Group actions on projective varieties
and
chains of rational curves on Fano varieties

射影多様体への群作用と
ファノ多様体上の有理曲線の鎖

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Chapter 1

Introduction

In this thesis, we study homogeneous varieties and chains of rational curves on Fano varieties. By a *homogeneous variety* we mean a projective variety acted by a group variety transitively. Projective spaces, smooth quadric hypersurfaces and abelian varieties are typical examples of it. Homogeneous varieties often appear in many different fields of mathematics such as differential geometry, Lie theory and representation theory. Moreover these have attracted attention over the years in physics. On the other hand, a smooth projective variety is called a *Fano variety* if its anticanonical divisor is ample. According to the minimal model program, it is conjectured that any algebraic variety is birationally equivalent to a minimal model or a Mori fiber space which is a fibration with the general fiber being a (possibly singular) Fano variety. From this viewpoint, Fano varieties play important roles in birational geometry. Moreover rational homogeneous varieties are Fano varieties.

This thesis consists of four chapters. Chapter 1 is devoted to an overview of homogeneous varieties and some known results of projective geometry. We start with a review of the correspondence between homogeneous varieties and marked Dynkin diagrams. We also survey Zak's works on projective geometry such as linear normality theorem and a classification of Severi varieties. The results appearing here play important roles in all of later chapters.

In Chapter 2, we study homogeneous varieties from the viewpoint of polarized varieties. By a *polarized variety* we mean a pair (X, L) consisting of a complete variety X and an ample line bundle L on it. One of the important problems in the study of polarized varieties is to classify the pairs (X, L) such that the linear system $|L|$ has a member A with preassigned properties. For instance, when A is the n-dimensional projective space $\mathbb{P}^n$ $(n \geq 2)$, (X, L) is isomorphic to $(\mathbb{P}^{n+1}, \mathscr{O}(1))$. If A is an n-dimensional smooth quadric hypersurface Q^n $(n \geq 3)$, then (X, L) is isomorphic to $(\mathbb{P}^{n+1}, \mathscr{O}(2))$ or $(Q^{n+1}, \mathscr{O}(1))$.

As is seen from these examples, the structure of X is imposed a strong restriction by the one of A. In this chapter, we deal with the following problem.

Problem 1.0.1. *Classify smooth polarized varieties (X, L) admitting a homogeneous variety A as a member of the complete linear system $|L|$.*

Remark that A. J. Sommese [Som76] studied the case where A is an abelian variety and T. Fujita [Fuj80I, Fuj81I, Fuj82] the case where A is a Grassmann variety. K. Konno [Kon88] solved it under the assumption that X and A are rational homogeneous varieties. As a natural generalization of these results, we consider the above problem.

Note that a 1-dimensional homogeneous variety is a projective line or an elliptic curve. So when the dimension of A is 1, an answer to the problem is derived from known results. For example, it follows from classifications of polarized varieties whose sectional genera are 0 and 1. So we may make the assumption that the dimension of homogeneous member A is at least 2. Then we provide a complete answer of the above problem.

Theorem 1.0.2. *Let (X, L) be a smooth polarized variety such that the linear system $|L|$ has a homogeneous member A. Assume that $\dim A \geq 2$. Then (X, L) is one of the following:*

(i) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(1))$.

(ii) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(2))$.

(iii) $(Q^{n+1}, \mathscr{O}_{Q^{n+1}}(1))$.

(iv) $(\mathbb{P}^l \times \mathbb{P}^l, \mathscr{O}_{\mathbb{P}^l \times \mathbb{P}^l}(1, 1))$, $2l = n + 1$.

(v) $(G(2, \mathbb{C}^{2l}), \mathscr{O}_{\text{Plücker}}(1))$, $4l - 4 = n + 1$.

(vi) $(E_6(\omega_1), \mathscr{O}_{E_6(\omega_1)}(1))$, $\varphi_{|\mathscr{O}_{E_6(\omega_1)}(1)|} : E_6(\omega_1) \hookrightarrow \mathbb{P}^{26}$ *is the projectivization of the highest weight vector orbit in the 27-dimensional irreducible representation of a simple algebraic group of Dynkin type E_6.*

(vii) $(\mathbb{P}(\mathcal{E}), H(\mathcal{E}))$, *$\mathcal{E}$ is a vector bundle on $\mathbb{P}^1$ of rank $n + 1$ and $a > 1$ with a non-splitting exact sequence:*

$$0 \to \mathscr{O}_{\mathbb{P}^1} \to \mathcal{E} \to \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus n} \to 0,$$

where $H(\mathcal{E})$ is the tautological line bundle on $\mathbb{P}(\mathcal{E})$.

(viii) $(\mathbb{P}(\mathcal{E}),\ H(\mathcal{E}))$, *$\mathcal{E}$ is a vector bundle on an elliptic curve E of rank $n+1$ and $\mathscr{L}$ an ample line bundle on E with a non-splitting exact sequence:*

$$0 \to \mathscr{O}_E \to \mathcal{E} \to \mathscr{L}^{\oplus n} \to 0.$$

Our proof consists of two parts. First we prove that most homogeneous varieties cannot be ample divisors in any smooth variety, using a result of Fujita. The assertion of the Fujita's result is a smooth variety satisfying the NS-condition cannot be an ample divisor in any smooth variety. Here we say X satisfies the NS-condition if $H^q(X, T_X[-L]) = 0$ for any ample line bundle L on X and $q = 0, 1$. So we show most homogeneous varieties satisfy the NS-condition in Proposition 3.2.3.

Second we determine all the possibilities of X for a fixed A which does not satisfy NS-condition. In this part, we shall use some results introduced in Chapter 1 such as a classication of Severi varieties. One of applications of this theorem will appear in the next chapter.

Chapter 3 deals with a classification problem of varieties from the viewpoint of actions of group varieties.

Let X be a smooth projective variety of dimension n and G a simple linear algebraic group acting regularly and non-trivially on X. The action of G strongly influences the structure of X. For example, such X is uniruled, that is, covered by rational curves. Hence it admits an extremal contraction in the sense of the minimal model program. Furthermore any extremal contraction is G-equivalent. These properties were first pointed out by Mukai and Umemura [MU83]. By investigating such contractions, they studied smooth projective 3-folds with a dense $SL(2)$-orbit. After that, such 3-folds were completely classified by T. Nakano [Nak89].

Let r_G be the minimum of the dimension of the homogeneous variety of a simple linear algebraic group G, that is, the minimum codimension of the maximal parabolic subgroup of G. M. Andreatta [And01] proved that if $r_G < n$ the only regular action of G on X is trivial, and if $r_G = n$ then X is homogeneous. In this chapter, we consider the following.

Problem 1.0.3. *Classify n-dimensional smooth projective varieties acted by a simple linear algebraic group with $n = r_G + 1$.*

For this problem, Andreatta gave a classification under the assumption that G is classical type. Since $r_{SL(n)} = n - 1$, his result contains a classification of smooth projective n-folds acted by $SL(n)$ which was obtained by T. Mabuchi [Mab79]. Now we may assume G is simply connected by replacing G with its universal cover. In this setting, we give an answer to this problem in the case where G is not classical type.

Theorem 1.0.4. *Let X be a smooth projective variety of dimension n and G a simple, simply connected and connected linear algebraic group of exceptional type acting regularly and non-trivially on X. Assume that $n = r_G + 1$. Then X is one of the following; the action of G is unique for each case:*

(i) $\mathbb{P}^6$,

(ii) Q^6,

(iii) $E_6(\omega_1)$,

(iv) $G_2(\omega_1 + \omega_2)$,

(v) $Y \times Z$, *where Y is $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$ and Z is a smooth projective curve,*

(vi) $\mathbb{P}(\mathscr{O}_Y \oplus \mathscr{O}_Y(m))$, *where Y is as in* (v) *and $m > 0$.*

As in the Andreatta's paper, we study extremal contractions, G-orbits and these relations. However our situation is rather complicated than Andreatta's. In fact, G-orbits on X are very simple (for example a projective space and a quadric) in the case where G is classical type, but they are not in our case. So we need other arguments than Andreatta's in several points. When the Picard number of X is 1, we obtain a classification of X by using Theorem 1.0.2. Hence this theorem is an application of Theorem 1.0.2.

As a consequence, by combining the Andreatta's result, we obtain a complete classification of n-dimensional smooth projective varieties acted by a simple linear algebraic group with $n = r_G + 1$.

In the last chapter, we discuss lengths of chains of rational curves on Fano varieties.

In the remarkable work [Mor79], S. Mori proved a smooth projective variety with ample tangent bundle is the projective space. It was called the Hartshorne conjecture. Mori's paper [Mor79] contains many important results of rational curves on Fano varieties. For example, it is the first paper the bend-and-break lemmas appear. Moreover, by using bend-and-break lemmas and modulo p reduction, it was proved that a Fano variety is uniruled. These results are the most fundamental and significant in the study of rational curves on Fano varieties. His basic idea of the proof of the Hartshorne conjecture is to study the family of rational curves of minimal degree. Based on the idea, N. Mok, J. M. Hwang and others have been studied families of rational curves of minimal degree on Fano varieties [Hwa01, KS06].

For a Fano variety, a *minimal rational component* $\mathscr{K}$ is defined to be a dominating irreducible component of the normalization of the parameter space of rational curves whose degree is minimal among such components and a *variety of minimal rational tangents* is the parameter space of the tangent directions of $\mathscr{K}$-curves at a general point.

On the other hand, chains of rational curves play an important role in the study of Fano varieties. For instance, Kollár-Miyaoka-Mori [KMM92II] and Nadel [Nad91] independently showed the boundedness of the degree of Fano varieties of Picard number 1 by using chains of rational curves. Furthermore it implies that there exists only finitely many deformation types of smooth Fano varieties with fixed dimension.

On the basis of these results, this chapter is devoted to consider the following problem.

Problem 1.0.5. *How many general rational curves in the family $\mathscr{K}$ are needed to join two general points on a Fano variety?*

We denote by $l_{\mathscr{K}}$ the minimal length of such chains of $\mathscr{K}$-curves. For example, it is easy to see that a projective space has $l_{\mathscr{K}} = 1$ and a smooth quadric hypersurface has $l_{\mathscr{K}} = 2$. However for the other examples, it is not easy to compute the length $l_{\mathscr{K}}$. In this direction, Hwang and S. Kebekus [HK05] developed an infinitesimal method to study the lengths of Fano varieties via the varieties of minimal rational tangents and computed the lengths in some cases such as complete intersections, Hermitian symmetric spaces and contact homogeneous spaces.

In this chapter, we compute the length $l_{\mathscr{K}}$ in the cases where the dimension of a Fano variety X is at most 5, the coindex of a Fano variety is at most 3 and X equips with structure of a double cover. For instance, we show the following.

Theorem 1.0.6 (Theorem 5.5.2, Theorem 5.5.7)**.** *Let X be a Fano n-fold of Picard number* 1*, $\mathscr{K}$ a minimal rational component of X and $p+2$ the anti-canonical degree of rational curves in $\mathscr{K}$. Then if $p = n-3 > 0$, we have $l_{\mathscr{K}} = 2$ and if $(n, p) = (5, 1)$, we have $l_{\mathscr{K}} = 3$.*

By combining this theorem and well-known or easy arguments, we obtain the following table.

n	p	$l_{\mathscr{K}}$	n	p	$l_{\mathscr{K}}$	n	p	$l_{\mathscr{K}}$
3	2	1	4	3	1	5	4	1
3	1	2	4	2	2	5	3	2
3	0	3	4	1	2	5	2	2
			4	0	4	5	1	3
						5	0	5

Theorem 1.0.7 (Theorem 5.6.4)**.** *Let X be a Fano variety of Picard number 1 with coindex 3 and $\mathscr{K}$ a minimal rational component of X. Assume that $n := \dim X \geq 6$. Then $l_{\mathscr{K}} = 2$ except the case X is a 6-dimensional Lagrangian Grassmannian $LG(3,6)$. In the case $X = LG(3,6)$, we have $l_{\mathscr{K}} = 3$.*

As a consequence, we obtain the following table.

X	i_X	$l_{\mathscr{K}}$
$\mathbb{P}^n$	$n+1$	1
Q^n	n	2
del Pezzo mfd. of dim. n	$n-1$	2
Mukai mfd. of dim. $n \geq 7$	$n-2$	2
Mukai mfd. of dim. 6	4	2 or 3

In Theorem 5.6.11, we give a classification of prime Fano n-folds satisfying $i_X = \frac{2}{3}n$ and $l_{\mathscr{K}} \neq 2$. These are extremal cases of Theorem 5.1.1. Except the case $n = 3$, these varieties are deeply related to Severi varieties which are classified by Zak [Zak93] (see Cororally 5.6.12). Furthermore, for prime Fano manifolds, we discuss a relation among 2-*connectedness by lines*, *conic-connectedness* and *defectiveness of the secant varieties* (Corollary 5.6.12 and Remark 5.6.13).

Notation and Convention

Throughout this paper we work over the complex number field $\mathbb{C}$, and employ the notation basically as in [Har71] or [Fuj90, Hwa01, Kol96]. In particular, *variety* means an integral separated scheme of finite type over $\mathbb{C}$ and sometimes we deal with a projective variety as a complex analytic space. For a vector bundle V, let $V^\vee$ be the total space of the dual bundle of V and o the zero section. Then $\mathbb{P}(V)$ denotes the quotient space of $V^\vee - o$ by the natural $\mathbb{G}_m$-action via the scalar multiplication. $\mathbb{P}_*(V)$ means $\mathbb{P}(V^\vee)$.

