h' + h''$ with some $h', h'' \in \pi_z^* I(X_z)$.*

Proof. (a) If $D_{\boldsymbol{i}} h$ has the variable x_0, then the polynomial h has the monomial $x_0 \cdot x^{\boldsymbol{i}}$, which means that $\boldsymbol{i} + \boldsymbol{\omega}_0 \leqslant_{\mathrm{lex}} \mathbf{mdeg}(h)$.

(b) Let $H \subset \mathbb{P}^N$ be the hypersurface of degree e defined by h. Since $\deg(h, x_0) = 1$, we have $\mu_{z,H} = e - 1$. Hence $\pi_{z|_H}$ is birational, and moreover, for each point $x \in \mathbb{P}^{N-1}$, it follows that either the intersection $\pi_z^{-1}(x) \cap (H \setminus \{ z \})$ consists of one point, or the line $\pi_z^{-1}(x)$ is contained in H. Suppose that $Y \not\subset H$. Then $\pi_z^{-1}(x) \not\subset H$ for general $x \in X_z$, and then $\pi_z^{-1}(x) \cap (X \setminus \{ z \})$ consists of one point since so is $\pi_z^{-1}(x) \cap (H \setminus \{ z \})$. Thus $\pi_{z|_X}$ is purely inseparable, which implies $z \in \mathrm{Str}(X)$, a contradiction. Hence Y

is contained in H. Since $I(Y) = S(\mathbb{P}^N) \cdot \pi_z^* I(X_z)$ and since $\deg(h, x_0) = 1$, we obtain that

$$h = \sum_j (\varphi_{j,1} x_0 + \varphi_{j,2}) \psi_j = \Big(\sum_j \varphi_{j,1} \psi_j \Big) x_0 + \Big(\sum_j \varphi_{j,2} \psi_j \Big)$$

with $\varphi_{j,1}, \varphi_{j,2} \in \pi_z^* S(\mathbb{P}^{N-1})$ and $\psi_j \in \pi_z^* I(X_z)$; hence the assertion follows. □

Now we come to the proof of the proposition, where recall that $Y := \mathrm{Cone}_z(X) \subset \mathbb{P}^N$, a cone of degree e_0 with maximal vertex M.

Proof of Proposition 2.4. Let $w \in \mathbb{P}^N$ be a point with $w \notin Y$ (equivalently, $w \notin M$ and $w_M \notin Y_M$). Then, from Lemma 1.18, there exists a polynomial $f \in \pi_M^* I(Y_M)_{e_0}$ such that $w \notin F$ (equivalently, $w_M \notin F_M$), where $F := (f = 0) \subset \mathbb{P}^N$.

Let $e \leqslant e_0$ be the smallest integer such that there exists a polynomial $f \in \pi_M^* I(Y_M)_e$ satisfying $w \notin F$. We take such a polynomial f of degree e. In the following steps (i-ii), by modifying $f \in \pi_M^* I(Y_M)_e$ with keeping the property $w \notin F$, we will find polynomials $h_{\bar r+1}, \dots, h_r$ and indices $\boldsymbol{i}_{\bar r+1}, \dots, \boldsymbol{i}_r$ satisfying the property (39) in Lemma 2.6.

By changing coordinates $(x_0, x_1, \dots, x_N)$ on $\mathbb{P}^N$, we may assume that

$$z = (1, 0, \dots, 0) \quad \text{and} \quad M = (x_{m+1} = \dots = x_N = 0) \text{ in } \mathbb{P}^N$$

with $m = \dim M$. Here a polynomial h is contained in $\pi_M^* S(\mathbb{P}^{N-m-1})$ if and only if h is of multidegree $\leqslant_{\mathrm{lex}} \deg(h) \cdot \boldsymbol{\omega}_{m+1}$. By changing coordinates $(x_{m+1}, \dots, x_N)$ on $\mathbb{P}^{N-m-1}$, we may assume that

$$w_M = (x_{m+2} = \dots = x_N = 0) \text{ in } \mathbb{P}^{N-m-1}.$$

Since $w_M \notin F_M$, we have $\mathbf{mdeg}(f) = e \boldsymbol{\omega}_{m+1}$.

