# 公聴会（渡辺究）

## 1 学位申請論文の構成

表題: Group actions on projective varieties and chains of rational curves on Fano varieties
(射影多様体への群作用とファノ多様体上の有理曲線の鎖)

第 1 章: Overview of homogeneous variety and results of projective geometry
第 2 章: Classification of polarized manifolds admitting homogeneous varieties as ample divisors
第 3 章: Actions of linear algebraic groups of exceptional type on projective varieties
第 4 章: Lengths of chains of minimal rational curves on Fano manifolds

以下，基礎体は全て複素数体 $\mathbb{C}$ とする．

## 2 Classification of polarized manifolds admitting homogeneous varieties as ample divisors

**Definition 2.1.** *$(X, L)$: sm polarized var $\Leftrightarrow$ $X$: sm proj var, $L$: ample line bundle on $X$.*

**Problem 2.2.** *Classify sm polarized var $(X, L)$ s.t. $\exists A \in |L|$: homogeneous var.*

**Theorem 2.3.** *Let $(X, L)$ be as in the above problem. Assume that $\dim A \geq 2$. Then $(X, L)$ is one of the following:*
(i)$(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(i))$, $i = 1, 2$; (ii)$(Q^{n+1}, \mathscr{O}_{Q^{n+1}}(1))$;
(iii)$(\mathbb{P}(\mathcal{E}), H(\mathcal{E}))$, *$\mathcal{E}$ is an ample vector bundle on a smooth curve $C$ with $g(C) = 0$ or $1$ and $\mathscr{L}$ an ample line bundle on $C$ with a exact sequence:*

$$0 \to \mathscr{O}_C \to \mathcal{E} \to \mathscr{L}^{\oplus n} \to 0;$$

(iv)$(\mathbb{P}^m \times \mathbb{P}^m, \mathscr{O}_{\mathbb{P}^m \times \mathbb{P}^m}(1, 1))$; (v)$(G(2, \mathbb{C}^{2m}), \mathscr{O}_{\text{Plücker}}(1))$; (vi)$(E_6(\omega_1), \mathscr{O}_{E_6(\omega_1)}(1))$.

## 3 Actions of linear algebraic groups of exceptional type on projective varieties

**Problem 3.1.** *$X$: sm proj var, $G$: simple linear algebraic group acting on $X$.*
*Then classify such pairs $(X, G)$.*

**Definition 3.2.** $r_G := \min\{\dim G/P \mid P \subset G\text{: parabolic subgroup}\}$.

**Theorem 3.3.** *$X$: a sm proj var of dim $n$,*
*$G$: a simple linear algebraic group of exceptional type acting on $X$.*
*Assume that $n = r_G + 1$. Then $X$ is one of the following:*
(i) $\mathbb{P}^6$, (ii) $Q^6$, (iii) $E_6(\omega_1)$, (iv) $G_2(\omega_1 + \omega_2)$,
(v) $Y \times Z$, *where $Y$ is $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$ and $Z$ is a sm proj curve,*
(vi) $\mathbb{P}(\mathscr{O}_Y \oplus \mathscr{O}_Y(m))$, *where $Y$ is as in $(5)$ and $m > 0$.*

# 4 Lengths of chains of minimal rational curves on Fano manifolds

**Definition 4.1.** *Fano var $\Leftrightarrow$ sm proj var whose anticanonical divisor is ample.*

Remark (Definition): $\mathscr{O}_X(K_X) \cong \bigwedge^n \Omega_X$, $\mathscr{O}_X(-K_X) \cong \bigwedge^n T_X$.

$X$: a Fano $n$-fold with $\mathrm{Pic}(X) \cong \mathbb{Z}[H]$ ($n \geq 3$), where $H$ is ample,
$\mathscr{K} \subset \mathrm{RatCurves}^n(X)$: a minimal rational component,
that is, a dominating irreducible component whose degree is minimal among such components,
$i_X$: Fano index $\Leftrightarrow -K_X = i_X H$,
$\mathrm{coindex}(X) := n + 1 - i_X$: coindex.

**Problem 4.2.** *How many $\mathscr{K}$-curves are needed to join two general points on $X$ ?*

$l_{\mathscr{K}}$: the minimal length of connected chains of general $\mathscr{K}$-curves joining two general points.

The speaker computed the lenghts in the following four cases:
(i) $n \leq 5$, (ii) $\mathrm{coindex}(X) \leq 3$,
(iii) $X$: prime and $i_X = \frac{2}{3}n$, (iv) $X$ admits a double cover structure and is covered by lines.

**Basic Property 4.3.** *For general $[C] \in \mathscr{K}$, $f^*T_X \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^p \oplus \mathscr{O}_{\mathbb{P}^1}^{n-1-p}$, where $f : \mathbb{P}^1 \to X$ is the normalization of $C$. We have $p = i_X d_{\mathscr{K}} - 2$. In particular, $0 \leq p \leq n-1$.*

**Theorem 4.4.** (i) *If $p = n-3 > 0$, then $l_{\mathscr{K}} = 2$.*

(ii) *If $(n, p) = (5, 1)$, then $l_{\mathscr{K}} = 3$.*

| $n$ | $p$ | $l_{\mathscr{K}}$ | $n$ | $p$ | $l_{\mathscr{K}}$ | $n$ | $p$ | $l_{\mathscr{K}}$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 4 | 3 | 1 | 5 | 4 | 1 |
| 3 | 1 | 2 | 4 | 2 | 2 | 5 | 3 | 2 |
| 3 | 0 | 3 | 4 | 1 | 2 | 5 | 2 | 2 |
| | | | 4 | 0 | 4 | 5 | 1 | 3 |
| | | | | | | 5 | 0 | 5 |

**Theorem 4.5.** *Let $X$ be a prime Fano manifold with $i_X = \frac{2}{3}n$. Then $l_{\mathscr{K}} = 2$ except the following cases:*

(i) $(3) \subset \mathbb{P}^4$ *a hypersurface of degree 3.*

(ii) $(2) \cap (2) \subset \mathbb{P}^5$ *a complete intersection of two hyperquadrics.*

(iii) $G(2, \mathbb{C}^5) \cap (1)^3 \subset \mathbb{P}^6$ *a 3-dimensional linear section of $G(2, \mathbb{C}^5)$.*

(iv) $LG(3, \mathbb{C}^6)$ *a Lagrangian Grassmann.*

(v) $G(3, \mathbb{C}^6)$ *a Grassmann.*

(vi) $S_5 := \mathbb{F}_5(Q^{10})^+$ *a 15-dimensional spinor variety.*

(vii) $E_7(\omega_1)$ *a 27-dimensional rational homogeneous manifold of type $E_7$.*

*Furthermore in the above exceptional cases, $l_{\mathscr{K}} = 3$.*