Chapter 2

Overview of homogeneous variety and results of projective geometry

2.1 Lie algebra

Briefly we recall basic facts of Lie algebra. For more detail on this chapter, see [Bou68, FH99, Hum72, Hum75].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ it's Cartan subalgebra. For $\alpha \in \mathfrak{h}^\vee$, we set

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \text{ for any } h \in \mathfrak{h}\},$$

$$\Phi := \{\alpha \in \mathfrak{h}^\vee | \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}.$$

Here $\alpha \in \Phi$ is called a *root* (relative to $\mathfrak{h}$), $\mathfrak{g}_\alpha$ a *root space* and Φ a *root system*. Then we have the *root decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

We denote by $< \Phi >_\mathbb{R}$ the $\mathbb{R}$-linear span of Φ. Then there exists a subset $\Delta := \{\alpha_1, \ldots, \alpha_l\} \subset \Phi$ which satisfies

(i) $\alpha_1, \cdots, \alpha_l$ is a basis of the $\mathbb{R}$-vector space $< \Phi >_\mathbb{R}$,

(ii) for $\alpha = \sum k_i \alpha_i \in \Phi$ ($k_i \in \mathbb{R}$), coefficients k_i are all positive integers or all negative ones.

This Δ is called a *base* of the root system Φ. In general, there are many bases on Φ. So we fix one Δ. Then the roots in Δ are called *simple.* We can define a partial order on $\mathfrak{h}^\vee$. In fact, for $\lambda, \mu \in \mathfrak{h}^\vee$, define

$$\lambda \geq \mu \Leftrightarrow \lambda - \mu = \sum k_i \alpha_i \text{ with } k_i \geq 0 \text{ and } \alpha_i \in \Delta.$$

When all coefficients of $\alpha \in \Phi$ are positive (resp. negative), such root α is called by *positive* (resp. *negative*). Also we denote by Φ^+ the set of positive roots.

Next we define an inner product on $\mathfrak{g}$ called by Killing form. For $u, v \in \mathfrak{g}$, we set $\mathrm{ad}(u)(v) := [u, v]$. $\mathrm{ad}(u) : \mathfrak{g} \to \mathfrak{g}$ is an endomorphism of the Lie algebra $\mathfrak{g}$. By using this notation, *Killing form* $<,>$: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is defined by $< u, v >:= \mathrm{Trace}(\mathrm{ad}(u) \circ \mathrm{ad}(v))$. As is well known, it is non-degenerate (Cartan's criterion). Hence we can identify $\mathfrak{h}$ with $\mathfrak{h}^\vee$ via the Killing form. For $\alpha \in \Phi$, one can set its *coroot* by $\alpha^\vee := \frac{2\alpha}{<\alpha,\alpha>}$. Then we can obtain the $l \times l$ matrix $C := (c_{ij})$, where $c_{ij} :=< \alpha_i, {\alpha_j}^\vee >$. We call it *Cartan matrix* of $\mathfrak{g}$ and c_{ij} *Cartan integer.* Remark that Cartan integers satisfy $c_{ij}c_{ji} = 0, 1, 2$ or 3. Here recall the Dynkin diagram associated to $\mathfrak{g}$:

Definition 2.1.1. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\Delta := \{\alpha_1, \ldots, \alpha_l\}$ a base of its root system. We define the Dynkin diagram associated to $\mathfrak{g}$ as follows:

(i) Draw l nodes labelled by the simple roots $\{\alpha_1, \ldots, \alpha_l\}$ and call the node labelled by α_i *i-th node.*

(ii) The i-th node and j-th one are joined by $c_{ij}c_{ji}$ edges.

(iii) If $|c_{ij}| < |c_{ji}|$, add an arrow from the j-th node to the i-th one.

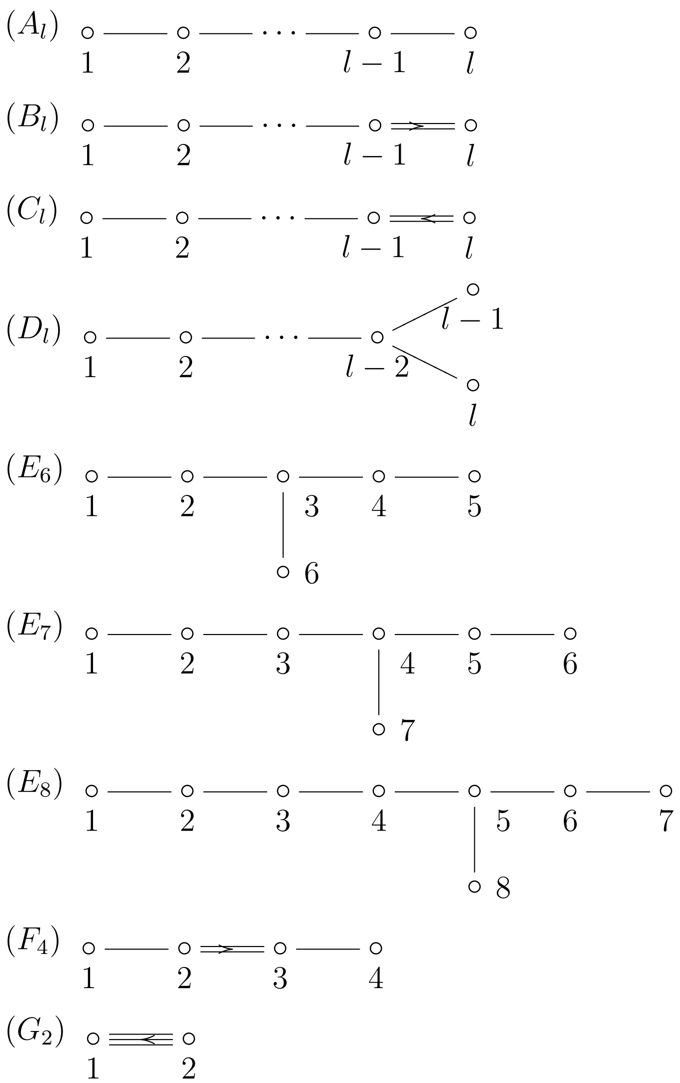
The diagram does not depend on a choice of a base Δ.

A semisimple Lie algebra is characterized by its Dynkin diagram:

Theorem 2.1.2. *Two semisimple Lie algebras are isomorphic to each other if and only if these Dynkin diagrams are the same.*

Proposition 2.1.3. *A semisimple Lie algebra $\mathfrak{g}$ is simple if and only if its Dynkin diagram is connected.*

Theorem 2.1.4. *A Dynkin diagram of a simple Lie algebra is one of the following.*



Throughout whole thesis, we use this numbering of the simple roots.

2.2 Homogeneous Varieties

Definition 2.2.1. A projective variety X is *homogeneous* if there exists a group variety which acts on X transitively.

Projective spaces, smooth quadric hypersurfaces and abelian varieties are fundamental examples of a homogeneous variety. First, applying the results reviewed in the previous section, recall a description of rational homogeneous varieties.

For a semisimple Lie algebra $\mathfrak{g}$, there exists a maximal solvable subalgebra which is unique up to conjugate. It is called a *Borel subalgebra* of $\mathfrak{g}$. In particular, we fix one as follows:

$$\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}, \text{ where } \mathfrak{n} := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ containing $\mathfrak{b}$ is called *parabolic*. Let $\Delta_{\mathfrak{p}}$ be a subset of Δ. Then we set

$$\mathfrak{p} := \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi^+_{\mathfrak{p}}} \mathfrak{g}_{-\alpha}, \text{ where } \Phi^+_{\mathfrak{p}} := \mathrm{span}\Delta_{\mathfrak{p}} \cap \Phi^+.$$

Relate the set of simple roots $\Delta \setminus \Delta_{\mathfrak{p}}$ to a parabolic subalgebra $\mathfrak{p}$ as above. Then it gives a one-to-one correspondence between parabolic subalgebras and sets of nodes of Dynkin diagrams. As a consequence, any parabolic subalgebra is expressed by a marked Dynkin diagram, that is, a pair consisting of a Dynkin diagram and a subset of its nodes.

On the other hand, let G be a simply-connected algebraic group associated to $\mathfrak{g}$ and P a subgroup of G associated to $\mathfrak{p}$. Then the quotient G/P is a projective variety, which is called a *rational homogeneous variety*. By the above correspondence, the following is obtained:

Theorem 2.2.2. *Any rational homogeneous variety can be expressed by a marked Dynkin diagram.*

Definition 2.2.3. By abuse of notation, we denote the Dynkin type of G simply by G. If $\Delta \setminus \Delta_{\mathfrak{p}} := \{\alpha_{i_1}, \cdots, \alpha_{i_j}\}$, we denote the rational homogeneous variety G/P by $G(\omega_{i_1} + \cdots + \omega_{i_j})$.

Remark 2.2.4. A projective variety X is a rational homogeneous variety if and only if X is a homogeneous variety which is birational to a projective space, that is, X is rational in the usual sense.

Theorem 2.2.5 ([BR61])**.** *Any homogeneous variety splits uniquely as a product of an abelian variety and a rational homogeneous variety.*

Definition 2.2.6. (i) The Grassmannian of r-planes is defined by

$$G(r, \mathbb{C}^m) := \{[V] | V \text{ is an } r\text{−dimensional subspace of } \mathbb{C}^m\}.$$

This is a homogeneous variety acted by the special linear group $SL(m, \mathbb{C})$ transitively.

(ii) For a non-degenerate symmetric bilinear form ω on $\mathbb{C}^m$, the orthogonal Grassmannian of isotropic r-planes is defined by

$$OG(r, \mathbb{C}^m) := \{[V] \in G(r, \mathbb{C}^m) | \omega(V, V) = 0\}.$$

This is a homogeneous variety acted by the special orthogonal group $SO(m, \mathbb{C})$ transitively if $m \neq 2r$. Remark that $OG(r, \mathbb{C}^{2r})$ has two components and these are isomorphic to each other as abstract varieties.

(iii) For a non-degenerate skew-symmetric bilinear form ω on $\mathbb{C}^{2m}$, the Lagrangian Grassmannian of isotropic r-planes is defined by

$$LG(r, \mathbb{C}^{2m}) := \{[V] \in G(r, \mathbb{C}^{2m}) | \omega(V, V) = 0\}.$$

This is a homogeneous variety acted by the symplectic group $Sp(2m, \mathbb{C})$ transitively.

Proposition 2.2.7. *Any rational homogeneous variety of classical type is one of the varieties appearing in the above Definition 2.2.6. More precisely,*

(i) $A_l(\omega_r)$ *is isomorphic to* $G(r, \mathbb{C}^{l+1})$,

(ii) $B_l(\omega_r)$ *is isomorphic to* $OG(r, \mathbb{C}^{2l+1})$,

(iii) $C_l(\omega_r)$ *is isomorphic to* $LG(r, \mathbb{C}^{2l})$,

(iv) $D_l(\omega_r)$ *is isomorphic to* $OG(r, \mathbb{C}^{2l})$ *if* $r \leq l - 2$,

(v) $D_l(\omega_r)$ *is isomorphic to one of components of* $OG(r, \mathbb{C}^{2r})$ *if* $r = l - 1$ *or* l.

Example 2.2.8. (i) $G_2(\omega_2)$ is a Mukai variety, that is, a Fano variety of coindex 3.

(ii) $A_2(\omega_1 + \omega_2)$ is isomorphic to $\mathbb{P}(T_{\mathbb{P}^2})$, where $T_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.

Proposition 2.2.9 ([BH58])**.** *A rational homogeneous variety is a Fano variety.*

Moreover, the cohomology ring $H^*(X, \mathbb{Z})$ and the total Chern class of a rational homogeneous variety X can be expressed in terms of the root system [BH58]. In particular, we can calculate the Fano index of rational homogeneous varieties. The list of the Fano index of a rational homogeneous variety of Picard number 1 is on [Sno89]. We have the following list:

Proposition 2.2.10. *Let $X = G(\omega_r)$ be a rational homogeneous variety of Picard number 1. Let n be the dimension of X and i_X the Fano index of X. Then the following holds.*

(i) $G = A_l$: $n = r(n+1-r)$, $i_X = l+1$.

(ii) $G = B_l$: $n = r(4l+1-3r)/2$,

$$i_X = \begin{cases} 2l - r & (r < l) \\ 2l & (r = l) \end{cases}$$

(iii) $G = C_l$: $n = r(4l+1-3r)/2$, $i_X = 2l - r + 1$.

(iv) $G = D_l$: $n = r(4l-1-3r)/2$,

$$i_X = \begin{cases} 2l - r - 1 & (r < l-1) \\ 2l - 2 & (r = l-1, l) \end{cases}$$

(v) $G = E_6$:

r	1	2	3	4	5	6
n	16	25	29	25	16	21
i_X	12	9	7	9	12	11

(vi) $G = E_7$:

r	1	2	3	4	5	6	7
n	27	42	50	53	47	33	42
i_X	18	13	10	8	11	17	14

(vii) $G = E_8$:

r	1	2	3	4	5	6	7	8
n	57	83	97	104	106	98	78	92
i_X	29	19	14	11	9	13	23	17

(viii) $G = F_4$:

r	1	2	3	4
n	15	20	20	15
i_X	8	5	7	11

(ix) $G = G_2$:

r	1	2
n	5	5
i_X	5	3

Proposition 2.2.11 ([Bot57]). *The complex structure of a rational homogeneous variety is locally rigid.*

Hence if X_t is a smooth deformation of the complex structure of a rational homogeneous variety X, X_t is biholomorphic to X_0 for sufficiently small t.

Remark 2.2.12. J. M. Hwang and N. Mok have studied the rigidity of rational homogeneous varieties (see [HM05]). They showed the following:

Let $\pi : \chi \to \Delta$ be a smooth and projective morphism from complex manifold χ to the unit disk Δ. If the fiber $X_t = \pi^{-1}(t)$ is biholomorphic to a rational homogeneous variety Y of Picard number 1 for any $t \in \Delta - \{0\}$, then X_0 is also biholomorphic to Y.

2.3 Secants of varieties

We introduce some of F. Zak's results. For details, see [Zak93].

Zak proved Hartshorne's conjecture on linear normality:

Theorem 2.3.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate smooth projective variety of dimension n. For $3n > 2(N-1)$, X is linearly normal, that is, the natural map $H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1)) \to H^0(X, \mathscr{O}_X(1))$ is surjective.*

Definition 2.3.2. For varieties $X, Y \subset \mathbb{P}^N$, we define the *join of X and Y* by the closure of the union of lines passing through distinct two points $x \in X$ and $y \in Y$ and denote by $S(X,Y)$. In the special case that $X = Y$, $\mathrm{Sec}(X) := S^1X := S(X,X)$ is called the *secant variety of X*. Furthermore $\delta_X := 2\dim X + 1 - \dim \mathrm{Sec}(X)$ is called the *secant defect* of $X \subset \mathbb{P}^N$.