Step (i) Let $h_{\bar r+1}, \dots, h_r \in I(X)_e$ be homogeneous polynomials which give a basis of the quotient space $I(X)_e / \pi_z^* I(X_z)_e$, where $r := h^0(\mathbb{P}^N, \mathcal{I}_X(e))$ and by $\bar r := h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_z}(e))$. Since $h_i \notin \pi_z^* I(X_z)$, we have $\deg(h_i, x_0) > 0$. By replacing h_i, we can assume that $h_{\bar r+1}, \dots, h_r$ are monic polynomials satisfying the following strictly descending sequence:

$$\mathbf{mdeg}(h_{\bar r+1}) >_{\mathrm{lex}} \mathbf{mdeg}(h_{\bar r+2}) >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} \mathbf{mdeg}(h_r). \tag{40}$$

Now we set $\boldsymbol{i}_s := \mathbf{mdeg}(h_s) - \boldsymbol{\omega}_0$, and set the column vector

$$E := {}^t \Big[D_{\mathbf{mdeg}(h_{\bar r+1}) - \boldsymbol{\omega}_0} \quad D_{\mathbf{mdeg}(h_{\bar r+2}) - \boldsymbol{\omega}_0} \quad \cdots \quad D_{\mathbf{mdeg}(h_r) - \boldsymbol{\omega}_0} \Big].$$

Then the determinant

$$\xi := \det \big(E \cdot \big[h_{\bar r+1} \quad h_{\bar r+2} \quad \cdots \quad h_r \big] \big)$$

is equal to

$$\begin{vmatrix} \deg(h_{\bar r+1}, x_0) \cdot x_0 + \sharp & \sharp & \cdots & \sharp & \sharp \\ * & \deg(h_{\bar r+2}, x_0) \cdot x_0 + \sharp & \cdots & \sharp & \sharp \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & \deg(h_{r-1}, x_0) \cdot x_0 + \sharp & \sharp \\ * & * & \cdots & * & \deg(h_r, x_0) \cdot x_0 + \sharp \end{vmatrix},$$

where linear polynomials ($\sharp$) are in $(x_1, \dots, x_N)$, and ($*$) are in $(x_0, \dots, x_N)$. This is because, it follows from $\mathbf{mdeg}(h_{\bar r+1}) >_{\mathrm{lex}} \mathbf{mdeg}(h_{\bar r+2})$ that $D_{\mathbf{mdeg}(h_{\bar r+1}) - \boldsymbol{\omega}_0} h_{\bar r+2} \in$

$(x_1, \dots, x_N)$ as in Lemma 2.7(a); similarly, each polynomial in the part ($\sharp$) is contained in $(x_1, \dots, x_N)$.

Thus ξ is equal to $\alpha x_0^{\bar{r}-r} + \xi'$ with

$$\alpha := \deg(h_{\bar{r}+1}, x_0) \cdot \deg(h_{\bar{r}+2}, x_0) \cdots \deg(h_{r-1}, x_0) \cdot \deg(h_r, x_0)$$

and $\xi' \in (x_1, \dots, x_N)$, where we obtain $\alpha \neq 0$ by $e \leqslant e_0$ and by the assumption of the characteristic p. Since $\xi \notin I(z) = (x_1, \dots, x_N)$ and since $I(Z) \subset I(z)$, we have $\xi \notin I(Z)$.

Step (ii) Next, let r_1 be the largest integer with $\bar{r} + 1 \leqslant r_1 \leqslant r$ such that the following inequality holds:

$$\mathbf{mdeg}(h_{r_1}) >_{\text{lex}} \boldsymbol{\omega}_0 + (e-1)\boldsymbol{\omega}_{m+1}.$$

Then, since $f \in \pi_M^* S(\mathbb{P}^{N-m-1})$ (i.e., f does not have the variables $x_0, \dots, x_m$), we find that $D_{\mathbf{mdeg}(h_i)-\boldsymbol{\omega}_0} f = 0$ for $i \leqslant r_1$. In order to obtain $D_{\mathbf{mdeg}(h_i)-\boldsymbol{\omega}_0} f = 0$ for $i > r_1$, by Lemma 1.5, we need to show

$$\text{coef}_{\mathbf{mdeg}(h_i)-\boldsymbol{\omega}_0+\boldsymbol{\omega}_j}(f) = 0 \tag{41}$$

for each $0 \leqslant j \leqslant N$, where we denote by $\text{coef}_{\boldsymbol{i}}(f)$ the coefficient of monomial $x^{\boldsymbol{i}}$ in f. Since $f \in \pi_M^* S(\mathbb{P}^{N-m-1})$, we immediately have that (41) holds for $j \leqslant m$. In the following, by modifying f and h_i with $i > r_1$, we find a new polynomial which satisfies (41) for $j > m$.

For $i > r_1$, since $\mathbf{mdeg}(h_i) \leqslant_{\text{lex}} \boldsymbol{\omega}_0 + (e-1)\boldsymbol{\omega}_{m+1}$, we have $\deg(h_i, x_0) \leqslant 1$, which implies $\deg(h_i, x_0) = 1$. Thus Lemma 2.7(b) implies that h_i is equal to $x_0 h_i' + h_i''$ with some $h_i', h_i'' \in \pi_z^* I(X_z)$. Here the multidegree of h_i and $x_0 h_i'$ coincide. By removing h_i'' for $i > r_1$, we can assume that

$$h_i = x_0 h_i' \quad \text{with} \quad h_i' \in \pi_z^* I(X_z) \ \ (i > r_1).$$

Then, polynomials $h_{\bar{r}+1}, \dots, h_{r_1}$ and new polynomials $h_{r_1+1} \dots, h_r$ give a basis of $I(X)_e / \pi_z^* I(X_z)_e$, and satisfies the condition (40). Therefore $\xi \notin I(Z)$ still holds.

For each $i > r_1$, since h_i is of multidegree $\leqslant_{\text{lex}} \boldsymbol{\omega}_0 + (e-1)\boldsymbol{\omega}_{m+1}$, it follows that h_i' is of multidegree $\leqslant_{\text{lex}} (e-1)\boldsymbol{\omega}_{m+1}$, i.e., $h_i' \in \pi_M^* S(\mathbb{P}^{N-m-1})$. This leads to $h_i' \in \pi_M^* I(Y_M)_{e-1}$ as in Remark 2.5. By the assumption of the degree e, the point w is contained in the hypersurface of degree $e-1$ in $\mathbb{P}^N$ defined by h_i', that is to say, $\mathbf{mdeg}(h_i') <_{\text{lex}} (e-1)\boldsymbol{\omega}_{m+1}$.

Let $\{ g_k \}_{1 \leqslant k \leqslant k_0}$ be a maximal subset of $\{ x_j h_i' \}_{m+1 \leqslant j \leqslant N,\, r_1+1 \leqslant i \leqslant r}$ which satisfies the following strictly descending sequence:

$$\mathbf{mdeg}(g_1) >_{\text{lex}} \mathbf{mdeg}(g_2) >_{\text{lex}} \cdots >_{\text{lex}} \mathbf{mdeg}(g_{k_0}).$$

Here g_k is contained in $\pi_M^* I(Y_M)$, since so is h_i' with $r_1 + 1 \leqslant i \leqslant r$. In addition, it follows that $\mathbf{mdeg}(g_k) <_{\text{lex}} e\boldsymbol{\omega}_{m+1}$. Inductively, we set $\varphi_0 := f$ and set $\varphi_k := \varphi_{k-1} - \text{coef}_{\mathbf{mdeg}(g_k)}(\varphi_{k-1}) \cdot g_k$ for each k with $0 < k \leqslant k_0$.

We replace f with φ_{k_0}. Then we still have $f \in \pi_M^* I(Y_M)_e$ and $\mathbf{mdeg}(f) = e\boldsymbol{\omega}_{m+1}$, i.e., $w \notin F$. Moreover, we have $\text{coef}_{\mathbf{mdeg}(g_k)}(f) = 0$ for any $1 \leqslant k \leqslant k_0$. This implies that (41) holds for any $i > r_1$ and $j > m$, since the multi-index $\mathbf{mdeg}(h_i) - \boldsymbol{\omega}_0 + \boldsymbol{\omega}_j =$

$\mathbf{mdeg}(h'_i) + \boldsymbol{\omega}_j$ is given by the multidegree of some g_k. Thus $D_{\mathbf{mdeg}(h_i)-\boldsymbol{\omega}_0} f = 0$ for $i > r_1$. As a result, we have $E \cdot f = \mathbf{0}$.

Now, the assumption of Lemma 2.6 is satisfied. Hence $Z \subset \mathrm{Vert}(F)$. □

Proof of Theorem 2.2. For general $x \in M$, we have that $\mu_{x,X} \leqslant \mu_{z,X}$ due to Lemma 1.11(c). Since $\deg(X_x) = \deg(Y_x) = e_0$, Lemma 1.11(a) implies $\deg(\pi_{x|X}) = (\deg(X) - \mu_{x,X})/e_0 \geqslant \deg(\pi_{z|X}) > 1$, that is, $x \in \mathfrak{S}^{\mathrm{tot}}(X)$. Hence we have $M \subset \mathfrak{S}^{\mathrm{tot}}(X)$.