Remark 2.3.3. In general, it is easy to see the dimension of the secant variety S^1X is at most $2n+1$, where $n := \dim X$. The expected dimension of the secant variety S^1X is $2n+1$. When the dimension of S^1X is less than $2n+1$, we say the secant variety S^1X *defective*.

Theorem 2.3.1 is equivalent to the following:

Theorem 2.3.4. *Let $X \subset \mathbb{P}^N$ be a non-degenerate smooth projective variety of dimension n. If $3n > 2(N-2)$, then $\mathrm{Sec}(X) = \mathbb{P}^N$.*

It is a natural question to classify projective varieties on the boundary of the above theorem. For $n = 2$, F. Severi classified such varieties.

Definition 2.3.5. Let $X \subset \mathbb{P}^N$ be a non-degenerate smooth projective variety of dimension n. X is a *Severi variety* if it satisfies that $3n = 2(N-2)$ and $\mathrm{Sec}(X) \neq \mathbb{P}^N$.

As we remarked above, a classification of the 2-dimensional Severi variety was studied by Severi. The 4-dimensional case was studied by T. Fujita and J. Roberts ([FR81]). In general case, Zak proved the following:

Theorem 2.3.6. *Each Severi variety is projectively equivalent to one of the following:*

(i) *The Veronese surface* $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$.

(ii) *The Segre variety* $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$.

(iii) *The Grassmann variety* $G(\mathbb{P}^1, \mathbb{P}^5) \subset \mathbb{P}^{14}$.

(iv) *The* E_6*-variety* $E_6(\omega_1) \subset \mathbb{P}^{26}$.

In particular, Severi varieties are homogeneous.

Proposition 2.3.7. *Let* $M(n, \delta)$ *be the maximal number* N *for which there exists a non-degenerate smooth projective variety* $X \subset \mathbb{P}^N$ *such that* $\dim X = n$ *and* $\delta_X = \delta$. *Then we have* $M(n, \delta) \leq f([n/\delta])$, *where* $f(k) = (k+1)(n+1) - k(k+1)\delta/2 - 1$ *and* $[n/\delta]$ *is the largest integer not exceeding* n/δ.

For $n, \delta \in \mathbb{N}$, a non-degenerate smooth projective variety $X \subset \mathbb{P}^N$ be called an *extremal variety* if $\delta_X = \delta$ and $N = M(n, \delta)$.

For $\delta > n/2$, we have $\mathrm{Sec(X)} = \mathrm{\mathbb{P}^N}$. So each variety is extremal. When we have $\delta = n/2$, X is extremal if and only if X is a Severi variety.

Definition 2.3.8. Let $X \subset \mathbb{P}^N$ be an n-dimensional non-degenerate smooth projective variety. We call X a *Scorza variety* if,

(i) $\mathrm{Sec(X)} \neq \mathbb{P}^N$;

(ii) $N = f([n/\delta])$, where $f(k) = (k+1)(n+1) - k(k+1)\delta/2 - 1$ and $[n/\delta]$ is the largest integer not exceeding n/δ.

Remark 2.3.9. X is a Scorza variety if and only if $n \geq 2\delta > 0, N = f([n/\delta])$.

Theorem 2.3.10. *Each Scorza variety is projectively equivalent to one of the following:*

(i) *The Veronese surface* $v_2(\mathbb{P}^N)$.

(ii) *The Segre variety* $\mathbb{P}^{[n/2]} \times \mathbb{P}^{[n/2]}$.

(iii) *The Grassmann variety* $G(\mathbb{P}^1, \mathbb{P}^{n/2+1})$.

(iv) *The* E_6*-variety* $E_6(\omega_1)$.

In particular, Scorza varieties are homogeneous.

Chapter 3

Classification of polarized manifolds admitting homogeneous varieties as ample divisors

3.1 Introduction

By a *polarized variety* we mean a pair (X, L) consisting of a complete variety X and an ample line bundle L on it.

One of the important problems in the study of polarized varieties is to classify the pairs (X, L) such that the linear system $|L|$ has a smooth member A with preassigned properties. The purpose of this chapter is to study the case where A is *homogeneous*, that is, where a group variety acts on A transitively, such as abelian varieties, Grassmann varieties, and so on. Note that A. J. Sommese [Som76] studied the case where A is an abelian variety, T. Fujita [Fuj80I, Fuj81I, Fuj82] the case where A is a Grassmann variety, and that the case of $\dim A = 1$ is easily classified (see [Som76, Proposition II, Remark I. B]).

Our result is

Theorem 3.1.1. *Let (X, L) be a smooth polarized variety such that the linear system $|L|$ has a homogeneous member A. Assume that $\dim A \geq 2$. Then (X, L) is one of the following:*

(i) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(1))$.

(ii) $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(2))$.

(iii) $(Q^{n+1}, \mathscr{O}_{Q^{n+1}}(1))$.

(iv) $(\mathbb{P}^l \times \mathbb{P}^l, \mathscr{O}_{\mathbb{P}^l \times \mathbb{P}^l}(1,1))$, $2l = n+1$.

(v) $(G(2, \mathbb{C}^{2l}), \mathscr{O}_{\text{Plücker}}(1))$, $4l - 4 = n+1$.

(vi) $(E_6(\omega_1), \mathscr{O}_{E_6(\omega_1)}(1))$, $\varphi_{|\mathscr{O}_{E_6(\omega_1)}(1)|} : E_6(\omega_1) \hookrightarrow \mathbb{P}^{26}$ *is the projectivization of the highest weight vector orbit in the 27-dimensional irreducible representation of a simple algebraic group of Dynkin type* E_6.

(vii) $(\mathbb{P}(\mathcal{E}), H(\mathcal{E}))$, $\mathcal{E}$ *is a vector bundle on* $\mathbb{P}^1$ *of rank* $n+1$ *and* $a > 1$ *with a non-splitting exact sequence:*

$$0 \to \mathscr{O}_{\mathbb{P}^1} \to \mathcal{E} \to \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus n} \to 0,$$

where $H(\mathcal{E})$ *is the tautological line bundle on* $\mathbb{P}(\mathcal{E})$.

(viii) $(\mathbb{P}(\mathcal{E}), H(\mathcal{E}))$, $\mathcal{E}$ *is a vector bundle on an elliptic curve* E *of rank* $n+1$ *and* $\mathscr{L}$ *an ample line bundle on* E *with a non-splitting exact sequence:*

$$0 \to \mathscr{O}_E \to \mathcal{E} \to \mathscr{L}^{\oplus n} \to 0.$$

The contents of this chapter are organized as follows: In section 5, we prove that most homogeneous varieties cannot be ample divisors in any smooth variety, using results of Fujita (see Proposition 3.2.2) and of S. Merkulov and L. Schwachhöfer [MS99]. In section 6 we give a proof of the main theorem, where one of the bottlenecks is to determine all the possibilities of X for a fixed A: For example, an n-dimensional smooth hyperquadric Q^n is not only a very ample divisor on $\mathbb{P}^{n+1}$ but also a hyperplane section of a hyperquadric Q^{n+1}.

3.2 Preliminaries

Definition 3.2.1 (Fujita, [Fuj82, Definition 1.4]). If $H^q(X, T_X[-L]) = 0$ for any ample line bundle L on a smooth variety X and $q = 0, 1$, we say that X satisfies the NS-condition. Here T_X is the tangent bundle of X.

Proposition 3.2.2 (Fujita, [Fuj82, Corollary 1.3]). *If a smooth variety* X *satisfies the NS-condition, then* X *cannot be an ample divisor in any smooth variety.*

Proposition 3.2.3. *Let* X *be a homogeneous variety with* $\dim X \geq 2$. *Then the following are equivalent:*

(i) *X does not satisfy the NS-condition.*

(ii) *X is one of the following:*

(a) $\mathbb{P}^n$, (b) Q^n, (c) $\mathbb{P}(T_{\mathbb{P}^l})$, (d) $C_l(\omega_2)$,

(e) $F_4(\omega_4)$, (f) $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, (g) $E \times \mathbb{P}^{n-1}$ *with an elliptic curve E.*

Proof. Let X be a homogeneous variety which splits as a product of an abelian variety X_1 and a rational homogeneous variety X_2.

If X is an abelian variety with $\dim X \geq 2$, X satisfies the NS-condition (see [Fuj82, Proposition 2.2]).

Next we assume that X is a non-trivial product of an abelian variety X_1 and a rational homogeneous variety X_2. Then we have $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_1) \times \mathrm{Pic}(X_2)$ (see [Har71, III, Exercises 12.6]); hence we may assume that any ample line bundle on X is of the form $p_1^*(L_1) \otimes p_2^*(L_2)$, where p_i are natural projections and L_1 (resp. L_2) is an ample line bundle on X_1 (resp. X_2).

By using the Künneth formula, we have

$$
\begin{aligned}
&h^1(X, T_X[p_1^*(-L_1) \otimes p_2^*(-L_2)]) \\
&= h^1(X, p_1^*(T_{X_1}[-L_1]) \otimes p_2^*(-L_2)) + h^1(X, p_1^*(-L_1) \otimes p_2^*(T_{X_2}[-L_2])) \\
&= h^1(X_1, \bigoplus[-L_1]) \cdot h^0(X_2, [-L_2]) + h^0(X_1, \bigoplus[-L_1]) \cdot h^1(X_2, [-L_2]) \\
&+ h^1(X_1, [-L_1]) \cdot h^0(X_2, T_{X_2}[-L_2]) + h^0(X_1, [-L_1]) \cdot h^1(X_2, T_{X_2}[-L_2]) \\
&= h^1(X_1, [-L_1]) \cdot h^0(X_2, T_{X_2}[-L_2]).
\end{aligned}
$$

$$
\begin{aligned}
&h^0(X, T_X[p_1^*(-L_1) \otimes p_2^*(-L_2)]) \\
&= h^0(X, p_1^*(T_{X_1}[-L_1]) \otimes p_2^*(-L_2)) + h^0(X, p_1^*(-L_1) \otimes p_2^*(T_{X_2}[-L_2])) \\
&= h^0(X_1, \bigoplus[-L_1]) \cdot h^0(X_2, [-L_2]) + h^0(X_1, [-L_1]) \cdot h^0(X_2, T_{X_2}[-L_2]) \\
&= 0.
\end{aligned}
$$

If $\dim X_1 \geq 2$, X_1 satisfies the NS-condition. Hence, so does X.

Next, consider the case where $\dim X_1 = 1$, that is, X_1 is an elliptic curve E. Note that $h^0(X_2, T_{X_2}[-L_2]) \neq 0$ for some ample line bundle L_2 on X_2 if and only if $X_2 \cong \mathbb{P}^{n-1}$. This follows from a result of Mori-Sumihiro (see [MS78] and [Sno89, Theorem 6.5]), one of J. M. Wahl [Wah83] or one of S. Merkulov and L. Schwachhöfer [MS99, Theorem B].

The above argument infers that X does not satisfy the NS-condition if and only if X is a product of an elliptic curve E and a projective space under the condition that X is a non-trivial product of an abelian variety and a rational homogeneous variety.

Finally assume that X is a rational homogeneous variety. Then we see that X does not satisfy the NS-condition if and only if X is one of the cases (a)-(f) in Proposition 5.1.2 by [MS99, Theorem B].

$\square$

Lemma 3.2.4. *Let (X, L) be a smooth polarized variety such that the linear system $|L|$ has a rational homogeneous member A. Assume that* $\mathrm{Pic}(A) \cong \mathbb{Z}[L_A]$ *and* $\dim A \geq 2$. *Then L is very ample.*

Proof. Let (X, L) be a smooth polarized variety which satisfies the condition of this lemma. Then L_A is a very ample line bundle on A (see [Sno89, Theorem 6.5]). Let $\varphi := \varphi_{|L_A|} : A \hookrightarrow \mathbb{P}^N$ be a closed embedding determined by the complete linear system $|L_A|$. Using [Ste84, P226 Remark and Theorem 1], we see $\varphi(A) \subset \mathbb{P}^N$ is factorial, that is, the homogeneous coordinate ring of $\varphi(A)$ is a unique factorization domain. Since a UFD is integrally closed, L_A is projectively normal. So L_A is simply generated.

On the other hand, $H^0(X, L) \to H^0(A, L_A)$ is surjective since X is a Fano variety. Hence L is simply generated by [Fuj90, Corollary 2.5]. This directly leads us to the conclusion that L is very ample (see [Fuj90, P27]).

$\square$

Lemma 3.2.5 ([Zak93, P114])**.** *Let $X \subset \mathbb{P}^N$ be a smooth variety and $H \subset \mathbb{P}^N$ a general hyperplane. Then $\delta_{X \cap H} = 0$ if $\delta_X = 0$, and $\delta_{X \cap H} = \delta_X - 1$ otherwise. Here δ_X (resp. $\delta_{X \cap H}$) is the secant defect of X (resp. $X \cap H$), that is, $\delta_X := 2 \dim X + 1 - \dim \mathrm{Sec}(X)$, where $\mathrm{Sec}(X)$ is the secant variety of X in $\mathbb{P}^N$.*

Proposition 3.2.6 ([Zak93, Chapter VI], Theorem 2.3.10)**.** $G(2, \mathbb{C}^{\frac{m}{2}+2})$ *is the only m-dimensional Scorza variety with the secant defect $\delta = 4$ for $m \geq 8$.*

3.3 Proof of the Main Theorem

Proof. Let (X, L) be a smooth polarized variety such that the linear system $|L|$ has a homogeneous member A. Assume that $\dim A \geq 2$. Using Proposition 3.2.2 and 3.2.3, we see that A is one of the varieties listed (ii) in Proposition 3.2.3. Hence it is sufficient to consider the cases where A is isomorphic to $\mathbb{P}^n$, Q^n in $\mathbb{P}^{n+1}$, $\mathbb{P}(T_{\mathbb{P}^l})$, $C_l(\omega_2)$, $F_4(\omega_4)$, $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ and $E \times \mathbb{P}^{n-1}$.

If $A \cong \mathbb{P}^n$, we have $(X, L) \cong (\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(1))$ (see [Fuj90, Theorem 7.18]). If $A \cong Q^n$ with $n \geq 3$, we have $(X, L) \cong (\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(2))$ or $(Q^{n+1}, \mathscr{O}_{Q^{n+1}}(1))$ (see [Som76, Proposition VI and its Corollary]). If $n = 2$, we have $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. So let us consider the following case where $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$. If $A \cong \mathbb{P}(T_{\mathbb{P}^l})$, we have two natural projections p_i from A to $\mathbb{P}^l$ $(i = 1, 2)$. Therefore X

has two bundle structures (see [BS95, Theorem 5.5.2 and 5.5.3]). Applying a result of E. Sato [Sat85], we see X is isomorphic to $\mathbb{P}^a \times \mathbb{P}^b$ or $\mathbb{P}(T_{\mathbb{P}^l})$. Since A is isomorphic to $\mathbb{P}(T_{\mathbb{P}^l})$, we have $(X, L) \cong (\mathbb{P}^l \times \mathbb{P}^l, \mathscr{O}_{\mathbb{P}^l \times \mathbb{P}^l}(1,1))$.

Next we deal with the remaining cases where A is isomorphic to $C_l(\omega_2)$, $F_4(\omega_4)$, $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ and $E \times \mathbb{P}^{n-1}$.

The case where $A \cong C_l(\omega_2)$.

Now A is a very ample divisor on X by Lemma 3.2.4. If $l = 2$, we have $C_2(\omega_2) \cong Q^3$. This is the case where $A \cong Q^n$. So we assume that $l \geq 3$.

If $(X, L) \cong (G(2, \mathbb{C}^{2l}), \mathscr{O}_{\text{Plücker}}(1))$, it is a well-known result that the linear system $|L|$ has $C_l(\omega_2)$ as a smooth member (see [Sak85]). What we have to show is the pair $(G(2, \mathbb{C}^{2l}), \mathscr{O}_{\text{Plücker}}(1))$ is the only case where the linear system $|L|$ has $A \cong C_l(\omega_2)$ as a smooth member.

Let X be a smooth variety containing A as a very ample divisor. According to [Sno89, 9.3], $A \cong C_l(\omega_2)$ is a Fano variety of $\dim A = 4l - 5$ and index $2l - 2$. By the Lefschetz theorem ([Fuj90, Theorem 7.1]), we have $\operatorname{Pic}(X) \cong \operatorname{Pic}(A) \cong \mathbb{Z}$. Furthermore, we see that $\mathscr{O}_A(A)$ (respectively, $\mathscr{O}_X(A)$) is a very ample generator of $\operatorname{Pic}(A)$ (respectively, $\operatorname{Pic}(X)$) by [MS99, Theorem B]. Because it follows from the adjunction formula that X is a Fano variety, we see $H^1(X, \mathscr{O}_X) = 0$. So we get $h^0(X, \mathscr{O}_X(A)) = h^0(A, \mathscr{O}_A(A)) + 1$. By the same argument, we see that $h^0(G(2, \mathbb{C}^{2l}), \mathscr{O}_{\text{Plücker}}(1)) = h^0(A, \mathscr{O}_A(A)) + 1$. Therefore we have

$$h^0(X, \mathscr{O}_X(A)) = h^0(G(2, \mathbb{C}^{2l}), \mathscr{O}_{\text{Plücker}}(1)).$$

This implies that X and $G(2, \mathbb{C}^{2l})$ can be embedded into the same projective space $\mathbb{P}^{N_1}$, where $N_1 = h^0(X, \mathscr{O}_X(A)) - 1$ by $\mathscr{O}_X(A)$ and $\mathscr{O}_{\text{Plücker}}(1)$, respectively.

Let δ_A (respectively, δ_X) be the secant defect of A (respectively, X), where $\operatorname{Sec}(A)$ is the secant variety of A in $\mathbb{P}^{N_2}$ and $N_2 = h^0(A, \mathscr{O}_A(A)) - 1$.

Assume that $\delta_X = 0$. We have $A = X \cap H$ for some hyperplane H in $\mathbb{P}^{N_X}$. If H is a general hyperplane, $\delta_A = 0$ (see Lemma 3.2.5). This contradicts the fact that $\delta_A = 3$. So H should not be a general hyperplane. We have a smooth deformation A_t of A by moving hyperplanes in $\mathbb{P}^{N_1}$. Then each A_t is isomorphic to A for all t near 0 since A is locally rigid (see Proposition 2.2.11). So we obtain a general hyperplane H' such that $X \cap H'$ is isomorphic to A. Consequently, we have a contradiction by the same argument as in the case where H is a general hyperplane.

This argument implies that $\delta_X > 0$. From this, it follows that $\delta_X = 4$ (see Lemma 3.2.5). Then we see that $\varphi_{|\mathscr{O}_X(A)|} : X \hookrightarrow \mathbb{P}^{N_1}$ is a Scorza variety. By Proposition 3.2.6, X is isomorphic to $G(2, \mathbb{C}^{2l})$.

The case where $A \cong F_4(\omega_4)$.

A is a very ample divisor on X by Lemma 3.2.4.

If $(X, L) \cong (E_6(\omega_1), \mathscr{O}_{E_6(\omega_1)}(1))$, it is a well-known result that the linear system $|L|$ has $F_4(\omega_4)$ as a smooth member. What we have to show is the pair $(E_6(\omega_1), \mathscr{O}_{E_6(\omega_1)}(1))$ is the only case where the linear system $|L|$ has $A \cong F_4(\omega_4)$ as a smooth member.

Let X be a smooth variety containing A as a very ample divisor. According to [Sno89, 9.3], $A \cong F_4(\omega_4)$ is a Fano variety of $\dim A = 15$ and index 11. The same argument as in the case where $A \cong C_l(\omega_2)$ implies that X and $E_6(\omega_1)$ can be embedded into the same projective space $\mathbb{P}^{26}$. Now $\dim \mathrm{Sec}(A) < 25$ (see [Zak93, P59]). Since A is locally rigid, we can assume that $A = X \cap H$ for some general hyperplane H in $\mathbb{P}^{26}$ by the same argument as in the case where $A \cong C_l(\omega_2)$. Using Lemma 3.2.5 below, we have $\dim \mathrm{Sec}(X) \leq 25$. Therefore we see that $\mathrm{Sec}(X) \neq \mathbb{P}^{26}$. So X is a Severi variety (see [Zak93, Chapter IV]). Hence X is isomorphic to $E_6(\omega_1)$ (see Theorem 2.3.6).

The case where $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$.

If $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$, we have $(X, L) \cong (\mathbb{P}(\mathcal{E}), H(\mathcal{E}))$ for some ample vector bundle $\mathcal{E}$ on $\mathbb{P}^1$, where a natural projection $p_1 : A \to \mathbb{P}^1$ is equal to the restriction to A of the bundle projection $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ (see [BS95, Theorem 5.5.2 and 5.5.3]). Then we have an exact sequence $0 \to \mathscr{O}_X \to \mathscr{O}_X(A) \to \mathscr{O}_A(A) \to 0$. This exact sequence is pushed down by π_* to an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^1} \to \mathcal{E} \to \pi_* H(\mathcal{E}) \to 0.$$

This exact sequence does not split, because $\mathcal{E}$ is an ample vector bundle. Furthermore, we have $\mathbb{P}^1 \times \mathbb{P}^{n-1} \cong A \in |H(\mathcal{E})|$. So we obtain that $\pi_* H(\mathcal{E}) = \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus n}$ and $a > 0$.

In the case $a = 1$, we have

$$\mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus n}, \mathscr{O}_{\mathbb{P}^1}) = \mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus n}) = H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus n}) = 0.$$

Hence we obtain that $a > 1$.

The case where $A \cong E \times \mathbb{P}^{n-1}$.

The same argument as in the case where $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ implies that (X, L) is isomorphic to $(\mathbb{P}(\mathcal{E}),\ H(\mathcal{E}))$ satisfying the condition (viii) as in Main Theorem 5.1.3.

□

Corollary 3.3.1. *Let X be a projective bundle over a smooth curve C. Assume that X is homogeneous. Then X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^n$ or $E \times \mathbb{P}^n$, where E is an elliptic curve.*

Proof. Let $\mathcal{E}$ be a vector bundle on a smooth curve C and $X = \mathbb{P}(\mathcal{E})$ a homogeneous variety. Then X splits as a product of an abelian variety and a rational homogeneous variety. By a result of Fujita [Fuj80I, Example 4.21], any projective bundle over a smooth curve is an ample divisor in some smooth variety. So X is one of the cases (a)-(g) in Proposition 5.1.2. Hence X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^n$ or $E \times \mathbb{P}^n$, where E is an elliptic curve by assumption. □

Corollary 3.3.2. *Let (X, L) be as in Theorem 3.1.1. Then X is a homogeneous variety if and only if (X, L) is isomorphic to one of polarized manifolds* (i) $-$ (vi) *in 3.1.1, $\mathbb{P}^1 \times \mathbb{P}^n$, or $E \times \mathbb{P}^n$, where E is an elliptic curve.*

Chapter 4

Actions of linear algebraic groups of exceptional type on projective varieties

4.1 Introduction

Let X be a smooth projective variety of dimension n and r_G the minimum of the dimension of a homogeneous variety of a simple linear algebraic group G, that is, the minimum codimension of a maximal parabolic subgroup of G. M. Andreatta [And01] proved that if $r_G < n$ the only regular action of G on X is trivial, and if $r_G = n$ then X is homogeneous. He also gives a classification of smooth projective varieties on which a simple linear algebraic group of classical type acts regularly and non-trivially in the case where $n = r_G + 1$. Our main purpose of this chapter is to prove the following:

Theorem 4.1.1. *Let X be a smooth projective variety of dimension n and G a simple, simply connected and connected linear algebraic group of exceptional type acting regularly and non-trivially on X. Assume that $n = r_G + 1$. Then X is one of the following; the action of G is unique for each case:*

(i) $\mathbb{P}^6$,

(ii) Q^6,

(iii) $E_6(\omega_1)$,

(iv) $G_2(\omega_1 + \omega_2)$,

(v) $Y \times Z$, *where Y is $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$ and Z is a smooth projective curve,*

(vi) $\mathbb{P}(\mathscr{O}_Y \oplus \mathscr{O}_Y(m))$, *where* Y *is as in* (v) *and* $m > 0$.

Note that G-orbits on X are very simple (for example a projective space and a quadric) in the case where G is classical type, but they are not in our case. So we need other arguments than Andreatta's in several points.

4.2 Preliminaries

Lemma 4.2.1 ([And01, Lemma 1.4, 1.5]). *Let X be a smooth projective variety on which a connected linear algebraic group G acts regularly and non-trivially. Then X has an extremal contraction $\phi : X \to Z$ which is G-equivariant, and G acts regularly on Z.*

Definition 4.2.2 ([And01, Definition 1.8]). Let G be a simple linear algebraic group. We define r_G to be the minimal codimension of parabolic subgroups of G.

Example 4.2.3 ([And01, Example 1.0.1]). If G is an exceptional linear algebraic group, we have $r_{E_6} = 16$, $r_{E_7} = 27$, $r_{E_8} = 57$, $r_{F_4} = 15$ and $r_{G_2} = 5$.

Proposition 4.2.4 ([And01, Proposition 2.1]). *Suppose that a connected reductive linear algebraic group G acts effectively on a complete normal variety Z. Then the followings are equivalent:*

(i) *There exists a fixed point z such that its projectivized tangent cone, that is the variety $P_z = \mathrm{Proj}(\bigoplus_k m_z^k/m_z^{k+1})$, is a G-homogeneous variety.*

(ii) *Z is a projective quasi-homogeneous cone over a homogeneous variety with respect to G.*

Proposition 4.2.5 ([And01, Lemma 2.2 and Proposition 3.1]). *Let X be a smooth projective variety of dimension n and G a simple, simply connected, connected linear algebraic group acting regularly and non-trivially on X. Then;*

(i) $n \geq r_G$;

(ii) *if moreover $n = r_G$, then X is homogeneous;*

(iii) *if G is exceptional and $n = r_G + 1$, X has no fixed points.*

Lemma 4.2.6 ([And01, Lemma 4.2]). *Let X and Y be smooth projective varieties on which a simple exceptional linear algebraic group G acts regularly and non-trivially. Assume that $r_G = \dim X - 1 = \dim Y - 1$. If X and Y each have a dense open orbit which is G-isomorphic, then we have a G-isomorphism $X \cong Y$.*

Proposition 4.2.7 (Theorem 3.1.1, [Wat08]). *Let X be a smooth projective variety and A a rational homogeneous variety $G(\omega)$, where G is exceptional. If A is an ample divisor on X, (X, A) is isomorphic to $(\mathbb{P}^6, Q^5)$, (Q^6, Q^5) or $(E_6(\omega_1), F_4(\omega_4))$.*

Remark that a 5-dimensional smooth quadric Q^5 is G_2-homogeneous.

4.3 Proof of the Main Theorem

By Lemma 4.2.1 we have a G-equivariant extremal contraction of a ray $\phi : X \to Z$.

Assume that $\rho(X) \geq 2$.

The case where ϕ is birational. Let ϕ be birational and E the exceptional locus of ϕ. Since r_G is equal to $n-1$ and X has no fixed points, ϕ is a divisorial contraction and E is contracted to a point z. Furthermore E is isomorphic to $E_6(\omega_1)(= E_6(\omega_5))$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)(= Q^5)$ and $G_2(\omega_2)$. The conormal bundle of the exceptional divisor is ${N_{E/X}}^* \cong \mathscr{O}(k)$ with $1 \leq k \leq i(E) - 1$, where $i(E)$ is the Fano index of E.

Applying Proposition 4.2.4, we see that X is a completion of an open orbit G/K (see [Akh77]). Here K is the kernel of the character map $\rho : P \to \mathbb{C}^*$ associated to the homogeneous line bundle ${N_{E/X}}^* \cong \mathscr{O}(k)$, where P is the parabolic subgroup which satisfies $E \cong G/P$.