From Proposition 2.4, there exist polynomials $\{ f_s \}_{s=1}^{s_0} \subset \pi_M^* I(Y_M)$ such that $Y = \bigcap_{s=1}^{s_0} F_s$ and that $Z \subset \mathrm{Vert}(F_s)$ for every s, where $F_s := (f_s = 0) \subset \mathbb{P}^N$. Then, it follows from Lemma 1.19 that we have $Z \subset M$. □

Here, we have the following result, which is a specific version of Theorem III.

Theorem 2.8. *Let $X \subset \mathbb{P}^N$ be as in Theorem III, and let $n := \dim(X) < N - 1$.*

(a) *Assume either $p \geqslant \deg(X)$ or $p = 0$. Then every irreducible component Z of $\mathfrak{S}^{\mathrm{tot}}(X)$ is a linear subspace of dimension $< n$. Moreover, the component Z coincides with the maximal vertex of an $(n+1)$-dimensional cone containing X, except when X satisfies $\mathfrak{S}^{\mathrm{out}}(X) = \emptyset$ and $\mathfrak{S}^{\mathrm{inn}}(X) = \mathrm{Vert}(X)$.*

(b) *Now let $p \geqslant 0$ be arbitrary. Let $Z \subset \mathbb{P}^N$ be a linear subspace not contained in X, and assume $\dim(Z) \geqslant \dim(Z \cap \mathrm{Vert}(X)) + 2$, where we regard $\dim(\emptyset) = -1$. Suppose that X lies on an $(n+1)$-dimensional cone with vertex Z. Then we have $Z \subset \mathfrak{S}^{\mathrm{out}}(X)$. In addition, if Z is the maximal vertex of the cone, then Z coincides with an irreducible component of $\mathfrak{S}^{\mathrm{out}}(X)$.*

Lemma 2.9 (cf. [**10**, Lemma 4(v)])**.** *Let $X \subset \mathbb{P}^N$ be a cone with maximal vertex $M = \mathrm{Vert}(X)$. Then $\mathfrak{S}^{\mathrm{tot}}(X)$ is equal to the closure of $\pi_M^{-1}(\mathfrak{S}^{\mathrm{tot}}(X_M))$.*

Proof. Let $z \in \mathbb{P}^N \setminus M$. Then $\mu_{z,X} = \mu_{z_M,X_M}$ and $\deg(X) = \deg(X_M)$. Let $M' \subset \mathbb{P}^N$ be the linear subspace spanned by M and z. Since $M_z \subset \mathrm{Vert}(X_z)$ and $(X_z)_{M_z} = X_{M'}$, we have $\deg(X_z) = \deg(X_{M'})$. Hence it follows from Lemma 1.11(a) that $\deg(\pi_{z|X}) = \deg(\pi_{z_M|X_M})$. In particular, $\pi_{z|X}$ is birational if and only if $\pi_{z_M|X_M}$ is so. □

Proof of Theorem 2.8. (a) From Lemma 2.9, we may assume that X is not a cone. Now we show $\mathrm{Str}(X) = \emptyset$, as follows: Suppose $z \in \mathrm{Str}(X)$. Then it follows that $p > 0$ and that $\pi_{z|X}$ is inseparable. Since $\deg(\pi_{z|X}) \geqslant p$ and $p \geqslant \deg(X)$, we have $\deg(\pi_{z|X}) = \deg(X)$. Then $X_z \subset \mathbb{P}^{N-1}$ is linear, which contradicts that X is non-degenerate and of codimension $\geqslant 2$.

Therefore, the result follows from Proposition 1.10 and Theorem 2.2.

(b) Let $Z \subset \mathbb{P}^N$ be a linear subspace not contained in X with

$$\dim(Z) \geqslant \dim(\mathrm{Vert}(X) \cap Z) + 2.$$

Let Y be the $(n+1)$-dimensional cone with vertex Z such that $X \subset Y$. For general $z \in Z \setminus X$, we have a line $L \subset Z$ such that $z \in L$ and $L \cap \mathrm{Vert}(X) = \emptyset$. Here we find that $X_z = Y_z \subset \mathbb{P}^{N-1}$ is a cone with vertex $v_1 := L_z$.