On the other hand, $X_k = \mathbb{P}({N_{E/X}}^* \oplus \mathscr{O})$ is also a completion of an open orbit G/K. By Lemma 4.2.6, X is isomorphic to $X_k = \mathbb{P}({N_{E/X}}^* \oplus \mathscr{O})$.

The case where ϕ is a fibering type. Let ϕ be a contraction of fibering type.

First we assume that the induced action of G on Z is trivial. In this case, any fiber of ϕ is isomorphic to $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$ and $\dim Z = 1$. Since rational homogeneous varieties are locally rigid, there is no ϕ which has both $F_4(\omega_1)$ and $F_4(\omega_4)$ (resp. $G_2(\omega_1)$ and $G_2(\omega_2)$) as fibers. So all fibers of ϕ are isomorphic to each other. Then we have $X = E_6(\omega_1) \times Z$, $E_7(\omega_1) \times Z$, $E_8(\omega_1) \times Z$, $F_4(\omega_1) \times Z$, $F_4(\omega_4) \times Z$, $G_2(\omega_1) \times Z$ or $G_2(\omega_2) \times Z$. This follows from [Mab79, Theorem 1.2.1].

Second we assume that the induced action of G on Z is not trivial. Then Z is isomorphic to $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$.

It follows that all fibers have dimension one. Moreover, all fibers of ϕ are isomorphic to each other. So ϕ is a conic bundle which fibers are isomorphic to $\mathbb{P}^1$. Since the Brauer group of Z is trivial, X is $\mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ a rank 2 vector bundle on Z.

The assumption that $n = r_G + 1$ implies that the dimension of any orbit of G in $\mathbb{P}(\mathcal{E})$ is at least $n-1$. If $\mathbb{P}(\mathcal{E})$ is G-homogeneous, then $\mathbb{P}(\mathcal{E})$ has another natural fibration structure $\mathbb{P}(\mathcal{E}) \to Z'$, where Z' is a G-homogeneous variety whose Picard number is 1 (see [BE89, 2.4]). Since $\dim Z + 1 = \dim X > \dim Z'$, (Z, Z') (or (Z', Z)) is $(E_6(\omega_1), E_6(\omega_5))$, $(F_4(\omega_1), F_4(\omega_4))$ or $(G_2(\omega_1), G_2(\omega_2))$ (see [Sno89, 9.3]). However, if (Z, Z') is $(E_6(\omega_1), E_6(\omega_5))$ or $(F_4(\omega_1), F_4(\omega_4))$, the fiber of $\mathbb{P}(\mathcal{E}) \to Z$ is not $\mathbb{P}^1$. Hence (Z, Z') is $(G_2(\omega_1), G_2(\omega_2))$ and we have $\mathbb{P}(\mathcal{E}) \cong G_2(\omega_1 + \omega_2)$.

If $\mathbb{P}(\mathcal{E})$ is not G-homogeneous, we have the G-orbit decomposition $\mathbb{P}(\mathcal{E}) = (\bigsqcup_{i \in I} Gx_i)$ or $\mathbb{P}(\mathcal{E}) = Gx \sqcup (\bigsqcup_{i \in I} Gx_i)$, where $x, x_i \in \mathbb{P}(\mathcal{E})$. Here, Gx is a G-orbit of dimension n and Gx_i is a rational homogeneous variety of dimension $n-1$ whose Picard number is 1. Since $\dim Gx_i = \dim Z$, $\phi_{Gx_i} : Gx_i \to Z$ is a finite morphism. If the ramification divisor R of ϕ_{Gx_i} is not empty, G acts on R. But this contradicts homogeneity of Gx_i. So ϕ_{Gx_i} is $\acute{e}$tale. Hence we see that $\phi_{Gx_i} : Gx_i \to Z$ is isomorphic, because a Fano variety is simply connected. So Gx_i is a section of ϕ. Since any G-homogeneous vector bundle has no a transitive action of G, we have $\sharp I \neq 1$. So $\mathbb{P}(\mathcal{E})$ has two sections which do not intersect each other. Hence $\mathcal{E}$ is decomposable. The uniqueness of action can be proved as above.

Assume that $\rho(X) = 1$. By using the list of parabolic subgroups of codimension n corresponding to one node of the Dynkin diagram, we see that X is not G-homogeneous. So X has a closed orbit H which is isomorphic to $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$. $\rho(X) = 1$ implies X is a Fano variety. Furthermore, Pic(X) $\cong \mathbb{Z}$. Hence H is an ample divisor of X. By Proposition 4.2.7, we see that (X, H) is $(\mathbb{P}^6, Q^5)$, (Q^6, Q^5) or $(E_6(\omega_1), F_4(\omega_4))$.

These X satisfy the assumption of the Theorem. In fact, we see that $F_4 \subset E_6$, $G_2 \subset SO(7) \subset SO(8)$. Here $SO(k)$ means the special orthogonal group.

At last, we shall prove the uniqueness of action. We only deal with the case where X is $E_6(\omega_1)$. We can prove other cases as the same.

Let V_{27} be the irreducible representation space of E_6 with highest weight ω_1. Then E_6 acts on V_{27}. If G whose Dynkin type is F_4 acts on $E_6(\omega_1)$, we obtain a 27-dimensional representation $G \to GL(V_{27})$. By the Weyl dimension theorem and our assumption, it is easy to see that V_{27} is a direct sum of a 26-dimensional irreducible representation space V_{26} and a 1-dimensional irreducible representation space V_1. Furthermore, we see that irreducible representations $G \to GL(V_{26})$ and $G \to GL(V_1)$ are unique. This implies that the action of G on $E_6(\omega_1)$ is unique.

Chapter 5

Lengths of chains of minimal rational curves on Fano manifolds

5.1 Introduction

For a Fano manifold, a *minimal rational component* $\mathscr{K}$ is defined to be a dominating irreducible component of the normalization of the parameter space of rational curves whose degree is minimal among such components and a *variety of minimal rational tangents* is the parameter space of the tangent directions of $\mathscr{K}$-curves at a general point. Nowadays these two objects often appear in the study of Fano manifolds [Hwa01, KS06]. On the other hand, chains of rational curves also play an important role in the study of Fano manifolds. For instance, Kollár-Miyaoka-Mori [KMM92II] and Nadel [Nad91] independently showed the boundedness of the degree of Fano manifolds of Picard number 1 by using chains of rational curves. From these viewpoints, it is a natural question how many rational curves in the family $\mathscr{K}$ are needed to join two general points. We denote by $l_{\mathscr{K}}$ the minimal length of such chains of general $\mathscr{K}$-curves. In this direction, Hwang and Kebekus [HK05] developed an infinitesimal method to study the lengths of Fano manifolds via the varieties of minimal rational tangents. They also dealt with some examples when the varieties of minimal rational tangents and those secant varieties are simple, such as complete intersections, Hermitian symmetric spaces and homogeneous contact manifolds. Furthermore the following was obtained.

Theorem 5.1.1 ([HK05, IR07]). *Let X be a prime Fano n-fold of Picard number 1. If the Fano index i_X satisfies $n+1 > i_X > \frac{2}{3}n$, then $l_{\mathscr{K}} = 2$.*

A Fano manifold is *prime* if the ample generator of the Picard group is very ample. Our original motivation of this chapter is to compute the lengths of Fano manifolds of coindex ≤ 3. By the above theorem, it is sufficient to consider the cases where $n \leq 5$, $(n, i_X) = (6, 4)$ and X is non-prime. Remark that non-prime Fano manifolds of coindex ≤ 3 admit double cover structures [Fuj80I, Fuj80II, Fuj81I, Muk89, Mel99]. First we show the following by using the method of Hwang and Kebekus (Precise definitions of notations are given in Section 2 and 4.):

Theorem 5.1.2 (Theorem 5.5.2, Theorem 5.5.7)**.** *Let X be a Fano n-fold of Picard number* 1*, $\mathscr{K}$ a minimal rational component of X and $p+2$ the anti-canonical degree of rational curves in $\mathscr{K}$. Then if $p = n-3 > 0$, we have $l_{\mathscr{K}} = 2$ and if $(n,p) = (5,1)$, we have $l_{\mathscr{K}} = 3$.*

By combining this theorem and well-known or easy arguments, we obtain the following table (see Remark 5.4.7, Theorem 5.5.4 and Theorem 5.5.8). In particular, when $n \leq 5$, $l_{\mathscr{K}}$ depends only on (n,p).

n	p	$l_{\mathscr{K}}$	n	p	$l_{\mathscr{K}}$	n	p	$l_{\mathscr{K}}$
3	2	1	4	3	1	5	4	1
3	1	2	4	2	2	5	3	2
3	0	3	4	1	2	5	2	2
			4	0	4	5	1	3
						5	0	5

On the other hand, the following shows $l_{\mathscr{K}}$ does not depend only on (n,p) in general.

Theorem 5.1.3 (Theorem 5.6.4)**.** *Let X be a Fano manifold of Picard number* 1 *with coindex* 3 *and $\mathscr{K}$ a minimal rational component of X. Assume that $n := \dim X \geq 6$. Then $l_{\mathscr{K}} = 2$ except the case X is a* 6*-dimensional Lagrangian Grassmannian $LG(3,6)$. In the case $X = LG(3,6)$, we have $l_{\mathscr{K}} = 3$.*

As a consequence, we obtain the following table.

X	i_X	$l_{\mathscr{K}}$
$\mathbb{P}^n$	$n+1$	1
Q^n	n	2
del Pezzo mfd. of dim. n	$n-1$	2
Mukai mfd. of dim. $n \geq 7$	$n-2$	2
Mukai mfd. of dim. 6	4	2 or 3

In Theorem 5.6.11, we give a classification of prime Fano n-folds satisfying $i_X = \frac{2}{3}n$ and $l_{\mathscr{K}} \neq 2$. These are extremal cases of Theorem 5.1.1. Except the case $n = 3$, these varieties are deeply related to Severi varieties which were classified by Zak Theorem 2.3.6, [Zak93] (see Corollary 5.6.12). Furthermore, for prime Fano manifolds, we discuss a relation among 2-*connectedness by lines*, *conic-connectedness* and *defectiveness of the secant varieties* (Corollary 5.6.12 and Remark 5.6.13). In the last section, we investigate Fano manifolds which equip with structures of double covers and are covered by rational curves of degree 1, by a geometric argument without using varieties of minimal rational tangents. In Proposition 5.7.1, we give a criterion for such Fano manifolds to be 2-connected. Remark that all Fano manifolds dealt in [HK05] as examples are prime. However our cases include some non-prime Fano manifolds.

5.2 Deformation theory of rational curves and varieties of minimal rational tangents

First we review some basic facts of deformation theory of rational curves and the definition of varieties of minimal rational tangents. For detail, we refer to [Hwa01, Kol96] and follow the conventions of them. In particular, a smooth projective variety with ample anti-canonical divisor is called a *Fano manifold*.

Throughout this chapter, unless otherwise noted, we always assume that X is a Fano manifold of $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathscr{O}_X(1)]$, where $\mathscr{O}_X(1)$ is the ample generator, and denote by $\mathrm{RatCurves}^n(X)$ the normalization of the space of integral rational curves on X. We also assume $n := \dim X \geq 3$. We denote by i_X the *Fano index of* X which is the integer satisfying $\omega_X \cong \mathscr{O}_X(-i_X)$, where ω_X is the canonical line bundle of X. We call $n + 1 - i_X$ the *coindex of* X.

As is well-known, a Fano manifold is uniruled. It is equivalent to the condition that there exists a free rational curve $f : \mathbb{P}^1 \to X$. Here we call a rational curve $f : \mathbb{P}^1 \to X$ *free* if f^*T_X is semipositive. An irreducible component $\mathscr{K}$ of $\mathrm{RatCurves}^n(X)$ is called a *minimal rational component* if it contains a free rational curve of minimal anti-canonical degree. We denote by $\mathscr{K}_x$ the normalization of the subscheme of $\mathscr{K}$ parametrizing rational curves passing through x. Since each member of $\mathscr{K}$ is numerically equivalent, we can define the $\mathscr{O}_X(1)$-degree of $\mathscr{K}$ which is denoted by $d_{\mathscr{K}}$. We will use the symbol p to denote $i_X d_{\mathscr{K}} - 2$. In this setting, the minimal rational component $\mathscr{K}$ satisfies the following fundamental properties.

Proposition 5.2.1 (see [Hwa01]). (i) *For a general point $x \in X$, $\mathscr{K}_x$ is a disjoint union of smooth projective varieties of dimension p.*

(ii) *For a general member $[f]$ of $\mathscr{K}$, $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^p \oplus \mathscr{O}^{n-1-p}$ which is called a standard rational curve. In particular, $p \leq n-1$.*

For a general point $x \in X$, we define the tangent map $\tau_x : \mathscr{K}_x \to \mathbb{P}_*(T_xX)$ by assigning the tangent vector at x to each member of $\mathscr{K}_x$ which is smooth at x. We denote by $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ the image of τ_x, which is called the *variety of minimal rational tangents* at x.