Suppose that $\pi_{z|_X} : X \to X_z$ is birational. Then there exists an open subset $U_1 \subset X_z$ such that $U := \pi_{z|_X}^{-1}(U_1) \to U_1$ is bijective. For general line $M_1 \subset X_z$ containing v_1, the subvariety

$$M := \overline{\pi_{z|_X}^{-1}(M_1 \cap U_1)} \subset X$$

is a line, because of $z \notin X$. Since L intersects M for infinitely many lines M_1, and since $\#(L \cap X) < \infty$, we find a point $v \in L \cap X$ such that $v \in M$ for general M_1; hence X is a cone with vertex v, which contradicts $L \cap \mathrm{Vert}(X) = \emptyset$. Hence $z \in \mathfrak{S}^{\mathrm{out}}(X)$. □

Remark 2.10. For the locus $\mathfrak{S}^{\mathrm{out}}(X)$, we can show the linearity under an assumption weaker than $p \geqslant \deg(X)$, as follows: Let $e < \deg(X)$ be the largest integer such that $e \mid \deg(X)$. Since $\deg(X) = \deg(\pi_{z|_X}) \cdot \deg(X_z)$ and $\deg(\pi_{z|_X}) > 1$ for general $z \in \mathfrak{S}^{\mathrm{out}}(X)$, we have $\mathfrak{S}^{\mathrm{out}}(X) \subset \mathrm{Loc}_e(X)$. Thus, from Theorem 2.2, the linearity of $\mathfrak{S}^{\mathrm{out}}(X)$ holds in the case $p > e$.

Remark 2.11. The linearity of $\mathfrak{S}^{\mathrm{tot}}(X)$ implies that of $\mathfrak{S}^{\mathrm{out}}(X)$, since we find that every irreducible component Z of $\mathfrak{S}^{\mathrm{out}}(X)$ is equal to an irreducible component of $\mathfrak{S}^{\mathrm{tot}}(X)$, as follows: Let Z' be an irreducible component of $\mathfrak{S}^{\mathrm{tot}}(X)$ containing Z. Then, since Z is not contained in X, so is Z'. Hence a general point $z \in Z'$ satisfies that $z \notin X$ and that π_z is not birational. Thus $Z' \subset \mathfrak{S}^{\mathrm{out}}(X)$, which implies $Z = Z'$.

In the following, we check the sharpness of Theorem 2.8(b). Here, Example 2.12(a) shows that the assumption of inequality "$\dim(Z) \geqslant \dim(\mathrm{Vert}(X) \cap Z) + 2$" is necessary. And (b) shows that, for a linear subspace $Z \subset X$ satisfying that X lies on an $(n+1)$-dimensional cone with vertex Z, the inclusion "$Z \subset \mathfrak{S}^{\mathrm{inn}}(X)$" does *not* hold in general.

Example 2.12. (a) Let $X \subset \mathbb{P}^N$ be an n-dimensional cone with vertex x, and let $z \in \mathbb{P}^N \setminus X$ be a point such that $\pi_{z|_X}$ is birational. Then the line $Z := \overline{xz}$ is equal to a vertex of the $(n+1)$-dimensional cone $\mathrm{Cone}_z(X)$, and is not contained in $\mathfrak{S}^{\mathrm{out}}(X)$.

(b) Let $X = (x_0x_1 - x_2^2 = x_1x_3 - x_2x_4 = x_0x_4 - x_2x_3 = 0) \subset \mathbb{P}^4$, a surface of degree 3 parametrized by $\mathbb{P}^2 \dashrightarrow \mathbb{P}^4 : (1, s, t) \mapsto (1, s^2, s, t, st)$. Then the line $Z = (x_0 = x_1 = x_2 = 0) \subset X$ is equal to the maximal vertex of the 3-dimensional cone $(x_0x_1 - x_2^2 = 0)$, and is not contained in $\mathfrak{S}^{\mathrm{inn}}(X)$.

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List of papers by Katsuhisa FURUKAWA

[F1] *Rational curves on hypersurfaces*, to appear in Journal für die reine und angewandte Mathematik.

[FFK] *Projective varieties admitting an embedding with Gauss map of rank zero*, Advances in Mathematics **224** (2010), 2645–2661 (with S. Fukasawa and H. Kaji).

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