Theorem 5.2.2 ([HM04, Keb02II]). *The tangent map $\tau_x : \mathscr{K}_x \to \mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is the normalization.*

Theorem 5.2.3 ([CMS02, Keb02I]). *If $p = n-1$, namely $\mathscr{C}_x = \mathbb{P}_*(T_xX)$, then X is isomorphic to $\mathbb{P}^n$.*

Theorem 5.2.4 ([HH08]). *Let $S = G/P$ a rational homogeneous variety corresponding to a long simple root and $\mathscr{C}_o \subset \mathbb{P}_*(T_oS)$ the variety of minimal rational tangents at a reference point $o \in S$. Assume $\mathscr{C}_o \subset \mathbb{P}_*(T_oS)$ and $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ are isomorphic as projective subvarieties. Then X is isomorphic to S.*

Theorem 5.2.5 ([Miy04]). *If X is a Fano manifold of $n := \dim X \geq 3$, the following are equivalent.*

(i) *X is isomorphic to a smooth quadric hypersurface Q^n.*

(ii) *The Picard number of X is 1 and the minimal value of the anti-canonical degree of rational curves passing through a very general point $x_0 \in X$ is equal to n.*

Corollary 5.2.6. *If $p = n-2$, namely $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is a hypersurface, X is isomorphic to Q^n.*

Proof. For a very general point $x_0 \in X$, any rational curve passing through x_0 is free. Let C_0 be a rational curve passing through x_0 whose degree is minimal among such rational curves and $\mathscr{H} \subset \mathrm{RatCurves}^n(X)$ an irreducible component containing $[C_0]$. Then $\mathscr{H}$ is a dominating family. It implies that the anticanonical degree of $\mathscr{H}$ is equal to one of $\mathscr{K}$. Furthermore the anticanonical degree of $\mathscr{K}$ is n from our assumption. Therefore X is isomorphic to Q^n by Theorem 5.2.5. $\square$

5.3 Varieties of minimal rational tangents in the cases $p = n - 3$ and $(n, p) = (5, 1)$

Proposition 5.3.1 ([Hwa01, Proposition 1.4, Proposition 1.5, Theorem 2.5], [Hwa03, Proposition 2, Proposition 5], [Hwa07, Proposition 2.2])**.** *Let X, $\mathscr{K}$ and p be as in Section 2 and $\mathscr{C}_x$ the variety of minimal rational tangents associated to $\mathscr{K}$ at a general point $x \in X$.*

(i) *The tangent map $\tau_x : \mathscr{K}_x \to \mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is an immersion at $[C] \in \mathscr{K}_x$ if C is a standard rational curve on X.*

(ii) *If $X \subset \mathbb{P}^N$ is covered by lines, the tangent map τ_x is an embedding. In particular, $\mathscr{C}_x$ is a disjoint union of smooth projective varieties of dimension p.*

(iii) *If $2p > n - 3$ and $\mathscr{C}_x$ is smooth, $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is non-degenerate.*

(iv) *If $\mathscr{C}_x$ is reducible, it has at least three components.*

(v) *If $\mathscr{C}_x$ is a union of linear subspaces of dimension $p > 0$, each component is disjoint.*

(vi) *$\mathscr{C}_x$ cannot be an irreducible linear subspace.*

Proposition 5.3.2 ([HM98, Proposition 9])**.** *Let X, $\mathscr{K}$ and $\mathscr{C}_x$ be as above, $\mathbb{P}_*(W_x)$ the linear span of $\mathscr{C}_x$, $\mathscr{T}_x \subset \mathbb{P}_*(\wedge^2 W_x)$ the subvariety parametrizing tangent lines of smooth locus of $\mathscr{C}_x$ and $[\ ,\ ]_x : \wedge^2 W_x \to T_xX/W_x$ is the Frobenius bracket tensor. Then $\mathscr{T}_x$ is contained in $\mathbb{P}_*(\mathrm{Ker}([\ ,\ ]_x)) \subset \mathbb{P}_*(\wedge^2 W_x)$.*

Lemma 5.3.3. *If $X \subset \mathbb{P}_*(V)$ is an irreducible hypersurface which is not linear, then its variety of tangential lines $\mathscr{T}_X \subset \mathbb{P}_*(\wedge^2 V)$ is non-degenerate.*

Proof. Assume that $\mathscr{T}_X \subset \mathbb{P}_*(\wedge^2 V)$ is degenerate. We denote by $C(X) \subset V$ the cone corresponding to $X \subset \mathbb{P}_*(V)$. Then there exists a nonzero $\omega \in \wedge^2 V^*$ such that $C(X)$ is isotropic with respect to ω. We set $Q := \{v \in V | \omega(v, w) = 0 \text{ for any } w \in V\}$. ω induces a nonzero symplectic form on V/Q. For the projection $\pi : V \to V/Q$, it follows $2 \dim \pi(C(X)) \leq \dim V/Q$. Remark that $\dim V - 1 = \dim C(X)$. Therefore we have an inequality $\dim V/Q \leq 2$. Since $\pi(C(X))$ is not V/Q and $\{0\}$, it implies that $\pi(C(X)) \subset V/Q$ is a line. Hence $C(X) \subset V$ is a hyperplane. This contradicts the non-linearity of X. □

Proposition 5.3.4 ([Hwa98, Proposition 2])**.** *Let X, $\mathscr{K}$, $\mathscr{C}_x$, W_x be as in Proposition 5.3.2 and W be the distribution defined by W_x for general $x \in X$. Then W is integrable if and only if W_x coincides with T_xX for general $x \in X$.*

Proposition 5.3.5 (cf. [Hwa03, Proposition 7]). *Let X be a Fano n-fold of Picard number 1 and $\mathscr{K}$ a minimal rational component of X with $p = n - 3 > 0$. Then the variety of minimal rational tangents $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ at a general point $x \in X$ is one of the following:*

(i) *a non-degenerate variety with no linear component, or*

(ii) *a disjoint union of at least three lines.*

Proof. First assume that $\mathscr{C}_x$ has a linear component. Then every component of $\mathscr{C}_x$ is linear. By Proposition 5.3.1, $\mathscr{C}_x$ is a disjoint union of at least three linear subspaces. Let $\mathscr{C}_{x,1}$ and $\mathscr{C}_{x,2}$ be distinct components of $\mathscr{C}_x$. Since $\dim \mathscr{C}_{x,1} + \dim \mathscr{C}_{x,2} - \dim \mathbb{P}_*(T_xX) = n - 5$, we have $\mathscr{C}_{x,1} \cap \mathscr{C}_{x,2} \neq \emptyset$ if $n \geq 5$. This implies $n = 4$ and $\mathscr{C}_x$ is a disjoint union of at least three lines.

Second assume that $\mathscr{C}_x$ has no linear components. We will show that $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is non-degenerate. Suppose $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is degenerate. Let $\mathbb{P}_*(W_x)$ be the linear span of $\mathscr{C}_x$ and $\mathscr{T}_x \subset \mathbb{P}_*(\wedge^2 W_x)$ be the subvariety parametrizing tangent lines of smooth locus of $\mathscr{C}_x$. By Proposition 5.3.2, we have $\mathscr{T}_x \subset \mathbb{P}_*(\mathrm{Ker}([\ ,\ ]_x)) \subset \mathbb{P}_*(\wedge^2 W_x)$ where $[\ ,\ ]_x : \wedge^2 W_x \to T_xX/W_x$ is the Frobenius bracket tensor. Lemma 5.3.3 implies that $\mathscr{T}_x \subset \mathbb{P}_*(\wedge^2 W_x)$ is non-degenerate. Therefore $\mathbb{P}_*(\mathrm{Ker}([\ ,\ ]_x))$ coincides with $\mathbb{P}_*(\wedge^2 W_x)$. Applying Frobenius Theorem, the distribution W defined by W_x is integrable. However, this contradicts Proposition 5.3.4. $\square$

By the same argument, we can show the following:

Proposition 5.3.6. *If $(n, p) = (5, 1)$, then the variety of minimal rational tangents $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ at a general point $x \in X$ satisfies one of the following:*

(i) *a curve with no linear component whose linear span $< \mathscr{C}_x >$ has dimension at least 3, or*

(ii) *a disjoint union of at least three lines.*

5.4 Spanning dimensions of loci of chains

Definition 5.4.1 ([HK05]). For a general point $x \in X$, we define $\mathrm{loc}^1(x) := \bigcup_{[C] \in \mathscr{K}_x} C$ and $\mathrm{loc}^{k+1}(x) := \overline{\bigcup_{[C] \in \mathscr{K}_y \text{ for general } y \in \mathrm{loc}^k(x)} C}$ inductively.

We denote the maximal value of the dimensions of irreducible components of $\mathrm{loc}^k(x)$ by d_k.

Definition 5.4.2 ([HK05]). If there exists an integer l such that $d_l = \dim X$ but $d_{l-1} < \dim X$, we say that X has *length l with respect to $\mathscr{K}$*, or X is *l-connected by $\mathscr{K}$*. We denote by $l_{\mathscr{K}}$ the length.

By our assumption that the Picard number of X is 1, we can define the length.

Proposition 5.4.3 ([KMM92I], [KMM92II, Lemma 1.3], [Nad91] and [Kol96, Corollary IV.4.14]). *Under our assumption that X is a Fano manifold of Picard number 1, $l_{\mathscr{K}} \leq \dim X$.*

Definition 5.4.4. For varieties $X, Y \subset \mathbb{P}^N$, we define the *join of X and Y* by the closure of the union of lines passing through distinct two points $x \in X$ and $y \in Y$ and denote by $S(X,Y)$. In the special case that $X = Y$, $S^1X := S(X,X)$ is called the *secant variety of X*.

In [HK05], Hwang and Kebekus computed the first spanning dimension d_1 and gave a lower bound of the second d_2 (resp. d_k) under the assumption $\mathscr{K}_x$ is irreducible for a general point $x \in X$ by using the secant variety of the variety of minimal rational tangents. However their proof works even if we drop the assumption on the irreducibility of $\mathscr{K}_x$.

Theorem 5.4.5 ([HK05, KS06]). *Without the assumption that $\mathscr{K}_x$ is irreducible for a general point $x \in X$,*

(i) $d_1 = p + 1$, $d_k \leq k(p+1)$,

(ii) $d_2 \geq \dim S^1\mathscr{C}_x + 1$, *if* $p > 0$.

Proof. The former follows from Mori's Bend and Break and an easy induction on k. For the later, there is a proof on [KS06] which is easier than one on [HK05].

□

Lemma 5.4.6. *If $p = 0$, we have $l_{\mathscr{K}} = n$.*

Proof. We have an inequality $d_{k+1} \leq d_k + 1$. In particular, $d_k \leq k$. By combining Proposition 5.4.3, we obtain our assertion. □

Remark 5.4.7. If X is a Fano 3-fold of Picard number 1 which is not isomorphic to $\mathbb{P}^3$ and Q^3, then $l_{\mathscr{K}} = 3$. Hence in the 3-dimensional case, we have the following table:

p	i_X	X	$l_{\mathscr{K}}$	$d_{\mathscr{K}}$
2	4	$\mathbb{P}^3$	1	1
1	3	Q^3	2	1
0	2	del Pezzo	3	1
0	1	Mukai	3	2

5.5 Lengths of Fano manifolds of dimension ≤ 5

Lemma 5.5.1. *Let $X \neq Y \subset \mathbb{P}^{n+2}$ be irreducible projective varieties of dimension n. Then*

(i) *$S(X, Y) = \mathbb{P}^{n+2}$ if $X \cup Y$ is non-degenerate.*

(ii) *$S^1 X = \mathbb{P}^{n+2}$ if $X \subset \mathbb{P}^{n+2}$ is a non-degenerate variety which is not linear.*

Proof. (i) Assume that $X \cup Y \subset \mathbb{P}^{n+2}$ is non-degenerate and $S(X, Y) \neq \mathbb{P}^{n+2}$. Then $\dim S(X, Y) = \dim X + 1 = \dim Y + 1$ holds. This implies that $X, Y \subset \mathrm{Vert}(S(X, Y))$. Here we denote the vertex of an irreducible projective variety $Z \subset \mathbb{P}^N$ by $\mathrm{Vert}(Z) := \{p \in Z | S(p, Z) = Z\}$. It is well-known that the vertex $\mathrm{Vert}(Z) \subset \mathbb{P}^N$ is a linear subspace (see [FOV99, Proposition 4.6.2]). Thus we have $X \cup Y \subset \mathrm{Vert}(S(X, Y)) = \mathbb{P}^{n+1} \subset \mathbb{P}^{n+2}$. This contradicts our assumption.

(ii) This is easy. For example, we can show this by the same argument as in (i). $\square$

Theorem 5.5.2. *Let X be a Fano manifold of Picard number 1 with $n = \dim X \geq 4$. Assume that X has a minimal rational component $\mathscr{K}$ with $p = n - 3 > 0$. Then X is 2-connected by $\mathscr{K}$. In particular, if the Fano index i_X is $n - 1$, then X is 2-connected by lines.*

Proof. By Proposition 5.3.5, the variety of minimal rational tangents $\mathscr{C}_x \subset \mathbb{P}_*(T_x X)$ is

(i) a non-degenerate variety with no linear component, or

(ii) a disjoint union of at least three lines.

Let $\mathscr{C}_x$ be as in (i). If $\mathscr{C}_x$ is irreducible, Lemma 5.5.1 implies $S^1\mathscr{C}_x = \mathbb{P}_*(T_x X)$. On the other hand, in the case where $\mathscr{C}_x$ is reducible, $S^1\mathscr{C}_x = \mathbb{P}_*(T_x X)$ also holds. In fact, for the irreducible decomposition $\mathscr{C}_x = \mathscr{C}_{x,1} \cup \cdots \cup \mathscr{C}_{x,m}$, we assume that $S(\mathscr{C}_{x,i}, \mathscr{C}_{x,j}) \neq \mathbb{P}_*(T_x X)$ for any i, j. Then we see $\dim S(\mathscr{C}_{x,i}, \mathscr{C}_{x,j}) = n - 2$. Hence $S(\mathscr{C}_{x,i}, \mathscr{C}_{x,j})$ is a linear subspace $\mathbb{P}^{n-2} \subset \mathbb{P}_*(T_x X)$ (see the proof of Lemma 5.5.1). It turns out from Proposition 5.3.1 that $m \geq 3$. Because $\mathscr{C}_x \subset \mathbb{P}_*(T_x X)$ is non-degenerate, there exists j such that $S(\mathscr{C}_{x,1}, \mathscr{C}_{x,2}) \neq S(\mathscr{C}_{x,1}, \mathscr{C}_{x,j})$. We may assume such j is 3. We have $\mathscr{C}_{x,1} \subset S(\mathscr{C}_{x,1}, \mathscr{C}_{x,2}) \cap S(\mathscr{C}_{x,1}, \mathscr{C}_{x,3})$. Furthermore since $S(\mathscr{C}_{x,1}, \mathscr{C}_{x,2})$ and $S(\mathscr{C}_{x,1}, \mathscr{C}_{x,3})$ are distinct linear subspaces of dimension $n - 2$, these intersection is a linear subspace of dimension $n-3$. Thus we have $\mathscr{C}_{x,1} = S(\mathscr{C}_{x,1}, \mathscr{C}_{x,2}) \cap S(\mathscr{C}_{x,1}, \mathscr{C}_{x,3})$. However this contradicts our assumption $\mathscr{C}_x$ has no linear components. If $\mathscr{C}_x$ is as in (ii), we also have $S^1\mathscr{C}_x = \mathbb{P}_*(T_xX)$.

As a consequence, in any case we have $S^1\mathscr{C}_x = \mathbb{P}_*(T_xX)$. This implies that $d_2 = n$. On the other hand, since $d_1 = p+1 = n-2 < n$, X is 2-connected by $\mathscr{K}$. If $i_X = n-1 \geq 3$, then it follows from the equation $p+2 = i_X d_{\mathscr{K}}$ that $p = n-3$. $\square$

Remark 5.5.3. If X is a prime Fano manifold with $i_X = n-1$ which is a del Pezzo manifold whose degree is at least 3, then the latter statement of the above theorem follows from Theorem 5.1.1.

Theorem 5.5.4. *Let X be a Fano 4-fold of Picard number 1. Then we have the following table:*

p	i_X	X	$l_{\mathscr{K}}$	$d_{\mathscr{K}}$
3	5	$\mathbb{P}^4$	1	1
2	4	Q^4	2	1
1	3	*del Pezzo*	2	1
1	1	*coindex* 4	2	3
0	2	*Mukai*	4	1
0	1	*coindex* 4	4	2

Proof. The computation of the length with respect to $\mathscr{K}$ is an immediate consequence of Theorem 5.2.3, Corollary 5.2.6, Lemma 5.4.6 and Theorem 5.5.2. The other parts follow from the relation $p+2 = i_X d_{\mathscr{K}}$.

$\square$

Lemma 5.5.5. *For an irreducible non-degenerate projective curve $C \subset \mathbb{P}^N$,* $\dim S^1C = \min\{3, N\}$.

Lemma 5.5.6 ([FOV99, Remark 4.6.10])**.** *For two distinct integral curves $C, D \subset \mathbb{P}^N$, $\dim S(C, D) = 2$ holds if and only if $C \cup D$ is a plane curve.*

Theorem 5.5.7. *Let X be a Fano 5-fold of Picard number 1. Assume that X has a minimal rational component $\mathscr{K}$ with $p = 1$. Then X is 3-connected by $\mathscr{K}$.*

Proof. By the same argument as in Theorem 5.5.2, we can prove this theorem. In fact, Proposition 5.3.6, Lemma 5.5.5 and Lemma 5.5.6 implies that $\dim S^1\mathscr{C}_x \geq 3$ for the variety of minimal rational tangents $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$. It turns out that $d_2 \geq 4$. Because $d_2 \leq 2(p+1) = 4$, $d_2 = 4$. Hence X is 3-connected by $\mathscr{K}$. $\square$

Theorem 5.5.8. *Let X be a Fano 5-fold of Picard number 1. Then we have the following table:*

p	i_X	X	$l_{\mathscr{K}}$	$d_{\mathscr{K}}$
4	6	$\mathbb{P}^5$	1	1
3	5	Q^5	2	1
2	4	*del Pezzo*	2	1
2	2	*coindex* 4	2	2
2	1	*coindex* 5	2	4
1	3	*Mukai*	3	1
1	1	*coindex* 5	3	3
0	2	*coindex* 4	5	1
0	1	*coindex* 5	5	2

Proof. The computation of the length with respect to $\mathscr{K}$ is an immediate consequence of Theorem 5.2.3, Corollary 5.2.6, Lemma 5.4.6, Theorem 5.5.2 and Theorem 5.5.7. The other parts follow from the relation $p + 2 = i_X d_{\mathscr{K}}$. □

5.6 Lengths of Fano manifolds of coindex 3

In this section, we study Fano manifolds of coindex 3. Because we already dealt with the case where $n := \dim X \leq 5$ in Theorem 5.1.2, we study the case where $n \geq 6$.

Proposition 5.6.1 ([KK04])**.** *Let X be a projective variety and $\mathscr{H}$ a proper dominating family of rational curves such that none of the associated curves has a cuspidal singularity.*

(i) *For general $x \in X$, all curves in $\mathscr{H}$ passing through x are smooth at x and no two of them share a common tangent direction at x.*

(ii) *Assume that for general $x \in X$ and any irreducible component $\mathscr{H}' \subset \mathscr{H}$, $\dim \mathscr{H}'_x \geq \frac{\dim X - 1}{2}$ holds. Then $\mathscr{H}_x$ is irreducible. In particular, $\mathscr{H}$ is irreducible.*

Lemma 5.6.2 ([BS95, Corollary 1.4.3])**.** *Let C be an integral curve and L a spanned line bundle of degree 1 on C. Then $(C, L) \cong (\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1))$.*

Notation 5.6.3. We denote by $(d_1) \cap \cdots \cap (d_k) \subset \mathbb{P}^n$ a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_k$, in particular, by $(d)^k$ if $d = d_1 = \cdots = d_k$. We denote by $G(k, n)$ a Grassmannian of k-planes in

$\mathbb{C}^n$, by $LG(k,n)$ a Lagrangian Grassmannian which is the variety of isotropic k-planes for a non-degenerate skew-symmetric bilinear form on $\mathbb{C}^n$, by S_k the spinor variety which is an irreducible component of the Fano variety of k-planes in Q^{2k}.

Theorem 5.6.4. *Let X be a Fano manifold of coindex 3 with $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathscr{O}_X(1)]$ and $\mathscr{K}$ a minimal rational component of X. Assume that $n := \dim X \geq 6$. Then $(l_{\mathscr{K}}, d_{\mathscr{K}}) = (2,1)$ except the case X is a Lagrangian Grassmannian $LG(3,6)$. In the case $X = LG(3,6)$ we have $(l_{\mathscr{K}}, d_{\mathscr{K}}) = (3,1)$.*

Proof. We have an inequality $n+1 \geq p+2 = (n-2)d_{\mathscr{K}}$. It follows from our assumption $n \geq 6$ that $(p, d_{\mathscr{K}}) = (n-4, 1)$.

By Iskovskikh Theorem [Isk80] or Mukai's classification result of Fano manifolds of coindex 3 [Muk89, Mel99], X is

(i) a prime Fano manifold, which means $\mathscr{O}_X(1)$ is very ample,

(ii) a double cover $\pi : X \to \mathbb{P}^n$ with a branch divisor $B \in |\mathscr{O}_{\mathbb{P}^n}(6)|$, or

(iii) a double cover $\pi : X \to Q^n$ with a branch divisor $B \in |\mathscr{O}_{Q^n}(4)|$.

Claim 5.6.5. *For a general point $x \in X$, the variety of minimal rational tangent $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is an equidimensional disjoint union of smooth projective varieties.*

When X is prime, this follows from Proposition 5.3.1. So we assume X is as in (ii) or (iii). We denote by Y the target of π which is $\mathbb{P}^n$ or Q^n. By Proposition 5.2.1 it is sufficient to show that the tangent map $\tau_x : \mathscr{K}_x \to \mathscr{C}_x$ is isomorphic.

Since $\mathscr{O}_X(1)$ is spanned and $d_{\mathscr{K}} = 1$, Lemma 5.6.2 implies that any l in $\mathscr{K}$ is isomorphic to $\mathbb{P}^1$. Furthermore Proposition 5.6.1 implies that τ_x is bijective. For $[l] \in \mathscr{K}_x$ we have $\mathscr{O}_X(1).l = 1$. Therefore $\pi(l) \subset Y$ is a standard line and $\pi_l : l \to \pi(l)$ is an isomorphism. According to the generality of $x \in X$, we may assume that l is free and the natural morphism between normal bundles $N_{l/X} \to N_{\pi(l)/Y}$ is generically surjective. Since $N_{l/X}$ is semipositive, $l \subset X$ is a standard rational curve. Hence by Proposition 5.3.1, τ_x is an immersion. As a consequence, we see τ_x is an embedding. So our claim holds.

Since $n \geq 6$, Proposition 5.3.1 implies that $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is non-degenerate. When $n \geq 7$, we see $\mathscr{C}_x$ is irreducible. In fact, if there are distinct irreducible components $\mathscr{C}_{x,1}, \mathscr{C}_{x,2}$ of $\mathscr{C}_x$, we see $\dim \mathscr{C}_{x,1} + \dim \mathscr{C}_{x,2} - \dim \mathbb{P}_*(T_xX) \geq 0$. This implies that $\mathscr{C}_{x,1} \cap \mathscr{C}_{x,2} \neq \phi$. This contradicts the above claim. According to Zak's theorem on linear normality [Zak93] and Theorem 5.4.5, we have $l_{\mathscr{K}} = 2$. So it remains to prove the case $n = 6$. If there exists an irreducible component of $\mathscr{C}_x$ whose secant variety coincides with $\mathbb{P}_*(T_xX)$, we have $l_{\mathscr{K}} = 2$. Therefore we assume that the secant variety of any irreducible component of $\mathscr{C}_x$ does not coincide with $\mathbb{P}_*(T_xX)$. If $\mathscr{C}_x$ is irreducible, then it is the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$. This follows from Zak's classification of Severi varieties [Zak93]. Here remark that the Veronese surface is the variety of minimal rational tangents of the Lagrangian Grassmannian $LG(3,6)$ at a general point (see [HM02, Eli02, LM03]). Thus in this case X is isomorphic to $LG(3,6)$ by Theorem 5.2.4. Because the secant variety of the Veronese surface is a hypersurface, it implies that $d_2 = 4$. Therefore we have $l_{\mathscr{K}} = 3$. If $\mathscr{C}_x$ is reducible, there exists disjoint irreducible components V_1 and V_2. Remark that we assumed that S^1V_i does not coincide with $\mathbb{P}_*(T_xX)$ for $i = 1,2$. If $\dim S(V_1, V_2) \leq 4$, we have a point $q \in \mathbb{P}^5 \setminus S(V_1, V_2) \cup S^1V_1 \cup S^1V_2$. For a projection π_q from a point q, $\pi_q(V_i) \subset \mathbb{P}^4$ is a surface. Hence it turns out that $\pi_q(V_1) \cap \pi_q(V_2) \subset \mathbb{P}^4$ is non-empty. This contradicts $q \in S(V_1, V_2)$. Therefore we have $S(V_1, V_2) = \mathbb{P}_*(T_xX)$. In particular, $S^1\mathscr{C}_x = \mathbb{P}_*(T_xX)$ and $l_{\mathscr{K}} = 2$. □

Here we remark a relation between 2-connectedness by lines and conic-connectedness.

Definition 5.6.6 ([KS02, IR07])**.** For a projective manifold $X \subset \mathbb{P}^N$, we call X *conic-connected* if there exists an irreducible conic passing through two general points on X.

Lemma 5.6.7 (cf. [IR07])**.** *Let $X \subset \mathbb{P}^N$ be a projective manifold which is covered by lines. Then*

(i) *if two general points on X are connected by two lines, X is conic-connected;*

(ii) *if X is conic-connected, then the Fano index i_X is at least $\frac{n+1}{2}$.*

(iii) *Assume that X is conic-connected. Then two general points on X are not connected by two lines if and only if $i_X = \frac{n+1}{2}$.*

Proof. (i) is well-known to the experts (see [KMM92I], [Deb01, Proof of Proposition 5.8]). Suppose that two general points $x_1, x_2 \in X$ are connected by two lines l_1, l_2. Then, without loss of generality, we may assume such two lines are free. By the gluing lemma, there exists a smoothing $(\pi : \mathscr{C} \to (T,0), F : \mathscr{C} \to X, s_1)$ of $l_1 \cup l_2 \subset X$ fixing x_1, where $s_1 : T \to \mathscr{C}$ is a section of π such that $s_1(0) = x_1 \in \pi^{-1}(0) \cong l_1 \cup l_2$ and $F \circ s_1(T) = \{x_1\}$ (see [Kol96, Chapter II.7]). According to a suitable base change, we may assume that there exists a section s_2 of π such that $s_2(0) = x_2 \in \pi^{-1}(0) \cong l_1 \cup l_2$. Let $Z \subset X \times X$ be the set of points $(y_1, y_2) \in X \times X$ which can be joined by an irreducible conic in X. Then for a point $t \neq 0$ in T, $(s_1(t), s_2(t)) \in Z$. It turns out that (x_1, x_2) is contained in the closure of Z. By the generality of $(x_1, x_2) \in X \times X$, we see Z is dense in $X \times X$. Consequently our assertion holds.

(ii) is in [IR07]. If X is conic-connected, then there exists a smooth conic C such that $T_X|_C$ is ample. This implies that $2i_X = \deg T_X|_C \geq n+1$. Hence (ii) holds.

(iii) Suppose that X is conic-connected and it is not 2-connected by lines. Then for general two points $x, y \in X$ there exists a smooth conic $f : \mathbb{P}^1 \cong C \subset X$ passing through x and y such that $T_X|_C$ is ample. This implies that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$. Hence there is no obstruction in the deformation of f fixing the marked points x, y. It turns out that

$$\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, X : f(0) = x, f(\infty) = y) = 2i_X - n. \tag{5.1}$$

If $2i_X - n \geq 2$, Mori's Bend and Break implies C degenerates into a union of two lines containing x and y. This is a contradiction. Hence $2i_X - n \leq 1$. By combining (ii), we have $i_X = \frac{n+1}{2}$. Conversely if the Fano index satisfies $i_X = \frac{n+1}{2}$, it turns out from the same argument as in Theorem 5.4.5 (i) that X is not 2-connected by lines. $\square$

Example 5.6.8. Let $S_4 \subset \mathbb{P}^{15}$ be the 10-dimensional spinor variety and let X be S_4 or its linear section of dimension $n \geq 6$. Then X is a Fano manifold of coindex 3 with the genus $g := \frac{H^n}{2} + 1 = 7$, where H is the ample generator of the Picard group of S_4. There exists a 6-dimensional smooth quadric passing through two general points on S_4 [ES89]. So X is conic-connected and 2-connected by lines. Hence two geneal points on X can be connected by a chain of two lines which is obtained as a degeneration of a conic.

Example 5.6.9. Let X be a Grassmaniann $G(2, 6) \subset \mathbb{P}^{14}$ or its linear section of dimension $n \geq 6$. Then X is a Fano manifold of coindex 3 with the genus $g = 8$. For two distinct points $x, y \in G(2, 6)$, they correspond to 2-dimensional vector subspaces $L_x, L_y \subset \mathbb{C}^6$. Then there exists a 4-dimensional vector subspace $V \subset \mathbb{C}^6$ which contains the join $< L_x, L_y >$. This implies that x, y is contained in a 4-dimensional quadric $Q^4 \cong G(2, 4) \subset G(2, 6)$. So X is conic-connected and 2-connected by lines.

Remark 5.6.10. $X := G(2, 6) \cap (1)^3 \subset \mathbb{P}^{14}$ is a 5-dimensional Fano manifold of index 3. According to Theorem 5.5.8, X is 3-connected by lines. However X is conic-connected. This example shows that our chain of minimal rational curves connecting two general points is not necessary a chain with minimal total degree.

Theorem 5.6.11. *Let X be a prime Fano n-fold with $i_X = \frac{2}{3}n$ and $\mathscr{K}$ a minimal rational component of X. Then $l_{\mathscr{K}} = 2$ except the following cases:*

(i) $(3) \subset \mathbb{P}^4$ *a hypersurface of degree* 3.

(ii) $(2) \cap (2) \subset \mathbb{P}^5$ *a complete intersection of two hyperquadrics.*

(iii) $G(2,5) \cap (1)^3 \subset \mathbb{P}^6$ *a* 3*-dimensional linear section of* $G(2,5)$.

(iv) $LG(3,6)$ *a Lagrangian Grassmannian.*

(v) $G(3,6)$ *a Grassmannian.*

(vi) S_5 *a spinor variety.*

(vii) $E_7(\omega_1)$ *a rational homogeneous manifold of type* E_7.

Furthermore in the cases (i) $-$ (vii) *we have* $l_{\mathscr{K}} = 3$.

Proof. According to the assumption that $3i_X = 2n$, n is 3, 6, or at least 9. If $n = 3$, X is a del Pezzo 3-fold. So Remark 5.4.7 implies that $(l_{\mathscr{K}}, d_{\mathscr{K}}) = (3,1)$. Hence X is isomorphic to one of the manifolds listed in (i), (ii) or (iii) by the Fujita-Iskovskikh's classification result [Fuj80I, Fuj80II, Fuj81I]. In the case where $n = 6$, we have $l_{\mathscr{K}} = 2$ or X is $LG(3,6)$ by Theorem 5.6.4.

From here, we make the assumption $n \geq 9$. In this case, we have $2i_X > n+1$. So $d_{\mathscr{K}} = 1$, that is, X is covered by lines. By Proposition 5.3.1 the variety of minimal rational tangents $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is smooth irreducible and non-degenerate. It follows from our assumption $2p \geq n-1$. Hence $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is a non-degenerate irreducible projective manifold of dimension $\frac{2}{3}n - 2$. By Zak's theorem on linear normality, a classification of Severi varieties [Zak93] and the assumption that $n \geq 9$, $S^1\mathscr{C}_x = \mathbb{P}_*(T_xX)$ or $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is isomorphic to the Segre product $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, the Grassmann variety $G(2,6) \subset \mathbb{P}^{14}$ or E_6-variety $E_6(\omega_1) \subset \mathbb{P}^{26}$. In the former case Theorem 5.4.5 implies that $l_{\mathscr{K}} = 2$. So we assume the latter holds. Remark that the above Segre variety, Grassmann variety and E_6-variety are varieties of minimal rational tangents of $G(3,6)$, S_5 and $E_7(\omega_1)$ respectively (For example, see [HM02, Eli02, LM03]). By Theorem 5.2.4, X is isomorphic to one of these varieties. In this case, since $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is a Severi variety, $S^1\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ is a hypersurface [Zak93]. This implies $d_2 = n-1$ [HK05, Theorem 3.14]. Hence $l_{\mathscr{K}} = 3$.

□

Corollary 5.6.12. *Let X be a prime Fano n-fold of Picard number* 1 *with $i_X = \frac{2}{3}n$ and $\mathscr{K}$ a minimal rational component of X. Assume that $n \geq 6$. Then the following are equivalent.*

(i) $l_{\mathscr{K}} \neq 2$.

(ii) $l_{\mathscr{K}} = 3$.

(iii) $X \subset \mathbb{P}(H^0(X, \mathscr{O}_X(1)))$ *is not conic-connected.*

(iv) *The dimension of the secant variety* $S^1X \subset \mathbb{P}(H^0(X, \mathscr{O}_X(1)))$ *is* $2n+1$.

(v) *The variety of minimal rational tangents* $\mathscr{C}_x \subset \mathbb{P}_*(T_xX)$ *at a general point is a Severi variety.*

(vi) $X \subset \mathbb{P}(H^0(X, \mathscr{O}_X(1)))$ *is projectively equivalent to one of the manifolds listed in Theorem 5.6.11* (iv) – (vii).

Proof. By the above theorem and its proof, (i), (ii), (v) and (vi) are equivalent to each other. In general, if $X \subset \mathbb{P}^N$ is conic-connected, then the dimension of the secant variety S^1X is smaller than $2n+1$ (see [IR08, Proposition 3.2] and [Rus09, Theorem 2.1]). Hence (iv) $\Rightarrow$ (iii) holds. (iii) $\Rightarrow$ (i) follows from Lemma 5.6.7. Finally, (vi) $\Rightarrow$ (iv) comes from [Kon88]. $\square$

Remark 5.6.13. Corollary 5.6.12 and Theorem 5.1.1 implies that $i_X = \frac{2}{3}n$ is also a boundary of conic-connectedness and defectiveness of the secant variety (c.f. Remark 2.3.3):

Property	$i_X > \frac{2}{3}n$	$i_X = \frac{2}{3}n$	$i_X = \frac{2}{3}n$
$l_{\mathscr{K}}$	2	2	3
Conic-connectedness	Yes	Yes	No
Defectiveness of the secant variety	Yes	Yes	No

5.7 Lengths of Fano manifolds admitting the structures of double covers

Let X be a Fano n-fold with $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathscr{O}_X(1)]$, where $\mathscr{O}_X(1)$ is ample and $n := \dim X \geq 3$. In this section, we assume that X is a double cover of a projective manifold $\pi : X \to Y$. Barth-type Theorem [Laz80] implies $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$ and $\pi^*\mathscr{O}_Y(1) \cong \mathscr{O}_X(1)$, where $\mathscr{O}_Y(1)$ is the ample generator of the Picard group of Y. It follows from the ramification formula of the branched cover that Y is a Fano manifold. We denote by $B \in |\mathscr{O}_Y(b)|$ the branch divisor of π and by $\mathscr{R}_1$ the family of rational curves of degree 1 $\mathrm{RatCurves}^n_1(X)$. We assume that $\mathscr{R}_1$ is a dominating family. Then we can define the k-th locus $\mathrm{loc}^k_{\mathscr{R}_1}(x)$ and the length with respect to $\mathscr{R}_1$ as in Definition 5.4.1 and Definition 5.4.2.

Proposition 5.7.1. *Let X and $\mathscr{R}_1$ be as in above. Then the following holds.*

(i) *For general $x_1, x_2 \in X$, $\pi(\mathrm{loc}^1_{\mathscr{R}_1}(x_1)) \cap \pi(\mathrm{loc}^1_{\mathscr{R}_1}(x_2)) \neq \phi$ if and only if X is 2-connected by $\mathscr{R}_1$.*

(ii) *Under the assumption $\mathscr{O}_Y(1)$ is spanned, X is 2-connected by $\mathscr{R}_1$ if and only if for general points $y_1, y_2 \in Y$ there exists curves $l_1 \ni y_1, l_2 \ni y_2$ on Y such that $l_1 \cap l_2 \neq \phi$, $\mathscr{O}_Y(1).l_i = 1$ and $\mathrm{length}_q(B \cap l_i) \equiv 0 \mod 2$ for any $q \in Y$ and $i = 1, 2$.*

Proof. (i) The "only if" part is trivial. We show the converse. Let x_1, x_2 be points on X which are not on the ramification locus of π and we set $y_2 := \pi(x_2)$. Then we have $\pi^{-1}(y_2) = \{x_2, x_2'\}$. We assume there exists a point $z \in \pi(\mathrm{loc}^1_{\mathscr{R}_1}(x_1)) \cap \pi(\mathrm{loc}^1_{\mathscr{R}_1}(x_2))$. Then there exists a curve $[l_{x_i}] \in \mathscr{R}_1$ such that $x_i \in l_{x_i}$ and $z \in \pi(l_{x_i})$ for $i = 1, 2$. Since $\pi(l_{x_2}) \subset Y$ is a curve of degree 1, $\pi^{-1}(\pi(l_{x_2}))$ is a curve of degree 2. It follows from the inclusion $l_{x_i} \subset \pi^{-1}(\pi(l_{x_i}))$ that there exists a curve $[l_{x_2'}] \in \mathscr{R}_{1,x_2'}$ such that $\pi^{-1}(\pi(l_{x_2})) = l_{x_2} \cup l_{x_2'}$. Our assumption implies that $l_{x_1} \cap l_{x_2} \neq \phi$ or $l_{x_1} \cap l_{x_2'} \neq \phi$. So x_2 or x_2' is contained in $\mathrm{loc}^2_{\mathscr{R}_1}(x_1)$. This means $\pi|_{\mathrm{loc}^2_{\mathscr{R}_1}(x_1)} : \mathrm{loc}^2_{\mathscr{R}_1}(x_1) \to Y$ is dominant. Since $\pi|_{\mathrm{loc}^2_{\mathscr{R}_1}(x_1)}$ is proper, it is surjective. Hence we see $X = \mathrm{loc}^2_{\mathscr{R}_1}(x_1)$.

(ii) Suppose that $\mathscr{O}_Y(1)$ is spanned. Let l be a rational curve on Y satisfying $\mathscr{O}_Y(1).l = 1$. $\pi^{-1}(l)$ is denoted by C. From (i), it is sufficient to show the following claim.

Claim 5.7.2. *C is reducible if and only if $\mathrm{length}_q(B \cap l) \equiv 0 \mod 2$ for any $q \in Y$.*

For the double cover $\pi : X \to Y$, we have $\pi_* \mathscr{O}_X \cong \mathscr{O}_Y \oplus L^{-1}$, where L is an ample line bundle on Y which satisfies $L^{\otimes 2} \cong \mathscr{O}_Y(B)$. Furthermore there exists a morphism $X \hookrightarrow \mathbb{L} := \mathrm{Spec}(\mathrm{Sym} L^{-1})$ over Y. Since X is a divisor on $\mathbb{L}$, we can obtain the defining equation of X on $\mathbb{L}$. In particular, we see that there exists a global section $s \in \Gamma(C, {\pi_C}^* L_l)$ such that $s^2 = {\pi_C}^* \phi$, where $\phi \in \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(b))$ and $(\phi = 0) = l \cap B$ as divisors of l. We may assume that π_C is unramified at $\infty \in \mathbb{P}^1$. Then we see C is reducible if and only if $\pi_C^{-1}(\mathbb{A}^1)$ is reducible. Without loss of generality, we may assume that $\phi = (x - a_1 y) \cdots (x - a_b y)$, where $a_i \in \mathbb{C}$ and $\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(b)) \cong \bigoplus_{i=0}^b \mathbb{C} x^i y^{b-i}$. Furthermore we may assume $\pi_C^{-1}(\mathbb{A}^1) = (s^2 = (x - a_1) \cdots (x - a_b)) \subset \mathbb{A}^2$. Thus C is reducible if and only if the cardinality $\#\{j | a_j = a_i\} \equiv 0 \mod 2$ for any i. Hence we obtain our assertion. $\square$

Corollary 5.7.3. *Let X, Y and $\mathscr{R}_1$ be as in above. If $Y = \mathbb{P}^n$ and $n \geq b$, then X is 2-connected by $\mathscr{R}_1$.*

Proof. There exists a standard rational curve $f : \mathbb{P}^1 \to X$ such that $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^p \oplus \mathscr{O}^{n-1-p}$. By the ramification formula, the Fano index i_X of X is equal to $n+1-\frac{b}{2}$. It follows from the assumption $n \geq b$ that $i_X > \frac{n+1}{2}$. Hence we have deg $f_*(\mathbb{P}^1) = 1$ and $p = n - \frac{b}{2} - 1$. For general two points $x_1, x_2 \in X$,

$$\dim \pi(\mathrm{loc}^1_{\mathscr{R}_1}(x_1)) + \dim \pi(\mathrm{loc}^1_{\mathscr{R}_1}(x_2)) - \dim \mathbb{P}^n = 2(n - \frac{b}{2}) - n = n - b \geq 0. \tag{5.2}$$

Hence Proposition 5.7.1 implies that X is 2-connected by $\mathscr{R}_1$.

□

Corollary 5.7.4. *Let X, Y and $\mathscr{R}_1$ be as in above. If $Y \subset \mathbb{P}^{n+1}$ is a hypersurface of degree d and $n \geq 2d + b - 1$, then X is 2-connected by $\mathscr{R}_1$.*

Proof. By the same argument as in Proposition 5.7.3, we see that there exists a standard rational curve $f : \mathbb{P}^1 \to X$ such that $\deg f_*(\mathbb{P}^1) = 1$ and $p = n - \frac{b}{2} - d$. For general two points $x_1, x_2 \in X$,

$$\dim \pi(\mathrm{loc}^1_{\mathscr{R}_1}(x_1)) + \dim \pi(\mathrm{loc}^1_{\mathscr{R}_1}(x_2)) - \dim \mathbb{P}^{n+1} \geq 0. \tag{5.3}$$

Thus X is 2-connected by $\mathscr{R}_1$.

□

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